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On the behavior of solutions of neutral impulsive difference equations of second order*

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Abstract. The work embodied in this paper is the study of oscillation properties of a class of second order neutral impulsive difference equations with constant coefficients of the form:

$$\begin{cases} \Delta^2[u(n) - pu(n-\alpha)] + qu(n-\beta) = 0, & n \neq m_j \\ \underline{\Delta}[\Delta(u(m_j-1) - pu(m_j-\alpha-1))] + ru(m_j-\beta-1) = 0, & j \in \mathbb{N} \end{cases}$$

for $p \in \mathbb{R}$. In addition, an effort has been made here to apply the constant coefficient results to nonlinear impulsive difference equations with variable coefficients of the form:

$$\begin{cases} \Delta^2[u(n) - p(n)f(u(n-\alpha))] + q(n)h(u(n-\beta)) = 0, & n \neq m_j \\ \underline{\Delta}[\Delta(u(m_j-1) - p(m_j-1)f(u(m_j-\alpha-1)))] & \\ \quad + r(m_j-1)h(u(m_j-\beta-1)) = 0, & j \in \mathbb{N} \end{cases}$$

for $p(n) \geq 1$. Our method suggests the explicit structure of the solution of impulsive difference equations.

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1. Introduction

Consider a second order neutral impulsive difference equations of the form:

$$(E_1)\begin{cases} \Delta^2[u(n) - pu(n-\alpha)] + qu(n-\beta) = 0, & n \neq m_j \quad (1) \\ \underline{\Delta}[\Delta(u(m_j-1) - pu(m_j-\alpha-1))] + ru(m_j-\beta-1) = 0, & j \in \mathbb{N}, \quad (2) \end{cases}$$

where α, β are positive integers, $p \in \mathbb{R} - \{0\}$, $q, r \in \mathbb{R}$ and $m_j, j \in \mathbb{N}$ are the discrete moments of impulsive effect such that $m_1 < m_2 < \cdots < m_j$ with the properties $\lim_{j\to\infty} m_j = \infty$ and $1 \leq \max\{m_j - m_{j-1}\} < \infty$. Here, Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, and $\underline{\Delta}$ is the difference operator defined

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by $\underline{\Delta} u(m_j - 1) = u(m_j) - u(m_j - 1)$. As in [13], we may expect possible solution of (E_1) as

$$u(n) = \lambda^n A^{i(n_0,n)},\ n_0 > \rho = \max\{\alpha, \beta\}, \tag{3}$$

where $i(n_0, n) = j$ = the number of impulsive points $m_j, j \in \mathbb{N}$ between n_0 and n and $A \neq 0$ is a real number called the *pulsatile constant*. In this work, our objective is to study (E_1) through (3).

In [12] and [11], Tripathy and Chhatria have discussed the oscillation properties of first order nonlinear neutral impulsive difference equations of the form:

$$(E_h) \begin{cases} \Delta[u(n) + p(n)u(n-\tau)] + q(n)h(u(n-\sigma)) = 0,\ n \neq m_j,\ j \in \mathbb{N} \\ \underline{\Delta}[u(m_j - 1) + p(m_j - 1)u(m_j - \tau - 1)] \\ \quad + r(m_j - 1)h(u(m_j - \sigma - 1)) = 0 \end{cases}$$

and

$$(E_{nh}) \begin{cases} \Delta[u(n) + p(n)u(n-\tau)] + q(n)h(u(n-\sigma)) = g(n),\ n \neq m_j,\ j \in \mathbb{N} \\ \underline{\Delta}[u(m_j - 1) + p(m_j - 1)u(m_j - \tau - 1)] \\ \quad + r(m_j - 1)h(u(m_j - \sigma - 1)) = e(m_j - 1), \end{cases}$$

where $h \in C(\mathbb{R}, \mathbb{R})$ with $xu(x) > 0$ for $0 \neq x \in \mathbb{R}$. The authors have studied (E_h) by considering the sublinear and superlinear properties of h. But, in the study of (E_{nh}), h could be linear, sublinear or superlinear upon a suitable choice of the forcing term $g(n)$. Often, we feel it is interesting to study neutral equations by means of their characteristic equations and the application of results to nonlinear equations with variable coefficients, and this fact has been established in [13].

The motivation of this work has come from the work of [13] and hence we aim to study (E_1) to establish the oscillation and nonoscillation properties by using pulsatile constants as defined by (3). Also, an effort has been made here to study a nonlinear impulsive system of the form:

$$(E_2) \begin{cases} \Delta^2[u(n) - p(n)f(u(n-\alpha))] + q(n)h(u(n-\beta)) = 0, \quad n \neq m_j \\ \underline{\Delta}[\Delta(u(m_j - 1) - p(m_j - 1)f(u(m_j - \alpha - 1)))] \\ \quad + r(m_j - 1)h(u(m_j - \beta - 1)) = 0, \qquad j \in \mathbb{N} \end{cases}$$

by using the characteristic equation of (E_1) with $f, h \in C(\mathbb{R}, \mathbb{R})$. Emphasis may be given to our state of the art that the nonlinear impulsive system (E_2) can be studied by using the characteristic equation of (E_1). In this direction, we refer the reader to some of the related works ([1], [3-8], [14], [15]) and monographs [2], [9], [10] and the references cited there in.

Unlike the works [5], [6], [7] and [8], our aim here is to represent the impulsive solution $u(m_j)$, $j \in \mathbb{N}$ satisfying another impulsive neutral difference equation but not by the impulsive conditions only. This is we believe inevitable when $u(n)$ is a solution of the so-called neutral equations without impulse. So, in our discussion, a neutral equation without impulse along with a neutral equation in impulse form impulsive neutral difference equations. Hence, the study of these types of impulsive system may lead and develop other directions for the researchers working in this area. The importance of this fact lies in the fact that our method suggests the structure of the solution of the system which we do not see in other methods in the literature.

Definition 1. *By a solution of* (E_1) *we mean a real valued function* $u(n)$ *defined on* $\mathbb{N}(n_0-\rho)=\{n_0-\rho,,\ldots,n_0,n_0+1,\ldots\}$ *satisfing* (E_1) *for* $n\geq n_0$ *with the initial conditions* $u(i)=\phi(i)$, $i=n_0-\rho,\ldots,n_0$, *where* $\phi(i)$, $i=n_0-\rho,\ldots,n_0$ *are given.*

Definition 2. *A nontrivial solution* $u(n)$ *of* (E_1) *is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.* (E_1) *is said to be oscillatory if all of its solutions are oscillatory.*

Definition 3. *A solution* $u(n)$ *of* (E_1) *is said to be oscillatory if there exists an integer* $N>0$ *such that* $u(n+1)u(n)\leq 0$ *for all* $n\geq N$*; otherwise,* $u(n)$ *is said to be nonoscillatory.*

2. Characterization of (E_1)

This section deals with the oscillation and nonoscillation properties of solutions of the system (E_1) through its associated characteristic equation when (3) holds. We need the following calculation for our purpose:

- $i(n_0,m_j)-i(n_0,m_j-1)=1$,
- $i(n-\beta,n)=l_1$ is the number of impulsive points between $n-\beta$ and n,
- $i(n-\alpha,n)=l_2$ is the number of impulsive points between $n-\alpha$ and n.

Theorem 1. *Let* $\alpha>\beta>0$ *and* $r\neq q\neq 0$. *Then* (E_1) *admits an oscillatory solution in the impulsive form* (3) *if and only if the algebraic equation*

$$\left[\frac{1}{\lambda}\left(1-\frac{r}{q}\right)+\frac{r}{q}\right]^{l_1}(\lambda-1)^2-p\lambda^{-\alpha}\left[\frac{1}{\lambda}\left(1-\frac{r}{q}\right)+\frac{r}{q}\right]^{l_1-l_2}(\lambda-1)^2+q\lambda^{-\beta}=0 \quad (4)$$

has at least one real root λ *with* $\lambda<1-\frac{q}{r}$ *for* $\frac{r}{q}>1$ *and* $\lambda>1-\frac{q}{r}$ *for* $\frac{r}{q}<1$.

Proof. Let $u(n)$ be a regular nontrivial solution of the system (E_1) such that $u(n)=\lambda^n A^{i(n_0,n)}$, $n>n_0>\rho$. Then (1) becomes

$$\Delta[u(n+1)-u(n)-pu(n+1-\alpha)+pu(n-\alpha)]+qu(n-\beta)=0,$$

that is,

$$\begin{aligned} &u(n+2)-2u(n+1)+u(n)-pu(n+2-\alpha)\\ &\quad+2pu(n+1-\alpha)-pu(n-\alpha)+qu(n-\beta)=0. \end{aligned}$$

Using (3) in the preceding equation, we obtain

$$\begin{aligned} &\lambda^{n+2}A^{i(n_0,n+2)}-2\lambda^{n+1}A^{i(n_0,n+1)}+\lambda^{n}A^{i(n_0,n)}-p\lambda^{n+2-\alpha}A^{i(n_0,n+2-\alpha)}\\ &\quad+2p\lambda^{n+1-\alpha}A^{i(n_0,n+1-\alpha)}-p\lambda^{n-\alpha}A^{i(n_0,n-\alpha)}+q\lambda^{n-\beta}A^{i(n_0,n-\beta)}=0, \end{aligned}$$

that is,

$$\lambda^2 A^{i(n_0,n+2)} - 2\lambda A^{i(n_0,n+1)} + A^{i(n_0,n)} - p\lambda^{2-\alpha} A^{i(n_0,n+2-\alpha)} \\ + 2p\lambda^{1-\alpha} A^{i(n_0,n+1-\alpha)} - p\lambda^{-\alpha} A^{i(n_0,n-\alpha)} + q\lambda^{-\beta} A^{i(n_0,n-\beta)} = 0.$$

Therefore,

$$\lambda^2 A^{i(n_0,n+2)-i(n_0,n-\beta)} - 2\lambda A^{i(n_0,n+1)-i(n_0,n-\beta)} + A^{i(n_0,n)-i(n_0,n-\beta)} \\ - p\lambda^{2-\alpha} A^{i(n_0,n+2-\alpha)-i(n_0,n-\beta)} + 2p\lambda^{1-\alpha} A^{i(n_0,n+1-\alpha)-i(n_0,n-\beta)} \\ - p\lambda^{-\alpha} A^{i(n_0,n-\alpha)-i(n_0,n-\beta)} + q\lambda^{-\beta} = 0. \tag{5}$$

We note that $i(n_0,n) - i(n_0,n-\beta) = i(n-\beta,n) = l_1$, $i(n_0,n+1) - i(n_0,n-\beta) = i(n-\beta,n+1) = l_1$ and $i(n_0,n+2) - i(n_0,n-\beta) = i(n-\beta,n+2) = l_1$. Also, it is true that $i(n_0,n-\alpha) - i(n_0,n-\beta) = -i(n-\alpha,n-\beta) = -[i(n-\alpha,n) - i(n-\beta,n)] = l_1 - l_2$, $i(n_0,n+1-\alpha) - i(n_0,n-\beta) = -i(n+1-\alpha,n-\beta) = -[i(n+1-\alpha,n) - i(n+1-\beta,n)] = l_1 - l_2$ and $i(n_0,n+2-\alpha) - i(n_0,n-\beta) = -i(n+2-\alpha,n-\beta) = -[i(n+2-\alpha,n) - i(n+2-\beta,n)] = l_1 - l_2$. Upon using the above relations in (5), we get

$$\lambda^2 A^{l_1} - 2\lambda A^{l_1} + A^{l_1} - p\lambda^{2-\alpha} A^{l_1-l_2} + 2p\lambda^{1-\alpha} A^{l_1-l_2} - p\lambda^{-\alpha} A^{l_1-l_2} + q\lambda^{-\beta} = 0,$$

that is,

$$(\lambda-1)^2 A^{l_1} - p\lambda^{-\alpha}(\lambda-1)^2 A^{l_1-l_2} + q\lambda^{-\beta} = 0. \tag{6}$$

Again, using (3) in (2), we get

$$\underline{\Delta}[(\lambda-1)\lambda^{m_j-1} A^{i(n_0,m_j-1)} - p(\lambda-1)\lambda^{m_j-\alpha-1} A^{i(n_0,m_j-\alpha-1)}] \\ + r\lambda^{m_j-\beta-1} A^{i(n_0,m_j-\beta-1)} = 0,$$

that is,

$$(\lambda-1)[\lambda^{m_j} A^{i(n_0,m_j)} - \lambda^{m_j-1} A^{i(n_0,m_j-1)} - p\lambda^{m_j-\alpha} A^{i(n_0,m_j-\alpha)} \\ + p\lambda^{m_j-\alpha-1} A^{i(n_0,m_j-\alpha-1)}] + r\lambda^{m_j-\beta-1} A^{i(n_0,m_j-\beta-1)} = 0$$

implies that

$$(\lambda-1)[A^{i(n_0,m_j)} - \lambda^{-1} A^{i(n_0,m_j-1)} - p\lambda^{-\alpha} A^{i(n_0,m_j-\alpha)} + p\lambda^{-\alpha-1} A^{i(n_0,m_j-\alpha-1)}] \\ + r\lambda^{-\beta-1} A^{i(n_0,m_j-\beta-1)} = 0.$$

Because $i(n_0,m_j) - i(n_0,m_j-1) = 1$, the above relation reduces to

$$(\lambda-1)[A^{1+i(n_0,m_j-1)} - \lambda^{-1} A^{i(n_0,m_j-1)} - p\lambda^{-\alpha} A^{1+i(n_0,m_j-\alpha-1)} \\ + p\lambda^{-\alpha-1} A^{i(n_0,m_j-\alpha-1)}] + r\lambda^{-\beta-1} A^{i(n_0,m_j-\beta-1)} = 0,$$

which is equivalent to

$$(\lambda-1)[\lambda A^{1+i(n_0,m_j-1)} - A^{i(n_0,m_j-1)} - p\lambda^{1-\alpha} A^{1+i(n_0,m_j-\alpha-1)} \\ + p\lambda^{-\alpha} A^{i(n_0,m_j-\alpha-1)}] + r\lambda^{-\beta} A^{i(n_0,m_j-\beta-1)} = 0$$

and hence

$$(\lambda-1)(\lambda A-1)A^{i(n_0,m_j-1)}-p(\lambda-1)\lambda^{-\alpha}(\lambda A-1)A^{i(n_0,m_j-\alpha-1)} \\ +r\lambda^{-\beta}A^{i(n_0,m_j-\beta-1)}=0,$$

that is,

$$(\lambda-1)(\lambda A-1)A^{i(n_0,m_j-1)-i(n_0,m_j-\beta-1)}-p(\lambda-1)\lambda^{-\alpha}(\lambda A-1) \\ \times A^{i(n_0,m_j-\alpha-1)-i(n_0,m_j-\beta-1)}+r\lambda^{-\beta}=0. \tag{7}$$

Since $i(n_0,m_j-1)-i(n_0,m_j-\beta-1)=i(m_j-\beta-1,m_j-1)=l_1$ and $i(n_0,m_j-\alpha-1)-i(n_0,m_j-\beta-1)=-i(m_j-\alpha-1,m_j-\beta-1)=-[i(m_j-\alpha-1,m_j-1)-i(m_j-\beta-1,m_j-1)]=l_1-l_2$, then (7) reduces to

$$(\lambda-1)(\lambda A-1)A^{l_1}-p(\lambda-1)\lambda^{-\alpha}(\lambda A-1)A^{l_1-l_2}+r\lambda^{-\beta}=0. \tag{8}$$

If we choose $A=\frac{1}{\lambda}(1-\frac{r}{q})+\frac{r}{q}$ for $\lambda\not\equiv 0$, then it is easy to see that (8) reduces to (6), which is the same as (4). Moreover, (4) is the required characteristic equation for (E_1). If $u(n)$ is an oscillatory solution of (E_1) with the pulsatile constant $A=\frac{1}{\lambda}(1-\frac{r}{q})+\frac{r}{q}<0$, where $\lambda<1-\frac{q}{r}$ for $\frac{r}{q}>1$ or $\lambda>1-\frac{q}{r}$ for $\frac{r}{q}<1$, then λ satisfies the characteristic equation (4). Conversely, let $\lambda=\lambda_1$ be a real root of (4) with $\lambda_1<1-\frac{q}{r}$ for $\frac{r}{q}>1$ or $\lambda_1>1-\frac{q}{r}$ for $\frac{r}{q}<1$. Then (E_1) admits an oscillatory solution $u(n)=\lambda_1^nA^{i(n_0,n)}$ with a pulsatile constant $A=\frac{1}{\lambda_1}(1-\frac{r}{q})+\frac{r}{q}<0$. This completes the proof of the theorem. $\square$

Theorem 2. *Let the assumptions of Theorem 1 hold. Then (E_1) admits an eventually positive solution in the form of (3) if and only if (4) has at least one real root λ with $\lambda>1-\frac{q}{r}$ for $\frac{r}{q}>1$ and $\lambda<1-\frac{q}{r}$ for $\frac{r}{q}<1$.*

Proof. The proof of the theorem follows from the proof of Theorem 1 and hence the details are omitted. $\square$

Corollary 1. *Let $q,r\in\mathbb{R}-\{0\}$, $\beta=\alpha\neq 0$ or $\beta=0\neq\alpha$ hold. Then the conclusions of Theorem 1 and Theorem 2 hold.*

Proof. Proceeding as in the proof of Theorem 1, we get (6) and (8). Assume that $\beta=\alpha\neq 0$. It is easy to calculate that $i(n_0,n)-i(n_0,n-\beta)=i(n-\beta,n)=l_1$ $=i(n_0,n+1)-i(n_0,n-\beta)=i(n-\beta,n+1)$ and $i(n_0,n+2)-i(n_0,n-\beta)=i(n-\beta,n+2)=l_1$. Also, $i(n_0,n-\alpha)-i(n_0,n-\beta)=-i(n-\alpha,n-\beta)=0=i(n_0,n+1-\alpha)-i(n_0,n-\beta)=-i(n+1-\alpha,n-\beta)$ and $i(n_0,n+2-\alpha)-i(n_0,n-\beta)=-i(n+2-\alpha,n-\beta)=0$. Therefore, (6) becomes

$$(\lambda-1)^2A^{l_1}-p(\lambda-1)^2\lambda^{-\beta}+q\lambda^{-\beta}=0. \tag{9}$$

Similarly, (8) gives

$$(\lambda-1)(\lambda A-1)A^{l_1}-p(\lambda-1)(\lambda A-1)\lambda^{-\beta}+r\lambda^{-\beta}=0. \tag{10}$$

If we choose $A = \frac{1}{\lambda}(1-\frac{r}{q})+\frac{r}{q}$, then (10) reduces to (9). The rest of the proof follows from Theorem 1.

Next, we assume that $\beta = 0 \neq \alpha$. In this case, $i(n_0,n) - i(n_0,n-\beta) = 0 = i(n_0,n+1) - i(n_0,n-\beta)$ and $i(n_0,n+2) - i(n_0,n-\beta) = 0$. Also, $i(n_0,n-\alpha) - i(n_0,n-\beta) = -i(n-\alpha,n) = -l_2 = i(n_0,n+1-\alpha) - i(n_0,n-\beta)$ and $i(n_0,n+2-\alpha) - i(n_0,n-\beta) = -l_2$. Hence, (6) becomes

$$(\lambda-1)^2 - p(\lambda-1)^2\lambda^{-\alpha}A^{-l_2} + q = 0. \tag{11}$$

Similarly, it follows from (8) that

$$(\lambda-1)(\lambda A-1) - p(\lambda-1)(\lambda A-1)\lambda^{-\alpha}A^{-l_2} + r = 0. \tag{12}$$

If we choose $A = \frac{1}{\lambda}(1-\frac{r}{q})+\frac{r}{q}$, then (12) reduces to (11). The rest of the proof is similar to Theorem 1. Hence the corollary is proved. □

Corollary 2. *In Theorem 1, let $q = r \neq 0$. Then*

(i) for $p \in (0,\infty)$ and $q \in (-\infty,0)$, (E_1) admits an oscillatory solution in the impulsive form (3) *if and only if $\lambda < 0$ is a root of the characteristic equation of (E_1);*

(ii) for $p \in (-\infty,0)$ and $q \in (0,\infty)$, (E_1) admits an eventually positive solution in the impulsive form (3) *if and only if $\lambda > 0$ is a root of the characteristic equation of (E_1).*

Proof. Proceeding as in the proof of Theorem 1, we obtain

$$\begin{aligned}(\lambda-1)^2A^{l_1} - p\lambda^{-\alpha}(\lambda-1)^2A^{l_1-l_2} + q\lambda^{-\beta} = 0,\\ (\lambda-1)(\lambda A-1)A^{l_1} - p(\lambda-1)\lambda^{-\alpha}(\lambda A-1)A^{l_1-l_2} + q\lambda^{-\beta} = 0,\end{aligned}$$

which is equivalent to saying that $A = 1$ and

$$(\lambda-1)^2 - p\lambda^{-\alpha}(\lambda-1)^2 + q\lambda^{-\beta} = 0. \tag{13}$$

Therefore, if $\lambda \in (-\infty,0)$ is a root of (13), then (E_1) admits an oscillatory solution of the form (3). Conversely, let $u(n) = \lambda^n A^{i(n_0,n)}$ be a solution of (E_1). If we take

$$f(\lambda) = (\lambda-1)^2(1-p\lambda^{-\alpha}) + q\lambda^{-\beta},$$

then it follows that $f(0) = -\infty$ for $p \in (0,\infty)$, $q \in (-\infty,0)$ and $f(-\infty) = \infty$. Thus, there exists a negative real root $\lambda \in (-\infty,0)$ and hence λ satisfies the characteristic equation (13).

If $\lambda \in (0,\infty)$ is a root of (13), then (E_1) admits an eventually positive solution of the form (3). Conversely, suppose that $u(n) = \lambda^n A^{i(n_0,n)}$ is a solution of (E_1). Clearly, $f(0) = \infty$, $f(1) = q > 0$ for $q \in (0,\infty)$ and $f(\infty) = \infty$ implies that there exists a real $\lambda \in (0,\infty)$ such that λ satisfies the characteristic equation (13). This completes the proof. □

Remark 1. *If we can write*

$$f(\lambda) = (\lambda - 1)^2 - \lambda^{-\alpha}[p(\lambda - 1)^2 - q\lambda^{\alpha-\beta}],$$

then it is easy to see that $f(0) = \infty$, $f(-a) > 0$, $a > 0$ and $f(-\infty) = \infty$ for $p, q \in (-\infty, 0)$ with α even and β odd. Therefore, $f(\lambda)$ has no real root in $(-\infty, 0]$.

Remark 2. *From Corollary 2, we may note that $\lambda = 1$ if and only if $A = 1$, that is, (E_1) admits an eventually positive solution.*

Theorem 3. *Let $\alpha > \beta > 0$ and $r = q = 0$. Then*

(i) for $p \in (-\infty, 0)$ with odd α, (E_1) admits an oscillatory solution if and only if $\lambda \in (-\infty, 0)$ is a root of the characteristic equation of (E_1);

(ii) for $p \in (0, \infty)$, (E_1) admits an eventually positive solution if and only if $\lambda \in (0, \infty)$ is a root of the characteristic equation of (E_1).

Proof. Let $u(n)$ be a nontrivial solution of (E_1) in the impulsive form (3). Proceeding as in the proof of Theorem 1, (1) becomes

$$\begin{aligned}&\lambda^2 A^{i(n_0,n+2)} - 2\lambda A^{i(n_0,n+1)} + A^{i(n_0,n)}\\ &\quad - p\lambda^{2-\alpha} A^{i(n_0,n+2-\alpha)} + 2p\lambda^{1-\alpha} A^{i(n_0,n+1-\alpha)} - p\lambda^{-\alpha} A^{i(n_0,n-\alpha)} = 0,\end{aligned}$$

that is,

$$(\lambda - 1)^2 A^{i(n_0,n)} - p\lambda^{-\alpha}(\lambda - 1)^2 A^{i(n_0,n-\alpha)} = 0.$$

Similarly, we from (2) obtain that

$$\begin{aligned}&(\lambda - 1)[A^{i(n_0,m_j)} - \lambda^{-1} A^{i(n_0,m_j-1)}\\ &\quad - p\lambda^{-\alpha} A^{i(n_0,m_j-\alpha)} + p\lambda^{-\alpha-1} A^{i(n_0,m_j-\alpha-1)}] = 0,\end{aligned}$$

that is,

$$\begin{aligned}&(\lambda - 1)[\lambda A^{1+i(n_0,m_j-1)} - A^{i(n_0,m_j-1)}\\ &\quad - p\lambda^{1-\alpha} A^{1+i(n_0,m_j-\alpha-1)} + p\lambda^{-\alpha} A^{i(n_0,m_j-\alpha-1)}] = 0\end{aligned}$$

implies that

$$(\lambda - 1)(\lambda A - 1)A^{i(n_0,m_j-1)} - p\lambda^{-\alpha}(\lambda - 1)(\lambda A - 1)A^{i(n_0,m_j-\alpha-1)} = 0.$$

Hence, the impulsive system reduces to

$$\begin{aligned}(\lambda - 1)^2 A^{i(n_0,n)} - p\lambda^{-\alpha}(\lambda - 1)^2 A^{i(n_0,n-\alpha)} &= 0,\\ (\lambda - 1)(\lambda A - 1)A^{i(n_0,m_j-1)} - p\lambda^{-\alpha}(\lambda - 1)(\lambda A - 1)A^{i(n_0,m_j-\alpha-1)} &= 0.\end{aligned}$$

Ultimately,

$$\begin{aligned}(\lambda - 1)^2 A^{j} - p\lambda^{-\alpha}(\lambda - 1)^2 A^{j-l_2} &= 0,\\ (\lambda - 1)(\lambda A - 1)A^{j} - p\lambda^{-\alpha}(\lambda - 1)(\lambda A - 1)A^{j-l_2} &= 0,\end{aligned}$$

where $i(n_0,n)=j$ and $i(n_0,n-\alpha)=i(n_0,n)-i(n-\alpha,n)=j-l_2$. Therefore, the above impulsive system can be viewed as

$$\begin{aligned}(\lambda-1)^2A^{l_2}-p\lambda^{-\alpha}(\lambda-1)^2&=0,\\(\lambda-1)(\lambda A-1)A^{l_2}-p\lambda^{-\alpha}(\lambda-1)(\lambda A-1)&=0,\end{aligned}$$

which is equivalent to saying that $A=1$ and hence the characteristic equation of the system (E_1) is

$$(\lambda-1)^2-p\lambda^{-\alpha}(\lambda-1)^2=0, \tag{14}$$

where either $\lambda=1$ or $\lambda=p^{\frac{1}{\alpha}}$.

Clearly, $p\in(-\infty,0)$ implies that $\lambda\in(-\infty,0)$ when α is odd. Therefore, (E_1) admits an oscillatory solution in the form of (3). Conversely, let us assume that $u(n)=\lambda_1^nA^{i(n_0,n)}$ is an oscillatory solution of (E_1). Since $A=1$, then $\lambda=\lambda_1<0$ and hence λ_1 satisfies the characteristic equation (14).

Similarly, $p\in(0,\infty)$ implies that $\lambda\in(0,\infty)$. Therefore, (E_1) admits an eventually positive solution in the form of (3). Conversely, assume that $u(n)=\lambda_1^nA^{i(n_0,n)}$ is an eventually positive solution of (E_1). Since $A=1$, then $\lambda>0$ and thus λ_1 satisfies the characteristic equation (14). Also, $p\in(0,\infty)$ and even α implies that $\lambda\in(0,\infty)$. Therefore, (E_1) admits an eventually positive solution in the form of (3). Conversely, assume that $u(n)=\lambda_1^nA^{i(n_0,n)}$ is an eventually positive solution of (E_1). Since $A=1$, then $\lambda>0$ and thus λ_1 satisfies the characteristic equation (14). This completes the proof of the theorem. □

Remark 3. *Indeed,* (14) *does not hold if* $p\in(-\infty,0),\lambda\in(-\infty,0)$ *and* α *is even.*

Theorem 4. *Let* $p,r\in\mathbb{R}-\{0\}$, $\alpha=\beta\neq0$, $q=0$ *and* $i(n-\alpha,n)=1$. *Then* (E_1) *admits an oscillatory solution if and only if the characteristic equation of* (E_1) *has complex roots.*

Proof. Let $u(n)$ be a nontrivial solution of (E_1) in the impulsive form (3). Proceeding as in the proof of Theorem 1 and from (1), it follows that

$$\begin{aligned}&\lambda^2A^{i(n_0,n+2)}-2\lambda A^{i(n_0,n+1)}+A^{i(n_0,n)}\\&\quad-p\lambda^{2-\alpha}A^{i(n_0,n+2-\alpha)}+2p\lambda^{1-\alpha}A^{i(n_0,n+1-\alpha)}-p\lambda^{-\alpha}A^{i(n_0,n-\alpha)}=0,\end{aligned}$$

which is equivalent to

$$(\lambda-1)^2A^{i(n_0,n)}-p\lambda^{-\alpha}(\lambda-1)^2A^{i(n_0,n-\alpha)}=0,$$

that is,

$$(\lambda-1)^2A^{i(n_0,n)-i(n_0,n-\alpha)}-p\lambda^{-\alpha}(\lambda-1)^2=0.$$

Similarly, we from (2) obtain that

$$\begin{aligned}&(\lambda-1)[A^{i(n_0,m_j)}-\lambda^{-1}A^{i(n_0,m_j-1)}-p\lambda^{-\alpha}A^{i(n_0,m_j-\alpha)}\\&\quad+p\lambda^{-\alpha-1}A^{i(n_0,m_j-\alpha-1)}]+r\lambda^{-1-\beta}A^{i(n_0,m_j-\beta-1)}=0,\end{aligned}$$

that is,

$$(\lambda-1)[\lambda A^{1+i(n_0,m_j-1)} - A^{i(n_0,m_j-1)} - p\lambda^{1-\alpha}A^{1+i(n_0,m_j-\alpha-1)} \\ + p\lambda^{-\alpha}A^{i(n_0,m_j-\alpha-1)}] + r\lambda^{-\beta}A^{i(n_0,m_j-\beta-1)} = 0.$$

As a result,due to $\alpha=\beta$,

$$(\lambda-1)(\lambda A-1)A^{i(n_0,m_j-1)-i(n_0,m_j-\alpha-1)} - p\lambda^{-\alpha}(\lambda-1)(\lambda A-1) \\ \times A^{i(n_0,m_j-\alpha-1)-i(n_0,m_j-\alpha-1)} + r\lambda^{-\alpha} = 0.$$

Therefore, the impulsive system becomes

$$(\lambda-1)^2A^{i(n_0,n)-i(n_0,n-\alpha)} - p\lambda^{-\alpha}(\lambda-1)^2 = 0,$$
$$(\lambda-1)(\lambda A-1)A^{i(n_0,m_j-1)-i(n_0,m_j-\alpha-1)} - p\lambda^{-\alpha}(\lambda-1)(\lambda A-1) + r\lambda^{-\alpha} = 0,$$

which is equivalent to

$$(E_4)\begin{cases} (\lambda-1)^2A - p\lambda^{-\alpha}(\lambda-1)^2 = 0 & (15)\\ (\lambda-1)(\lambda A-1)A - p\lambda^{-\alpha}(\lambda-1)(\lambda A-1) + r\lambda^{-\alpha} = 0, & (16)\end{cases}$$

where we have used the fact that $i(n-\alpha,n)=1$. For (15), if we consider that $\lambda\neq 1$, then $A=p\lambda^{-\alpha}$.Upon using this in (16), we find that $r=0$, which leads to a contradiction. Ultimately, $\lambda=1$. Following the same argument, we find a contradiction to $r=0$. Therefore, the characteristic equation of (E_1) has no real roots. Conversely, if $a+ib$ and $a-ib$ are the two complex roots of the characteristic of (E_1), then in the computation of $c_1(a+ib)^n+c_2(a-ib)^n$, we notice that

$$c_1(a+ib)^n + c_2(a-ib)^n = r^n[c_3\cos(n\theta)+c_4\sin(n\theta)],$$

when we put

$$a=r\cos\theta,\ b=r\sin\theta,\ r=\sqrt{a^2+b^2},\ \theta=\tan^{-1}(\frac{b}{a}),$$

and $c_3=c_1+c_2$, $c_4=ic_1-ic_2$. Therefore, a solution in the impulsive form oscillates. This completes the proof of the theorem. $\square$

Corollary 3. *Let $p,r\in\mathbb{R}-\{0\}$, $\beta=0=q$ and $i(n-\alpha,n)=1$. Then (E_1) admits an oscillatory solution if and only if the characteristic equation of (E_1) has complex roots.*

Remark 4. *If we denote*

$$H(\lambda)=\Big[\frac{1}{\lambda}\Big(1-\frac{r}{q}\Big)+\frac{r}{q}\Big]^{l_1}(\lambda-1)^2 - p\lambda^{-\alpha}\Big[\frac{1}{\lambda}\Big(1-\frac{r}{q}\Big)+\frac{r}{q}\Big]^{l_1-l_2}(\lambda-1)^2 + q\lambda^{-\beta},$$

then it is easy to verify that $H(1)=q>0$ and $H(\lambda)\to\infty$ as $\lambda\to\infty$ for $qr>0$. Further, if $\lambda=\lambda_0=1-\frac{q}{r}$, then

$$H(\lambda_0)=\Big(\frac{q}{r}\Big)^2\Big[\Big((\frac{r}{r-q})(\frac{q-r}{q})+\frac{r}{q}\Big)^{l_1} - p\Big(\frac{r-q}{q}\Big)^{-\alpha}\Big((\frac{r}{r-q})(\frac{q-r}{q})+\frac{r}{q}\Big)^{l_1-l_2}\Big] \\ + q\Big(\frac{r-q}{r}\Big)^{-\beta} > 0$$

for $r > q > 0$ implies that $H(\lambda)$ has no positive real root in $[1 - \frac{q}{r}, \infty)$. Hence, we have proved the following result:

Theorem 5. *Let $q, r > 0$ such that $r > q$. Then there exists an oscillatory solution for (E_1) in the form of (3) if and only if (4) has no positive real root in $[1 - \frac{q}{r}, \infty)$.*

Example 1. *Consider the system of impulsive difference equations*

$$(E_5)\begin{cases}\Delta^2[u(n) - pu(n-2)] + qu(n-1) = 0, & n \neq 3j,\ n > 2\\ \underline{\Delta}[\Delta(u(m_j - 1) - pu(m_j - 3))] + ru(m_j - 2) = 0, & j \in \mathbb{N},\end{cases}$$

where $p = -4$, $q = -6$, $r = 20$ and $m_j = 3j, j \in \mathbb{N}$. Let $l_1 = 1$ and $l_2 = 5$. Then from the characteristic equation of (E_5), we get $\lambda = 1.857142 > 1 - \frac{q}{r} = 1.3$ and $A = \frac{1}{\lambda}\left(1 - \frac{r}{q}\right) + \frac{r}{q} = -1$. Therefore, all assumptions of Theorem 1 are satisfied and hence (E_5) has an oscillatory solution $u(n) = (1.857142)^n(-1)^{i(2,n)}$. Clearly, when (E_5) is without impulse then it has a nonoscillatory solution $y(n) = (1.857142)^n$. Thus, impulse plays an important role.

Example 2. *Consider the system of impulsive difference equations*

$$(E_6)\begin{cases}\Delta^2[u(n) - pu(n-2)] + qu(n-1) = 0, & n \neq m_j, n > 2\\ \underline{\Delta}[\Delta(u(m_j - 1) - pu(m_j - 3))] + ru(m_j - 2) = 0, & j \in \mathbb{N},\end{cases}$$

where $p = 6$, $q = 0.75$, $r = -1$ and $m_j = 3j, j \in \mathbb{N}$. Let $l_1 = 1$ and $l_2 = 2$. From the characteristic equation of (E_6), we get $\lambda = 0.75 < 1 - \frac{q}{r} = 1.5$ and $A = \frac{1}{\lambda}\left(1 - \frac{r}{q}\right) + \frac{r}{q} = 2$. By Theorem 2, (E_6) has a nonoscillatory solution $u(n) = (0.75)^n 2^{i(2,n)}$.

Example 3. *Consider the system of impulsive difference equations*

$$(E_6')\begin{cases}\Delta^2[u(n) - pu(n-2)] + qu(n-2) = 0, & n \neq m_j, n > 2\\ \underline{\Delta}[\Delta(u(m_j - 1) - pu(m_j - 3))] + ru(m_j - 3) = 0, & j \in \mathbb{N},\end{cases}$$

where $p = -4$, $q = 0$, $r = -18$ and $m_j = 3j$. Let $l_1 = 1$ and $l_2 = 1$. Clearly, $\lambda = 2i$ or $\lambda = -2i$ satisfies the first equation of (E_6'). Also, $A = 1.96539$ satisfies the second equation of (E_6'). By Theorem 4, (E_6') has an oscillatory solution.

3. Application

In this section, we are applying some results of Section 2 while we go for the linearized oscillation method. We consider the nonlinear impulsive system:

$$(E_7)\begin{cases}\Delta^2[u(n) - p(n)f(u(n-\alpha))] + q(n)h(u(n-\beta)) = 0, & n \neq m_j\\ \underline{\Delta}[\Delta(u(m_j - 1) - p(m_j - 1)f(u(m_j - \alpha - 1)))] & \\ \qquad + r(m_j - 1)h(u(m_j - \beta - 1)) = 0, & j \in \mathbb{N},\end{cases}$$

where α, $\beta > 0$ are constants, p, q, $r > 0$ are real valued sequences and $f, h \in C(\mathbb{R}, \mathbb{R})$. Throughout our discussion, we assume the following hypotheses:

(H_1) $p(n) \geq 1$ for $n \geq n_0$ and $\limsup_{n\to\infty} p(n) = p_0 \in (1, \infty)$;

(H_2) $\lim_{n\to\infty} q(n) = q_0 \in (0, \infty)$ and $\liminf_{n\to\infty} r(n) = r_0 \in (0, \infty)$;

(H_3) $uf(u) > 0$, $\frac{f(u)}{u} \geq 1$ for $u \neq 0$ and $\lim_{|u|\to\infty} \frac{f(u)}{u} = 1$;

(H_4) $vh(v) > 0$ for $v \neq 0$, $|h(v)| \geq h_0 > 0$ and $\lim_{|v|\to\infty} \frac{h(v)}{v} = 1$.

With the system (E_7), we associate the linear system of equations

$$(E_8)\begin{cases}\Delta^2[u(n) - p_0 u(n-\alpha)] + q_0 u(n-\beta) = 0, & n \neq m_j\\ \underline{\Delta}[\Delta(u(m_j-1) - p_0 u(m_j-\alpha-1))] + r_0 u(m_j-\beta-1) = 0, & j \in \mathbb{N}.\end{cases}$$

Here, our aim is to establish conditions for the oscillation of solutions of the system (E_7) in terms of the oscillation of solutions of (E_8) by means of its associated characteristic equation

$$\left[\frac{1}{\lambda}\left(1-\frac{r_0}{q_0}\right)+\frac{r_0}{q_0}\right]^{l_1}(\lambda-1)^2 - p_0\lambda^{-\alpha}\left[\frac{1}{\lambda}\left(1-\frac{r_0}{q_0}\right)+\frac{r_0}{q_0}\right]^{l_1-l_2}(\lambda-1)^2 + q_0\lambda^{-\beta} = 0. \tag{17}$$

By Theorem 2, (E_8) admits a nonoscillatory solution in the form (3) if and only if (17) has at least one real root λ with $\lambda > 1 - \frac{q_0}{r_0}$ for $\frac{r_0}{q_0} > 1$.

Theorem 6. *In addition to $(H_1) - (H_4)$, assume that*

$$(H_5)\quad \sum_{s=n^*}^{\infty} q(s) + \sum_{j=1}^{\infty} r(m_j - 1) = \infty,$$

$$(H_6)\quad \sum_{s=n^*}^{\infty}\Big[\sum_{t=n^*}^{s-1} q(t) + \sum_{n^* \leq m_j - 1 \leq s-1} r(m_j-1)\Big] = \infty,\ s > n^*+1$$

hold. If (17) has no positive real root in $[1 - \frac{q_0}{r_0}, \infty)$ for $r_0 > q_0$, then every solution of the system (E_7) oscillates.

Proof. Suppose on the contrary that $u(n)$ is a nonoscillatory solution of (E_7). Without loss of generality and due to (H_3) and (H_4), we may assume that $u(n) > 0$, $u(n-\alpha) > 0$ and $u(n-\beta) > 0$ for $n \geq n_0 > \max\{\alpha, \beta\}$. Set

$$z(n) = u(n) - p(n) f(u(n-\alpha)),$$
$$z(m_j - 1) = u(m_j - 1) - p(m_j - 1) f(u(m_j - \alpha - 1)).$$

Then (E_7) reduces to

$$(E_9)\begin{cases}\Delta^2 z(n) = -q(n) h(u(n-\beta)) \leq 0, & n \neq m_j\\ \underline{\Delta}(\Delta z(m_j-1)) = -r(m_j-1) h(u(m_j-\beta-1)) \leq 0, & j \in \mathbb{N}.\end{cases}$$

Therefore, $\Delta z(n)$ is nonincreasing for $n \geq n_1 > n_0 + \beta$. Thus, either $\Delta z(n) < 0$ or $\Delta z(n) > 0$ for $n \geq n_1$. We claim that $\Delta z(n) < 0$ for $n \geq n_1$. If not, then there exists

$n_2 > n_1$ such that $\Delta z(n) > 0$ for $n \geq n_2$. Then summing the impulsive system (E_9) from n_2 to $n-1$ $(n > n_2+1)$, we get

$$\Delta z(n) - \Delta z(n_2) - \sum_{n_2 \leq m_j - 1 \leq n-1} \underline{\Delta}(\Delta z(m_j - 1)) = -\sum_{s=n_2}^{n-1} q(s)h(u(n-\beta)),$$

that is,

$$\sum_{s=n_2}^{n-1} q(s)h(u(n-\beta)) + \sum_{n_2 \leq m_j - 1 \leq n-1} r(m_j - 1)h(u(m_j - \beta - 1)) = -\Delta z(n) + \Delta z(n_2).$$

Using (H_4), it follows that

$$\begin{aligned} h_0 \Big[\sum_{s=n_2}^{n-1} q(s) + \sum_{n_2 \leq m_j - 1 \leq n-1} r(m_j - 1) \Big] &\leq -\Delta z(n) + \Delta z(n_2) \\ &\leq \Delta z(n_2), \end{aligned}$$

which is a contradiction to (H_5). So, our claim holds and hence $z(n)$ is nonincreasing for $n \geq n_2$. Ultimately, either $z(n) < 0$ or $z(n) > 0$ for $n \geq n_3 > n_2$. We assert that $z(n) < 0$ for $n \geq n_2$. If not, then there exists $n_4 > n_3$ such that $z(n) > 0$ for $n \geq n_4$. Indeed, $z(n) > 0$ implies that $u(n) > p(n)f(u(n-\alpha)) > u(n-\alpha)$ for $n \geq n_4$ due to (H_1) and (H_3). Therefore,

$$u(n) > u(n-\alpha) > u(n-2\alpha) > \cdots > u(n_4),$$

that is, $u(n)$ is bounded below by a positive constant M (say). Also, we encounter

$$u(m_j - 1) > u(m_j - \alpha - 1) > u(m_j - 2\alpha - 1) > \cdots > u(n_4)$$

for the nonimpulsive points $m_j - 1, m_j - \alpha - 1, \ldots$. Consequently, (E_9) becomes

$$\begin{aligned} \Delta^2 z(n) &< -q(n)h(M), \\ \underline{\Delta}(\Delta z(m_j - 1)) &< -r(m_j - 1)h(M) \end{aligned}$$

for $n \geq n_4$. Summing the preceding impulsive system from n_4 to $n-1$, we get

$$\sum_{s=n_4}^{n-1} q(s)h(M) + \sum_{n_4 \leq m_j - 1 \leq n-1} r(m_j - 1)h(M) = \Delta z(n_4) - \Delta z(n) \leq -\Delta z(n).$$

Further, we sum the last inequality from n_4 to $n-1$ to find

$$\begin{aligned} h(M) \sum_{s=n_4}^{n-1} \Big[\sum_{t=n_4}^{s-1} q(t) + \sum_{n_4 \leq m_j - 1 \leq t-1} r(m_j - 1) \Big] &\leq -\sum_{s=n_4}^{n-1} \Delta z(s) \\ &= z(n_4) - z(n) \\ &< z(n_4), \end{aligned}$$

which contradicts (H_6). So, our assertation holds. Therefore, we can find $n_5 > n_4+1$ and a constant $C > 0$ such that $z(n) \leq -C$ and $z(m_j - 1) \leq -C$ for $n \geq n_5$, that is,

$$\begin{aligned} u(n) - p(n)f(u(n-\alpha)) &\leq -C, \\ u(m_j - 1) - p(m_j - 1)f(u(m_j - \alpha - 1)) &\leq -C. \end{aligned}$$

We claim that $\liminf_{n\to\infty} u(n) = \gamma > 0$. If $\gamma = 0$, then we can find $\{n_k\} \subset \{n\}$ such that $n_k \to \infty$ as $k \to \infty$ and $u(n_k) \to 0$ as $k \to \infty$. Therefore,

$$u(n_k + \alpha) - p(n_k + \alpha)f(u(n_k)) \leq -C$$

implies that

$$\lim_{k\to\infty} u(n_k + \alpha) \leq \lim_{k\to\infty} (p(n_k + \alpha)f(u(n_k)) - C),$$

that is,

$$0 \leq \lim_{k\to\infty} u(n_k + \alpha) \leq -C < 0,$$

a contradiction due to $f(u(n_k)) \to 0$ as $k \to \infty$. So, our claim holds. Consequently, (E_9) becomes

$$\begin{aligned} \Delta^2 z(n) &\leq -q(n)h(\gamma),\ n \neq m_j, \\ \underline{\Delta}(\Delta z(m_j - 1)) &\leq -r(m_j - 1)h(\gamma),\ j \in \mathbb{N}. \end{aligned}$$

Summing the last impulsive system twice from n_5 to $n - 1$ $(n > n_5 + 1)$ and then proceeding as above, we get $z(n) \to -\infty$ as $n \to \infty$, which is if and only if $u(n) \to \infty$ as $n \to \infty$. Also, it is true for nonimpulsive points $u(m_j - 1) \to \infty$ as $j \to \infty$. Let us set

$$\begin{aligned} P(n) &= p(n)\frac{f(u(n-\alpha))}{u(n-\alpha)}, \\ Q(n) &= q(n)\frac{h(u(n-\beta))}{u(n-\beta)}, \\ R(m_j - 1) &= r(m_j - 1)\frac{h(u(m_j - \beta - 1))}{u(m_j - \beta - 1)}. \end{aligned}$$

Then due to (H_2) and (H_4), it follows that

$$\begin{aligned} \lim_{n\to\infty} Q(n) &= \lim_{n\to\infty} q(n) \lim_{n\to\infty} \frac{h(u(n-\beta))}{u(n-\beta)} = q_0, \\ \liminf_{j\to\infty} R(m_j - 1) &\geq \liminf_{j\to\infty} r(m_j - 1) \lim_{j\to\infty} \frac{h(u(m_j - \beta - 1))}{u(m_j - \beta - 1)} = r_0 \end{aligned}$$

and

$$\limsup_{n\to\infty} P(n) \leq \limsup_{n\to\infty} p(n) \lim_{n\to\infty} \frac{f(u(n-\alpha))}{u(n-\alpha)} = p_0.$$

Now, (E_7) can be written as

$$(E_{10})\begin{cases}\Delta^2[u(n)-P(n)u(n-\alpha)]+Q(n)u(n-\beta)=0, & n\neq m_j\\ \underline{\Delta}[\Delta(u(m_j-1)-P(m_j-1)u(m_j-\alpha-1))] & \\ \quad +R(m_j-1)u(m_j-\beta-1)=0, & j\in\mathbb{N}.\end{cases}$$

Summing (E_{10}) from n_5 to $n-1$, we obtain

$$\Delta Z(n)-\Delta Z(n_5)+\sum_{s=n_5}^{n-1}Q(s)u(s-\beta)+\sum_{n_5\leq m_j-1\leq n-1}R(m_j-1)u(m_j-\beta-1)=0,$$

where $Z(n)=u(n)-P(n)u(n-\alpha)$. Therefore,

$$\Delta Z(n)+\sum_{s=n_5}^{n-1}Q(s)u(s-\beta)+\sum_{n_5\leq m_j-1\leq n-1}R(m_j-1)u(m_j-\beta-1)\leq 0$$

due to decreasing nature of $\Delta Z(n)$. Consequently,

$$Z(n)-Z(n_5)+\sum_{s=n_5}^{n-1}\left[\sum_{t=n_5}^{s-1}Q(t)u(t-\beta)+\sum_{n_5\leq m_j-1\leq s-1}R(m_j-1)u(m_j-\beta-1)\right]\leq 0,$$

which is equivalent to

$$\begin{aligned}u(n)\geq&\frac{1}{P(n+\alpha)}\Big[u(n+\alpha)+\sum_{s=n_5}^{n+\alpha-1}\Big[\sum_{t=n_5}^{s+\alpha-1}Q(t)u(t-\beta)\\ &+\sum_{n_5\leq m_j-1\leq s+\alpha-1}R(m_j-1)u(m_j-\beta-1)-z(n_5)\Big]\Big].\end{aligned}\tag{18}$$

Let $\varepsilon\in(0,q_0)$ and $\beta>1$ be such that $(\beta-1)p_0<\varepsilon$. Suppose there exists $n_6>n_5+1$ such that

$$P(n)<\frac{p_0+\varepsilon}{\beta},\qquad Q(n)>q_0-\varepsilon,\ R(m_j-1)>r_0.$$

Then for $n\geq n_6$, (18) reduces to

$$\begin{aligned}u(n)\geq&\frac{\beta}{p_0+\varepsilon}\Big[u(n+\alpha)+\sum_{s=n_6}^{n+\alpha-1}\Big[(q_0-\varepsilon)\sum_{s=n_6}^{s+\alpha-1}u(t-\beta)\\ &+r_0\sum_{n_6\leq m_j-1\leq s+\alpha-1}u(m_j-\beta-1)-z(n_5)\Big]\Big].\end{aligned}\tag{19}$$

Let $X=l_\infty^{n^*}$ be the Banach space of all real valued bounded functions $y(n)$ for $n\geq n^*$ with sup norm defined by

$$\|y\|=\sup\{|y(n)|:n\geq n^*\}.$$

Set

$$\Omega = \{y \in X : 0 \le y(n) \le 1, n \ge n^*\}.$$

Since Ω is a closed subset of X, then Ω is a complete metric space. For $y \in \Omega$ and $n \ge n^* > n_6$, define

$$(Ty)(n) = \begin{cases} Ty(n^*+\rho),\ n^* \le n \le n^*+\rho, \\ \dfrac{1}{(p_0+\varepsilon)u(n)}\Big[u(n+\alpha)y(n+\alpha) \\ \quad + \displaystyle\sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1}u(t-\beta)y(t-\beta) \\ \quad + r_0 \displaystyle\sum_{n^* \le m_j-1 \le s+\alpha-1} u(m_j-\beta-1)y(m_j-\beta-1) - z(n_5)\big]\Big],\ n > n^*+\rho. \end{cases}$$

For $y \in \Omega$ and using (19) we have

$$\begin{aligned} Ty(n) \le & \frac{1}{(p_0+\varepsilon)u(n)}\Big[u(n+\alpha) + \sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1}u(t-\beta) \\ & + r_0 \sum_{n^* \le m_j-1 \le s+\alpha-1} u(m_j-\beta-1) - z(n_5)\big]\Big] \\ \le & \frac{1}{\beta} < 1, \end{aligned}$$

and $Ty(n) \ge 0$ implies that $Ty(n) \in \Omega$ for every $n \ge n^*$. On the other hand, $y_1, y_2 \in \Omega$ implies that

$$\begin{aligned} & |Ty_1(n) - Ty_2(n)| \\ & \quad \le \frac{1}{(p_0+\varepsilon)u(n)}\Big[u(n+\alpha)|y_1(n+\alpha) - y_2(n+\alpha)| \\ & \quad + \sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1}u(t-\beta)|y_1(t-\beta) - y_2(t-\beta)| \\ & \quad + r_0 \sum_{n^* \le m_j-1 \le s+\alpha-1} u(m_j-\beta-1)|y_1(m_j-\beta-1) - y_2(m_j-\beta-1)|\big]\Big], \end{aligned}$$

that is,

$$\begin{aligned} |Ty_1(n) - Ty_2(n)| \le & \frac{1}{(p_0+\varepsilon)u(n)}\Big[u(n+\alpha) + \sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1}u(t-\beta) \\ & + r_0 \sum_{n^* \le m_j-1 \le s+\alpha-1} u(m_j-\beta-1)\big]\Big]\|y_1 - y_2\| \\ \le & \frac{1}{\beta}\|y_1 - y_2\|, \end{aligned}$$

that is,

$$\|Ty_1 - Ty_2\| \leq \frac{1}{\beta}\|y_1 - y_2\|$$

and hence T is a contraction with the contraction constant $\frac{1}{\beta} < 1$. By Banach's fixed point theorem, T has a unique fixed point $y \in \Omega$ such that $Ty = y$, that is,

$$y(n) = \begin{cases} y(n^* + \rho), & n^* \leq n \leq n^* + \rho, \\ \dfrac{1}{(p_0+\varepsilon)u(n)}\Big[u(n+\alpha)y(n+\alpha) & \\ \quad + \displaystyle\sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1} u(t-\beta)y(t-\beta) & \\ \quad + r_0 \displaystyle\sum_{n^* \leq m_j - 1 \leq s+\alpha-1} u(m_j-\beta-1)y(m_j-\beta-1) - z(n_5)\big]\Big], & n > n^* + \rho. \end{cases}$$

Setting $w(n) = u(n)y(n)$ for $n \geq n^* + \rho$, we obtain

$$\begin{aligned} w(n) =& \frac{1}{(p_0+\varepsilon)}\Big[w(n+\alpha) + \sum_{s=n^*}^{n+\alpha-1}\big[(q_0-\varepsilon)\sum_{t=n^*}^{s+\alpha-1} w(t-\beta) \\ &+ r_0 \sum_{n^* \leq m_j-1 \leq s+\alpha-1} w(m_j-\beta-1) - z(n_5)\big]\Big], \end{aligned}$$

which is a positive solution of the impulsive system

$$(E_{11}) \begin{cases} \Delta^2[w(n) - (p_0+\varepsilon)w(n-\alpha)] + (q_0-\varepsilon)w(n-\beta) = 0 \\ \underline{\Delta}[\Delta(w(m_j-1) - (p_0+\varepsilon)w(m_j-\alpha-1))] + r_0 w(m_j-\beta-1) = 0. \end{cases}$$

Indeed, its characteristic equation is given by

$$\begin{aligned} &\Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0-\varepsilon)}) + \frac{r_0}{(q_0-\varepsilon)}\Big]^{l_1}(\lambda-1)^2 \\ &\quad - (p_0+\varepsilon)\lambda^{-\alpha}\Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0-\varepsilon)}) + \frac{r_0}{(q_0-\varepsilon)}\Big]^{l_1-l_2}(\lambda-1)^2 + (q_0-\varepsilon)\lambda^{-\beta} = 0. \end{aligned}$$

Due to Theorem 2, $w(n)$ is a positive solution of (E_{11}) if and only if

$$\lambda > 1 - \frac{(q_0-\varepsilon)}{r_0} > 1 - \frac{q_0}{r_0}$$

for $\frac{r_0}{q_0} > 1$, a contradiction due to Theorem 5. This completes the proof of the theorem. □

Example 4. *Consider the neutral impulsive difference equations of the form*

$$(E_{12}) \begin{cases} \Delta^2[u(n) - p(n)f(u(n-1))] + q(n)h(u(n-1)) = 0, \quad n \neq 3j,\ j \in \mathbb{N}, n > 2 \\ \underline{\Delta}[\Delta(u(m_j-1) - p(m_j-1)f(u(m_j-2)))] \\ \quad + r(m_j-1)h(u(m_j-2)) = 0, \end{cases}$$

where $p(n) = 2 + e^{-(n+1)}$, $q(n) = 4.59 + e^{-(n^2+1)}$, $r(m_j - 1) = 9(2 + \cos(m_j - 1))$, $m_j = 3j, j \in \mathbb{N}$, $f(u) = u$ *and* $h(u) = ue^{\frac{1}{1+|u|}}$. *The limiting equation of* (E_{12}) *is given by*

$$(E_{13}) \begin{cases} \Delta^2[y(n) - p_0 y(n-1)] + q_0 y(n-1) = 0, & n \neq 3j, n > 2 \\ \underline{\Delta}[\Delta(y(m_j - 1) - p_0 y(m_j - 2))] + r_0 y(m_j - 2) = 0, & j \in \mathbb{N}, \end{cases}$$

where $p_0 = 2$, $q_0 = 4.59$, $r_0 = 9$. *Let* $l_1 = 1$ *and* $l_2 = 2$. *By Remark 4,* (E_{12}) *has no positive real roots in* $[1 - \frac{q_0}{r_0}, \infty) = [0.49, \infty)$ *and hence by Theorem 6, every solution of* (E_{13}) *oscillates. We may note that* (E_{12}) *has an oscillatory solution* $y(n) = (0.24265)^n(-2)^{i(2,n)}$ *due to Theorem 1.*

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