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Steiner's Hat: a Constant-Area Deltoid Associated with the Ellipse

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ABSTRACT

The Negative Pedal Curve (NPC) of the Ellipse with respect to a boundary point M is a 3-cusp closed-curve which is the affine image of the Steiner Deltoid. Over all M the family has invariant area and displays an array of interesting properties.

Key words: curve, envelope, ellipse, pedal, evolute, deltoid, Poncelet, osculating, orthologic

MSC2010: 51M04 51N20 65D18

Steinerova krivulja: deltoide konstantne površine pridružene elipsi

SAŽETAK

Negativno nožišna krivulja elipse s obzirom na neku njezinu točku M je zatvorena krivulja s tri šiljka koja je afina slika Steinerove deltoide. Za sve točke M na elipsi krivulje dobivene familije imaju istu površinu i niz zanimljivih svojstava.

Ključne riječi: krivulja, envelopa, elipsa, nožišna krivulja, evoluta, deltoida, Poncelet, oskulacija, ortologija

1 Introduction

Given an ellipse $\mathcal{E}$ with non-zero semi-axes a, b centered at O, let M be a point in the plane. The Negative Pedal Curve (NPC) of $\mathcal{E}$ with respect to M is the envelope of lines passing through points $P(t)$ on the boundary of $\mathcal{E}$ and perpendicular to $[P(t)-M]$ [4, pp. 349]. Well-studied cases [7, 14] include placing M on (i) the major axis: the NPC is a two-cusp "fish curve" (or an asymmetric ovoid for low eccentricity of $\mathcal{E}$); (ii) at O: this yielding a four-cusp NPC known as Talbot's Curve (or a squashed ellipse for low eccentricity), Figure 1.

As a variant to the above, we study the family of NPCs with respect to points M on the *boundary* of $\mathcal{E}$. As shown in Figure 2, this yields a family of asymmetric, constant-area 3-cusped deltoids. We call these curves "Steiner's Hat" (or Δ), since under a varying affine transformation, they are the image of the Steiner Curve (aka. Hypocycloid), Figure 3. Besides these remarks, we've observed:

Main Results:

- The triangle T' defined by the 3 cusps P_i' has invariant area over M, Figure 7.
- The triangle T defined by the pre-images P_i of the 3 cusps has invariant area over M, Figure 7. The P_i are the 3 points on $\mathcal{E}$ such that the corresponding tangent to the envelope is at a cusp.
- The T are a Poncelet family with fixed barycenter; their caustic is half the size of $\mathcal{E}$, Figure 7.
- Let C_2 be the center of area of Δ. Then M, C_2, P_1, P_2, P_3 are concyclic, Figure 7. The lines $P_i - C_2$ are tangents at the cusps.
- Each of the 3 circles passing through M, P_i, P_i', $i = 1,2,3$, osculate $\mathcal{E}$ at P_i, Figure 8. Their centers define an area-invariant triangle T'' which is a half-size homothety of T'.

The paper is organized as follows. In Section 3 we prove the main results. In Sections 4 and 5 we describe properties of the triangles defined by the cusps and their pre-images, respectively. In Section 6 we analyze the locus of the cusps. In Section 6.1 we characterize the tangencies and intersections of Steiner's Hat with the ellipse. In Section 7 we describe properties of 3 circles which osculate the ellipse at the cusp pre-images and pass through M. In Section 8 we describe relationships between the (constant-area) triangles with vertices at (i) cusps, (ii) cusp pre-images, and (iii) centers of osculating circles. In Section 9 we analyze a fixed-area deltoid obtained from a "rotated" negative pedal curve. The paper concludes in Section 10 with a table of illustrative videos. Appendix A provides explicit coordinates for cusps, pre-images, and osculating circle centers. Finally, Appendix B lists all symbols used in the paper.

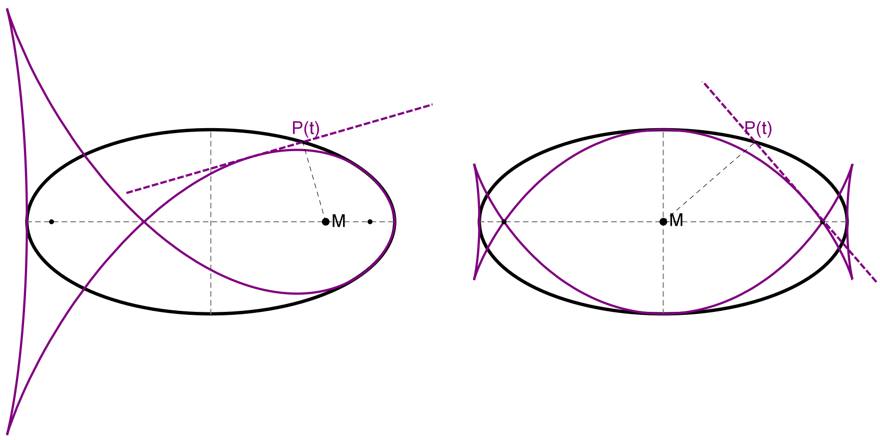

Figure 1: *The Negative Pedal Curve (NPC) of an ellipse $\mathcal{E}$ with respect to a point M on the plane is the envelope of lines passing through $P(t)$ on the boundary, and perpendicular to $P(t)-M$.* ***Left:*** *When M lies on the major axis of $\mathcal{E}$, the NPC is a two-cusp "fish" curve.* ***Right:*** *When M is at the center of $\mathcal{E}$, the NPC is 4-cusp curve with 2-self intersections known as Talbot's Curve [12]. For the particular aspect ratio $a/b=2$, the two self-intersections are at the foci of $\mathcal{E}$.*

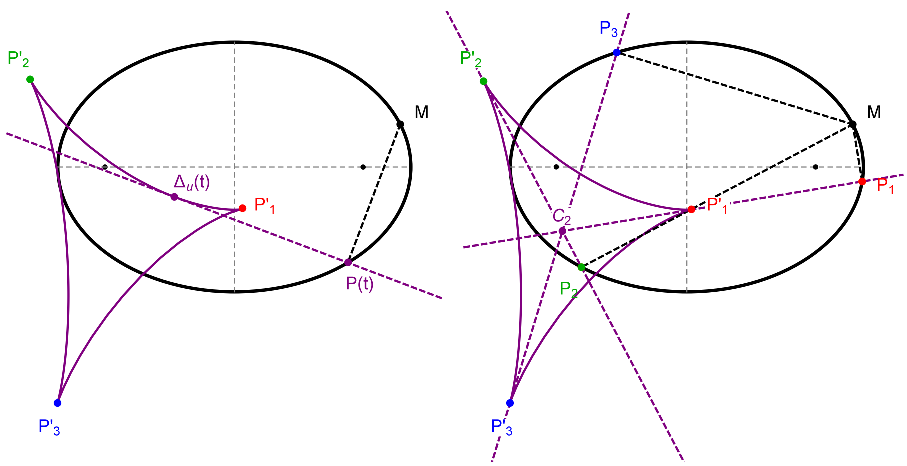

Figure 2: ***Left:*** *The Negative Pedal Curve (NPC, purple) of $\mathcal{E}$ with respect to a boundary point M is a 3-cusped (labeled P_i') asymmetric curve (called here "Steiner's Hat"), whose area is invariant over M, and whose asymmetric shape is affinely related to the Steiner Curve [12]. $\Delta_u(t)$ is the instantaneous tangency point to the Hat.* ***Right****: The tangents at the cusps points P_i' concur at C_2, the Hat's center of area, furthermore, P_i, P_i', C_2 are collinear.* ***Video:*** *[10, PL#01]*

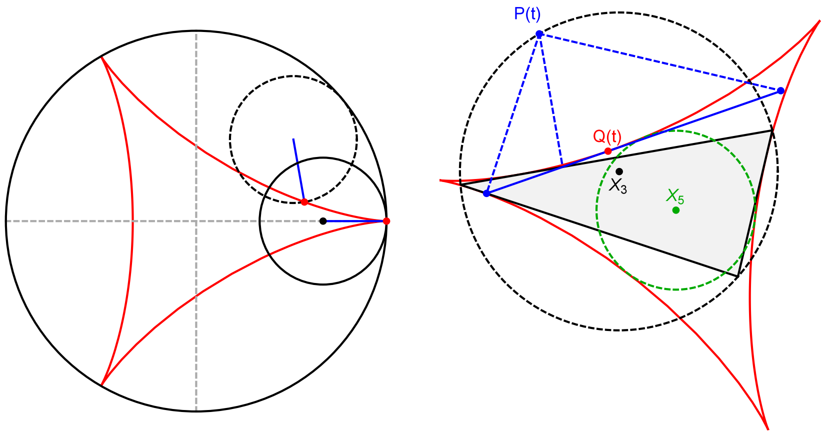

Figure 3: *Two systems which generate the 3-cusp Steiner Curve (red), see [2] for more methods.* ***Left:*** *The locus of a point on the boundary of a circle of radius* 1 *rolling inside another of radius* 3. ***Right:*** *The envelope of Simson Lines (blue) of a triangle T (black) with respect to points $P(t)$ on the Circumcircle [12]. $Q(t)$ denotes the corresponding tangent. Nice properties include (i) the area of the Deltoid is half that of the Circumcircle, and (ii) the 9-point circle of T (dashed green) centered on X_5 (whose radius is half that of the Circumcircle) is internally tangent to the Deltoid [13, p.231].*

2 Preliminaries

Let the ellipse $\mathcal{E}$ be defined implicitly as:

$$\mathcal{E}(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad c^2 = a^2 - b^2$$

where $a > b > 0$ are the semi-axes. Let a point $P(t)$ on its boundary be parametrized as $P(t) = (a\cos t, b\sin t)$.

Let $P_0 = (x_0, y_0) \in \mathbb{R}^2$. Consider the family of lines $L(t)$ passing through $P(t)$ and orthogonal to $P(t) - P_0$. Its envelope Δ is called *antipedal* or *negative pedal curve* of $\mathcal{E}$.

Consider the spatial curve defined by

$$\mathcal{L}(P_0) = \{(x,y,t) : L(t,x,y) = L'(t,x,y) = 0\}.$$

The projection $\mathcal{E}(P_0) = \pi(\mathcal{L}(P_0))$ is the envelope. Here $\pi(x,y,t) = (x,y)$. In general, $\mathcal{L}(P_0)$ is regular, but $\mathcal{E}(P_0)$ is a curve with singularities and/or cusps.

Lemma 1 *The envelope of the family of lines $L(t)$ is given by:*

$$\begin{aligned} x(t) =& \frac{1}{w}[(ay_0\sin t - ab)x_0 - by_0^2\cos t - c^2 y_0 \sin(2t) \\ &+ \frac{b}{4}((5a^2 - b^2)\cos t - c^2\cos(3t))] \\ y(t) =& \frac{1}{w}[-ax_0^2 \sin t + (by_0\cos t + c^2\sin(2t))x_0 - aby_0 \\ &- \frac{a}{4}((5a^2 - b^2)\sin t - c^2\sin(3t)] \end{aligned} \quad (1)$$

where $w = ab - bx_0\cos t - ay_0\sin t$.

Proof. The line $L(t)$ is given by:

$$\begin{aligned} &(x_0 - a\cos t)x + (y_0 - b\sin t)y + a^2\cos^2 t \\ &\quad + b^2\sin^2 t - ax_0\cos t - by_0\sin t = 0. \end{aligned}$$

Solving the linear system $L(t) = L'(t) = 0$ in the variables x, y leads to the result. □

Triangle centers will be identifed below as X_k following Kimberling's Encyclopedia [6], e.g., X_1 is the Incenter, X_2 Barycenter, etc.

3 Main Results

Proposition 1 *The NPC with respect to $M_u = (a\cos u, b\sin u)$ a boundary point of $\mathcal{E}$ is a 3-cusp closed curve given by $\Delta_u(t) = (x_u(t), y_u(t))$, where*

$$\begin{aligned} x_u(t) =& \frac{1}{a}\left(c^2(1+\cos(t+u))\cos t - a^2\cos u\right) \\ y_u(t) =& \frac{1}{b}\left(c^2\cos t\sin(t+u) - c^2\sin t - a^2\sin u\right) \end{aligned} \quad (2)$$

Proof. It is direct consequence of Lemma 1 with $P_0 = M_u$. □

Expressions for the 3 cusps P_i' in terms of u appear in Appendix A.

Remark 1 *As $a/b \to 1$ the ellipse becomes a circle and Δ shrinks to a point on the boundary of said circle.*

Remark 2 *Though Δ can never have three-fold symmetry, for M_u at any ellipse vertex, it has axial symmetry.*

Remark 3 *The average coordinates $\bar{C} = [\bar{x}(u), \bar{y}(u)]$ of Δ_u w.r.t. this parametrization are given by:*

$$\bar{x}(u) = \frac{1}{2\pi}\int_0^{2\pi} x_u(t)\,dt = -\frac{(a^2+b^2)}{2a}\cos u$$
$$\bar{y}(u) = \frac{1}{2\pi}\int_0^{2\pi} y_u(t)\,dt = -\frac{(a^2+b^2)}{2b}\sin u \tag{3}$$

Theorem 1 *Δ_u is the image of the 3-cusp Steiner Hypocycloid $\mathcal{S}$ under a varying affine transformation.*

Proof. Consider the following transformations in $\mathbb{R}^2$:

$$\begin{aligned} \text{rotation:} \quad & R_u(x,y) = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ \text{translation:} \quad & U(x,y) = (x,y) + \bar{C} \\ \text{homothety:} \quad & D(x,y) = \tfrac{1}{2}(a^2-b^2)(x,y). \\ \text{linear map:} \quad & V(x,y) = (\tfrac{x}{a}, \tfrac{y}{b}) \end{aligned}$$

The hypocycloid of Steiner is given by $\mathcal{S}(t) = 2(\cos t, -\sin t) + (\cos 2t, \sin 2t)$ [7]. Then:

$$\Delta_u(t) = [x_u(t), y_u(t)] = (U \circ V \circ D \circ R_u)\mathcal{S}(t) \tag{4}$$

Thus, Steiner's Hat is of degree 4 and of class 3 (i.e., degree of its dual). □

Corollary 1 *The area of $A(\Delta)$ of Steiner's Hat is invariant over M_u and is given by:*

$$A(\Delta) = \frac{(a^2-b^2)^2\pi}{2ab} = \frac{c^4\pi}{2ab} \tag{5}$$

Proof. The area of $\mathcal{S}(t)$ is $\int_S x dy = 2\pi$. The Jacobian of $(U \circ S \circ D \circ R_u)$ given by Equation 4 is constant and equal to $c^4/4ab$. □

Noting that the area of $\mathcal{E}$ is πab, Table 1 shows the aspect ratios a/b of $\mathcal{E}$ required to produce special area ratios.

a/b	approx. a/b	$A(\Delta)/A(\mathcal{E})$
$\sqrt{2+\sqrt{3}}$	1.93185	1
$\varphi = (1+\sqrt{5})/2$	1.61803	1/2
$\sqrt{2}$	1.41421	1/4
1	1	0

Table 1: *Aspect ratios yielding special area ratios of main ellipse $\mathcal{E}$ to Steiner's Hat Δ.*

It is well known that if M is interior to $\mathcal{E}$ then the NPC is a 2-cusp or 4-cusp curve with one or two self-intersections.

Remark 4 *It can be shown that when M is interior to $\mathcal{E}$ the iso-curves of signed area of the NPC are closed algebraic curves of degree 10, concentric with $\mathcal{E}$ and symmetric about both axes, see Figure 4.*

It is remarkable than when M moves from the interior to the boundary of $\mathcal{E}$, the iso-curves transition from a degree-10 curve to a simple conic.

Remark 5 *It can also be shown that when M is exterior to $\mathcal{E}$, the NPC is a two-branched open curve, see Figure 5.*

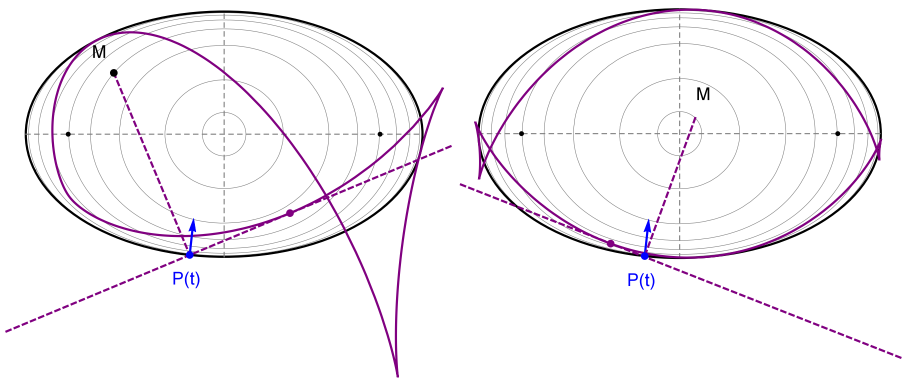

Figure 4: *The isocurves of signed area for the negative pedal curve when M is interior to the ellipse are closed algebraic curves of degree 10. These are shown in gray for an NPC with two cusps (left), and 4 cusps (right).*

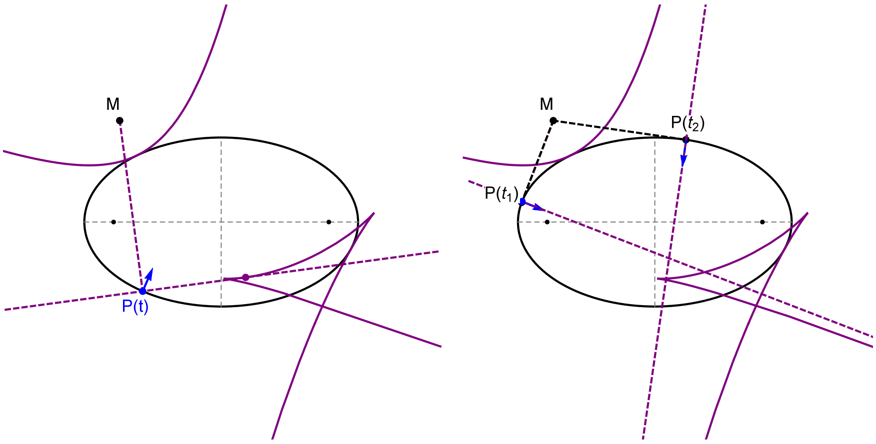

Figure 5: ***Left:*** *When M is exterior to $\mathcal{E}$ the NPC is a two-branched open curve. One branch is smooth and non-self-intersecting, and the other has 2 cusps and one self-intersection.* ***Right:*** *Let t_1, t_2 be the parameters for which $MP(t)$ is tangent to $\mathcal{E}$. At these positions, the NPC intersects the line at infinity in the direction of the normal at $P(t_1), P(t_2)$, i.e., the lines through $P(t)$ perpendicular to $P(t) - M$ are asymptotes.*

Proposition 2 *Let C_2 be the center of area of Δ_u. Then $C_2 = \bar{C}$.*

Proof.
The center of area is defined by

$$C_2 = \frac{1}{A(\Delta)} \left(\int_{int(\Delta)} x\,dx\,dy, \int_{int(\Delta)} y\,dx\,dy \right).$$

Using Green's Theorem, evaluate the above using the parametric in Equation (1). This yields the expression for $\bar{C}$ in Equation 3. Alternatively, one can obtain the same result from the affine transformation defined in Theorem 1. □

Referring to Figure 6(left):

Corollary 2 *The locus of C_2 is an ellipse always exterior to a copy of $\mathcal{E}$ rotated $90°$ about O.*

Proof. Equation 3 describes an ellipse. Since $a^2 + b^2 \geq 2ab$ the claim follows directly. □

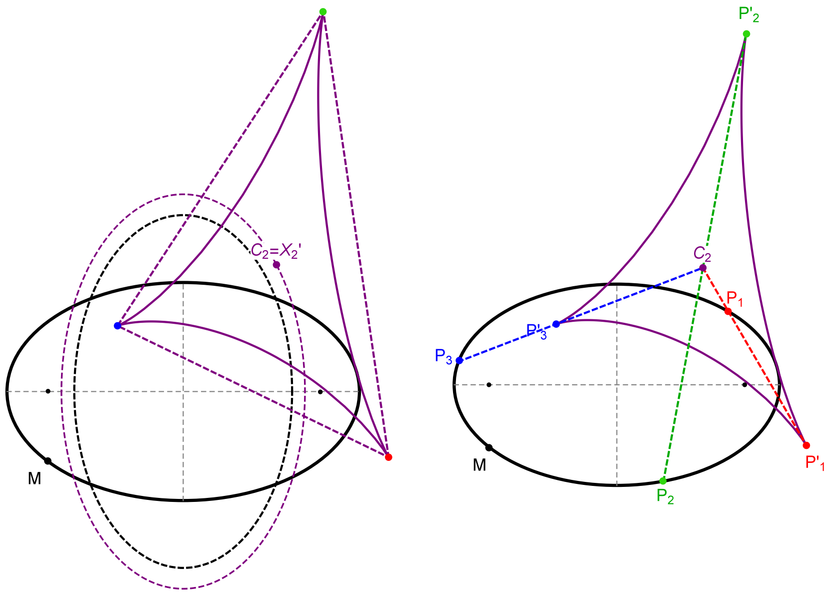

Figure 6: ***Left:*** *The area center C_2 of Steiner's Hat coincides with the barycenter X_2' of the (dashed) triangle T' defined by the cusps. Over all M, both the Hat and T' have invariant area. C_2's locus (dashed purple) is elliptic and exterior to a copy of $\mathcal{E}$ rotated $90°$ about its center (dashed black).* ***Right:*** *Let P_i' (resp. P_i), $i = 1,2,3$ denote the Hat's cusps (resp. their* pre-images *on $\mathcal{E}$), colored by i. Lines P_iP_i' concur at C_2.*

Let T denote the triangle of the *pre-images* P_i on $\mathcal{E}$ of the Hat's cusps, i.e. $P(t_i)$ such that $\Delta_u t_i$ is a cuspid. Explicit expressions for the P_i appear in Appendix A. Referring to Figure 7:

Theorem 2 *The points M, C_2, P_1, P_2, P_3 are concyclic.*

Proof. $\Delta_u(t)$ is singular at $t_1 = -\frac{u}{3}$, $t_2 = -\frac{u}{3} - \frac{2\pi}{3}$ and $t_3 = -\frac{u}{3} - \frac{4\pi}{3}$. Let $P_i = [a\cos t_i, b\sin t_i], i = 1,2,3$. The circle $\mathcal{K}$ passing through these is given by:

$$\mathcal{K}(x,y) = x^2 + y^2 - \frac{c^2\cos u}{2a}x + \frac{c^2\sin u}{2b}y - \frac{1}{2}(a^2+b^2) = 0. \tag{6}$$

Also, $\mathcal{K}(M) = \mathcal{K}(a\cos u, b\sin u) = 0$.
The center of $\mathcal{K}$ is $(M + C_2)/2$. It follows that $C_2 \in \mathcal{K}$ and that MC_2 is a diameter of $\mathcal{K}$. □

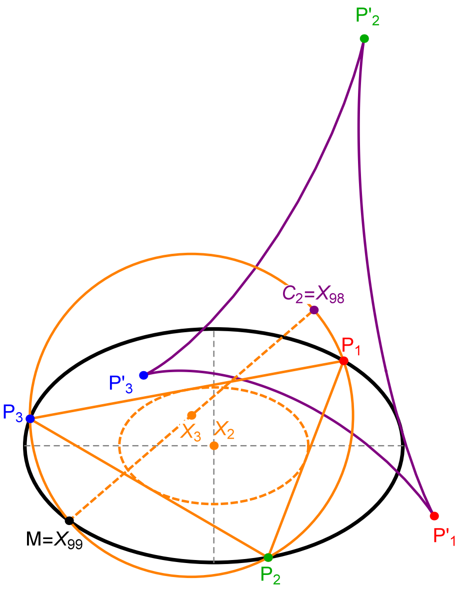

Figure 7: *The cusp pre-images P_i define a triangle T (orange) whose area is invariant over M. Its barycenter X_2 is stationary at the center of $\mathcal{E}$, rendering the latter its Steiner Ellipse. Let C_2 denote the center of area of Steiner's Hat. The 5 points M, C_2, P_1, P_2, P_3 lie on a circle (orange), with center at X_3 (circumcenter of T). Over all M, the T are a constant-area Poncelet family inscribed on $\mathcal{E}$ and tangent to a concentric, axis-aligned elliptic caustic (dashed orange), half the size of $\mathcal{E}$, i.e., the latter is the (stationary) Steiner Inellipse of the T. Note also that M is the Steiner Point X_{99} of T since it is the intersection of its Circumcircle with the Steiner Ellipse. Furthermore, the Tarry Point X_{98} of T coincides with C_2, since it is the antipode of $M = X_{99}$ [6]. **Video:** [10, PL#02,#05].*

In 1846, Jakob Steiner stated that given a point M on an ellipse $\mathcal{E}$, there exist 3 other points on it such that the osculating circles at these points pass through M [8, page 317]. This property is also mentioned in [4, page 97, Figure 3.26].

It turns out the cusp pre-images are said special points! Referring to Figure 8:

Proposition 3 *Each of the 3 circles $\mathcal{K}_i$ through M, P_i, P'_i, $i = 1,2,3$, osculate $\mathcal{E}$ at P_i.*

Proof. The circle $\mathcal{K}_1$ passing through M, P_1 and P'_1 is given by

$$\begin{aligned}\mathcal{K}_1(x,y) =& 2ab(x^2+y^2) - 4bc^2\cos^3\left(\frac{u}{3}\right)x - 4ac^2\sin^3\left(\frac{u}{3}\right)y \\ &+ ab\left(3c^2\cos\left(\frac{2u}{3}\right) - a^2 - b^2\right) = 0.\end{aligned}$$

Recall a circle osculates an ellipse if its center lies on the evolute of said ellipse, given by [4]:

$$\mathcal{E}^*(t) = \left[\frac{c^2\cos^3 t}{a}, -\frac{c^2\sin^3 t}{b}\right] \tag{7}$$

It is straightforward to verify that the center of $\mathcal{K}_1$ is $P''_1 = \mathcal{E}^*(-\frac{u}{3})$. A similar analysis can be made for $\mathcal{K}_2$ and $\mathcal{K}_3$. □

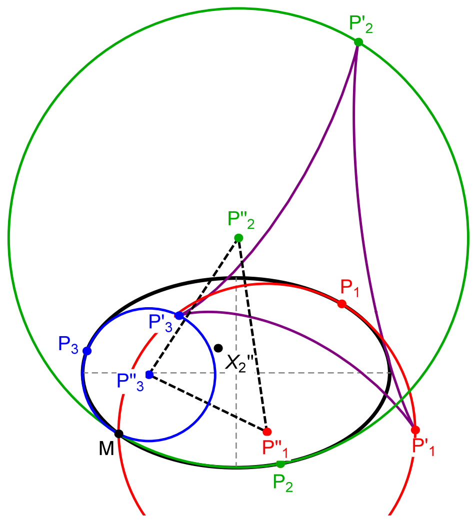

Figure 8: *The circles passing through a cusp P'_i, its pre-image P_i, and M osculate $\mathcal{E}$ at the P_i. The centers P''_i of said circles define a triangle T'' (dashed black) whose area is constant for all M. X''_2 denotes its (moving) barycenter. **Video:** [10, PL#03,#05].*

Since the area of $\mathcal{E}^*$ is $A(\mathcal{E}^*) = \frac{3\pi c^4}{8ab}$, and the area of Δ is given in Equation 5:

Remark 6 *The area ratio of Δ and the interior of $\mathcal{E}^*$ is equal to $4/3$.*

3.1 Why is Δ affine to Steiner's Curve

Up to projective transformations, there is only one irreducible curve of degree 4 with 3 cusps. In a projective coordinate frame $(x_0 : x_1 : x_2)$ with the cusps as base points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ and the common point of the cusps' tangents as unit point (1:1:1), the quartic has the equation

$$x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - 2x_0x_1x_2(x_0+x_1+x_2) = 0$$

At Steiner's three-cusped curve, the cusps form a regular triangle with the tangents passing through the center. Hence, whenever a three-cusped quartic has the meeting point of the cusps' tangents at the center of gravity of the cusps, it is affine to Steiner's curve, since there is an affine transformation sending the four points into a regular triangle and its center.

4 The Cusp Triangle

Recall $T' = P_1'P_2'P_3'$ is the triangle defined by the 3 cusps of Δ.

Proposition 4 *The area A' of the cusp triangle T' is invariant over M and is given by:*

$$A' = \frac{27\sqrt{3}}{16}\,\frac{c^4}{ab}$$

Proof. The determinant of the Jacobian of the affine transformation in Theorem 1 is $|J| = \frac{c^4}{4ab}$. Therefore, the area of T' is simply $|J|A_e$, where A_e is the area of an equilateral triangle inscribed in a circle of radius 3 with side $3\sqrt{3}$. □

Referring to Figure 6:

Proposition 5 *The barycenter X_2' of T' coincides with the center of area C_2 of Δ.*

Proof. Direct calculations yield $X_2' = C_2$. □

Referring to Figure 9:

Proposition 6 *The Steiner Ellipse $\mathcal{E}'$ of T' has constant area and is a scaled version of $\mathcal{E}$ rotated 90° about O.*

Proof. $\mathcal{E}'$ passes through the vertices of T' and is centered on $C_2 = X_2'$. Direct calculations yield the following implicit equation for it:

$$a^2x^2 + b^2y^2 + \left(a^2+b^2\right)\left(a\cos ux + b\sin uy\right) - \left(a^2-2b^2\right)\left(2a^2-b^2\right) = 0$$

Its semi-axes are $b' = \frac{3c^2}{2a}$ and $a' = \frac{3c^2}{2b}$. Therefore $\mathcal{E}'$ is similar to a 90°-rotated copy of $\mathcal{E}$. □

Remark 7 *This proves that T' can never be regular and Δ has never a three-fold symmetry.*

Corollary 3 *The ratio of area of $\mathcal{E}'$ and $\mathcal{E}$ is given by $\frac{9}{4}\frac{c^4}{a^2b^2}$, and at $a/b = (1+\sqrt{10})/3$, the two ellipses are congruent.*

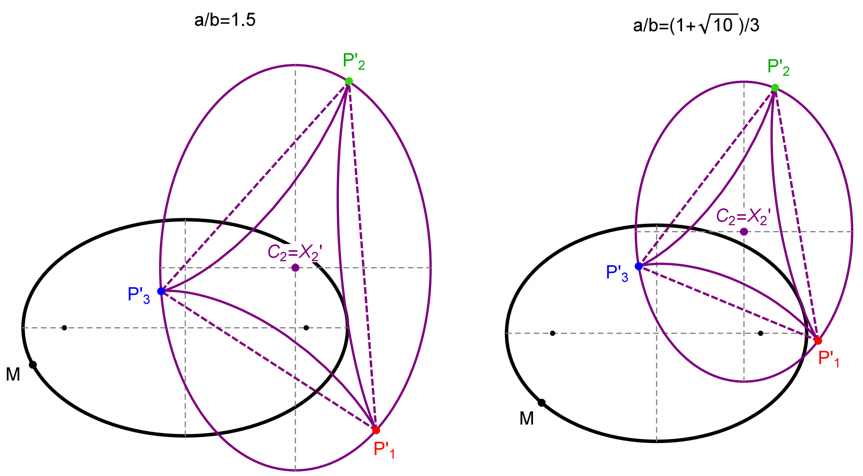

Figure 9: ***Left***: *The Steiner Ellipse $\mathcal{E}'$ of triangle T' defined by the P_i' is a scaled-up and 90°-rotated copy of $\mathcal{E}$.* ***Right:*** *At $a/b = (1+\sqrt{10})/3 \simeq 1.38743$, $\mathcal{E}'$ and $\mathcal{E}$ have the same area.*

Proposition 7 *The Steiner Point X'_{99} of T' is given by:*

$$X'_{99}(u) = \left[\frac{(a^2-2b^2)}{a}\cos u, -\frac{(2a^2-b^2)}{b}\sin u\right]$$

Proof. By definition X_{99} is the intersection of the circumcircle of T ($\mathcal{K}$) with the Steiner ellipse. The Circumcircle $\mathcal{K}'$ of the triangle $T' = \{P'_1, P'_2, P'_3\}$ is given by:

$$\begin{aligned}\mathcal{K}'(x,y) =& 8a^2b^2\left(x^2+y^2\right) + 2a\cos u\left(3a^4-2a^2b^2+7b^4\right)x\\ &+2b\sin u\left(7a^4-2a^2b^2+3b^4\right)y\\ &-\left(a^2+b^2\right)\left(c^2\left(a^2+b^2\right)\cos 2u+5a^4-14a^2b^2+5b^4\right)\\ =&0.\end{aligned}$$

With the above, straightforward calculations lead to the coordinates of X'_{99}. □

5 The Triangle of Cusp Pre-Images

Recall $T = P_1P_2P_3$ is the triangle defined by pre-images on $\mathcal{E}$ to each cusp of Δ.

Proposition 8 *The barycenter X_2 of T is stationary at O, i.e., $\mathcal{E}$ is is Steiner Ellipse [12].*

Proof. The triangle T is an affine image of an equilateral triangle with center at 0 and $P_i = \mathcal{E}(t_i) = \mathcal{E}(-\frac{u}{3} - (i-1)\frac{2\pi}{3})$. So the result follows. □

Remark 8 *M is the Steiner Point X_{99} of T.*

Proposition 9 *Over all M, the T are an $N = 3$ Poncelet family with external conic $\mathcal{E}$ with the Steiner Inellipse of T as its caustic [12]. Futhermore the area of these triangles is invariant and equal to $\frac{3\sqrt{3}ab}{4}$.*

Proof. The pair of concentric circles of radius 1 and $1/2$ is associated with a Poncelet 1d family of equilaterals. The image of this family by the map $(x,y) \to (ax,by)$ produces the original pair of ellipses, with the stated area. Alternatively, the ratio of areas of a triangle to its Steiner Ellipse is known to be $3\sqrt{3}/(4\pi)$ [12, Steiner Circumellipse] which yields the area result. □

6 Locus of the Cusps

We analyze the motion of the cusps P'_i of Steiner's Hat Δ with respect to continuous revolutions of M on $\mathcal{E}$. Referring to Figure 10:

Remark 9 *The locus $C(u)$ of the cusps of Δ is parametrized by:*

$$C(u): \frac{3c^2}{2}\left[\frac{1}{a}\cos\frac{u}{3}, \frac{1}{b}\sin\frac{u}{3}\right] - \frac{a^2+b^2}{2}\left[\frac{1}{a}\cos u, \frac{1}{b}\sin u\right] \quad (8)$$

This is a curve of degree 6, with the following implicit equation:

$$\begin{aligned}&-4a^6x^6-4b^6y^6-12a^2x^2b^2y^2\left(a^2x^2+b^2y^2\right)\\ &+12a^4\left(a^4-a^2b^2+b^4\right)x^4+12b^4\left(a^4-a^2b^2+b^4\right)y^4\\ &+24a^2b^2\left(a^4-a^2b^2+b^4\right)x^2y^2\\ &-3a^2\left(2a^2-b^2\right)\left(a^2+b^2\right)\left(2a^4-5a^2b^2+5b^4\right)x^2\\ &+3b^2\left(a^2-2b^2\right)\left(a^2+b^2\right)\left(5a^4-5a^2b^2+2b^4\right)y^2\\ &+\left(2a^2-b^2\right)^2\left(a^2-2b^2\right)^2\left(a^2+b^2\right)^2=0\end{aligned}$$

Proposition 10 *It can be shown that over one revolution of M about $\mathcal{E}$, C_2 will cross the ellipse on four locations $W_j, j = 1,\cdots,4$ given by:*

$$W_j = \frac{1}{2\sqrt{a^2+b^2}}\left(\pm a\sqrt{a^2+3b^2}, \pm b\sqrt{3a^2+b^2}\right)$$

At each such crossing, C_2 coincides with one of the pre-images.

Proof. From the coordinates of C_2 given in equation (3) in terms of the parameter u, one can derive an equation that is a necessary and sufficient condition for $C_2 \in \mathcal{E}$ to happen, by substituting those coordinates in the ellipse equation $x^2/a^2 + y^2/b^2 - 1 = 0$. Solving for $\sin u$ and substituting back in the coordinates of C_2, one easily gets the four solutions W_1, W_2, W_3, W_4.

Now, assume that $C_2 \in \mathcal{E}$. The points M, P_1, P_2, P_3, C_2 must all be in both the ellipse $\mathcal{E}$ and the circumcircle $\mathcal{K}$ of $P_1P_2P_3$. Since the two conics have at most 4 intersections (counting multiplicity), 2 of those 5 points must coincide. It is easy to verify from the previously-computed coordinates that M can only coincide with the preimages P_1, P_2, P_3 at the vertices of $\mathcal{E}$. In such cases, owing to the symmetry of the geometry about either the x- or y-axis, the circle $\mathcal{K}$ must be tangent to $\mathcal{E}$ at M. Thus, that intersection will count with multiplicity (of at least) 2, so another pair of those 5 points must also coincide. Since P_1, P_2, P_3 must all be distinct, C_2 will coincide with one of the preimages. However, this can never happen, since if M is on one of the vertices of $\mathcal{E}$, C_2 won't be in $\mathcal{E}$. In any other case, since P_1, P_2, P_3 must be distinct and C_2 is diametrically opposed to M in $\mathcal{K}$, C_2 must coincide with one of the preimages. □

Remark 10 *C_2 will visit each of the preimages cyclically. Moreover, upon 3 revolutions (with 12 crossings in total), each P_i will have been visited four times and the process repeats.*

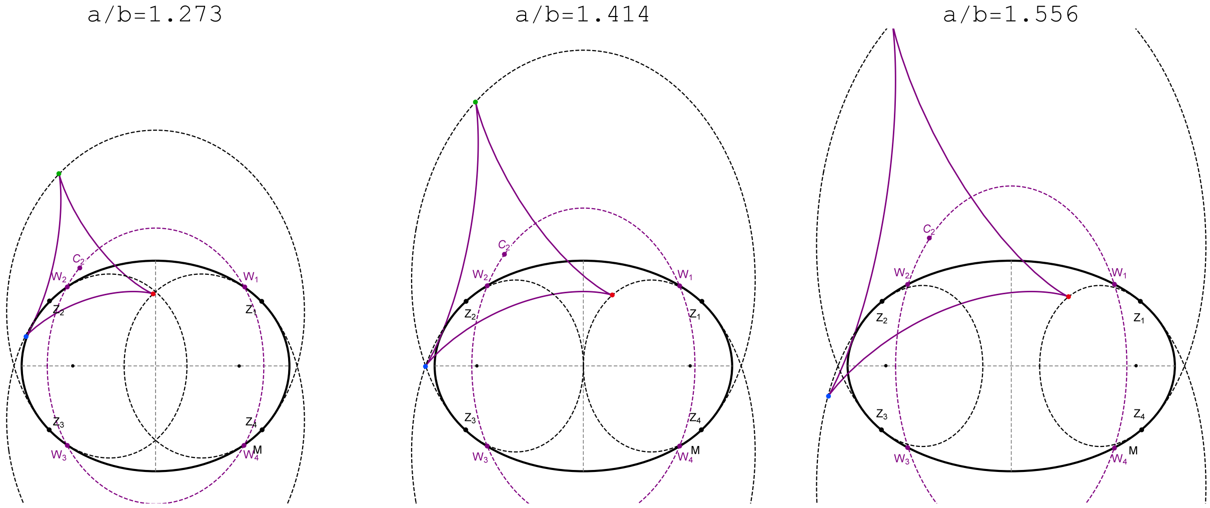

Figure 10: *The loci of the cusps of* Δ *(dashed line) is a degree-6 curve with 2 internal lobes with either 2, 3, or 4 self-intersections. From left to right,* $a/b = \{1.27, \sqrt{2}, 1.56\}$*. Note that at* $a/b = \sqrt{2}$ *the two lobes touch, i.e., the cusps pass through the center of* $\mathcal{E}$*. Also shown is the elliptic locus of* C_2 *(purple). Points* Z_i *(resp.* W_i*) mark off the intersection of the locus of the cusps (resp. of* C_2*) with* $\mathcal{E}$*. These never coincide* ***Video:*** *[10, PL#04].*

6.1 Tangencies and Intersections of the Deltoid with the Ellipse

Definition 1 (Apollonius Hyperbola) *Let* M *be a point on an ellipse* $\mathcal{E}$ *with semi-axes* a, b*. Consider a hyperbola* $\mathcal{H}$*, known as the Apollonius Hyperbola of* M *[5]:*

$$\mathcal{H} : \langle (x,y) - M, (y/b^2, -x/a^2) \rangle = 0.$$

Notice that for P on $\mathcal{E}$, only the points for which the normal at P points to M will lie on $\mathcal{H}$. See also [4, page 403]. Additionally, $\mathcal{H}$ passes through M and O, and its asymptotes are parallel to the axes of $\mathcal{E}$.

Proposition 11 Δ *is tangent to* $\mathcal{E}$ *at* $\mathcal{E} \cap \mathcal{H}$*, at 1, 2 or 3 points* Q_i *depending on whether* M *is exterior or interior to the evolute* $\mathcal{E}^*$*.*

Proof. Δ is tangent to $\mathcal{E}$ at some Q_i if the normal of $\mathcal{E}$ at Q_i points to $M = (M_x, M_y)$, i.e., when $\mathcal{H}$ intersects with $\mathcal{E}$. It can be shown that their x coordinate is given by the real roots of:

$$Q(x) = c^4x^3 - c^2M_x(a^2+b^2)x^2 - a^4(a^2-2b^2)x + a^6M_x = 0 \tag{9}$$

The discriminant of the above is:

$$-4c^4a^6(a^2 - M_x^2)[(a^2-b^2)(a^2+b^2)^3M_x^2 + a^4(a^2-2b^2)^3].$$

Let $\pm x^*$ denote the solutions to $(a^2-b^2)(a^2+b^2)^3M_x^2 + a^4(a^2-2b^2)^3 = 0$. Assuming $a > b$, Equation 9 has three real solutions when $|x| < x^*$. The intersections of the evolute $\mathcal{E}^*$ with the ellipse $\mathcal{E}$ are given by the four points $(\pm x^*, \pm y^*)$, where:

$$x^* = \frac{a^2\sqrt{a^4-b^4}\,(a^2-2b^2)^{\frac{3}{2}}}{(a^4-b^4)\,(a^2+b^2)}$$

$$y^* = \frac{b^2\sqrt{a^4-b^4}\,(2a^2-b^2)^{\frac{3}{2}}}{(a^4-b^4)\,(a^2+b^2)} \tag{10}$$

For $M \in \mathcal{E} \cap \mathcal{E}^*$ two coinciding roots result in a 4-point contact between Δ and the ellipse. □

Let $M = (M_x, M_y)$ be a point on $\mathcal{E}$ and $\mathcal{J}(x)$ denote the following cubic polynomial:

$$\mathcal{J}(x) = (a^2+b^2)^2x^2 - 2M_xc^2(a^2+b^2)x - 4a^4b^2 + M_x^2(a^2+b^2)^2 \tag{11}$$

Proposition 12 Δ *intersects* $\mathcal{E}$ *at* $Q(x)\mathcal{J}(x) = 0$*, in at least 3 and up to 5 locations locations, where* Q *is as in Equation 9.*

Proof. As before, $M = (M_x, M_y) = (a\cos u, b\sin u) \in \mathcal{E}$ and $P = (x,y) = (a\cos t, b\sin t)$. The intersection $\Delta_u(t)$ with $\mathcal{E}$ is obtained by setting $\mathcal{E}(\Delta_u(t)) = 0$. Using Equation 1, obtain the following system:

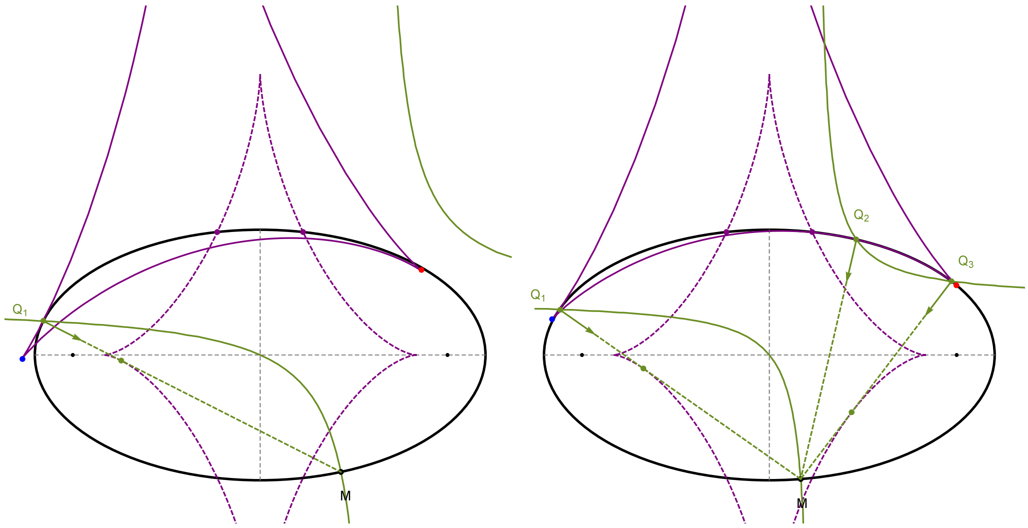

Figure 11: *Steiner's Hat Δ (purple, top cusp not shown) is tangent to $\mathcal{E}$ at the intersections Q_i of the Apollonius Hyperbola $\mathcal{H}$ (olive green) with $\mathcal{E}$, excluding M. Notice $\mathcal{H}$ passes through the center of $\mathcal{E}$.* ***Left:*** *When M is exterior to the evolute $\mathcal{E}^*$ (dashed purple), only one tangent Q_1 is present.* ***Right:*** *When M is interior to $\mathcal{E}^*$, three tangent points $Q_i, i = 1,2,3$ arise. The intersections of $\mathcal{E}^*$ are given in Equation 10. Note: the area ratio of Δ-to-$\mathcal{E}^*$ is always $4/3$.*

$$\begin{aligned}F(x,y) =& b^2\left(a^2-2M_x^2\right)\left(a^2+b^2\right)c^4x^4\\ &+2a^2M_xM_y\left(a^2+b^2\right)c^4x^3y\\ &+2a^2b^2M_xc^2\left(a^4+b^4\right)x^3-2a^4M_yc^2\left(a^4+b^4\right)x^2y\\ &+[-a^4b^2\left(a^2+b^2\right)\left(3a^4-4a^2b^2+2b^4\right)\\ &\quad+a^2b^2c^2M_x^2\left(3a^2-b^2\right)\left(a^2+b^2\right)]x^2\\ &-2a^6M_xM_yc^2\left(a^2+b^2\right)xy\\ &-2b^2M_xa^6\left(a^4-a^2b^2+b^4\right)x\\ &+2a^{12}M_yy+a^8b^2\left(2a^4-(a^2+b^2)M_x^2\right)=0\\ \mathcal{E}(x,y) =& \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\end{aligned}$$

By Bézout's theorem, the system $\mathcal{E}(x,y) = F(x,y) = 0$ has eight solutions, with algebraic multiplicities taken into account. Δ has three points of tangency with ellipse, some of which may be complex, which by Proposition 11 are given by the zeros of $Q(x)$. Eliminating y by computing the resultant we obtain an Equation $G(x) = 0$ of degree 8 over x. Manipulation with a Computer Algebra System yields a compact representation for $G(x)$:

$$G(x) = Q(x)^2\mathcal{J}(x)$$

with $\mathcal{J}$ as in Equation 11. If $|M_x| \le a$, the solutions of $\mathcal{J}(x) = 0$ are real and given by $J_x = [c^2M_x \pm 2ab\sqrt{a^2-M_x^2}]/(a^2+b^2)$ and $|J_x| \le a$.

□

Referring to Figure 10:

Proposition 13 *When a cusp P_i' crosses the boundary of $\mathcal{E}$, it coincides with its pre-image P_i at $Z_i = \frac{1}{\sqrt{a^2+b^2}}(\pm a^2, \pm b^2)$.*

Proof. Assume P_i' is on $\mathcal{E}$. Since P_i' is on the circle $\mathcal{K}_i$ defined by M, P_i and P_i' which osculates $\mathcal{E}$ at P_i, this circle intersects $\mathcal{E}$ at P_i with order of contact 3 or 4. By construction, we have $MP_i \perp P_iP_i'$, so M and P_i' are diametrically opposite in $\mathcal{K}_i$. Thus, M and P_i' must be distinct. Since two conics have at most 4 intersections (counting multiplicities), we either have $P_i' = P_i$ or $P_i = M$. The second case will only happen when M is on one of the four vertices of the ellipse $\mathcal{E}$, in which case the osculating circle $\mathcal{K}_i$ has order of contact 4, so P_i' could not also be in the ellipse in the first place. Thus, $P_i' = P_i$ as we wanted.
Substituting the parameterization of P_i in the equation of $\mathcal{E}$, we explicitly find the four points Z_i at which P_i' can intersect the ellipse $\mathcal{E}$. □

7 A Triad of Osculating Circles

Recall $\mathcal{K}_i$ are the circles which osculate $\mathcal{E}$ at the pre-images P_i, see Figure 8. Define a triangle T'' by the centers P_i'' of the $\mathcal{K}_i$. These are given explicit coordinates in Appendix A. Referring to Figure 12:

Proposition 14 *Triangles T' and T'' are homothetic at ratio $2:1$, with M as the homothety center.*

Proof. From the construction of Δ, for each $i = 1,2,3$ we have $MP_i \perp P_iP_i'$, that is, $\angle MP_iP_i' = 90°$. Hence, MP_i' is a diameter of the osculating circle $\mathcal{K}_i$ that goes through M, P_i, and P_i' as proved in Proposition 3. Thus, the center P_i'' of $\mathcal{K}_i$ is the midpoint of MP_i' and therefore P_i' is the image of P_i'' under a homothety of center M and ratio 2. □

Corollary 4 *The area A'' of T'' is invariant over all M and is $1/4$ that of T'.*

Proof. This follows from the homothety, and the fact that the area of T' is invariant from Proposition 4. □

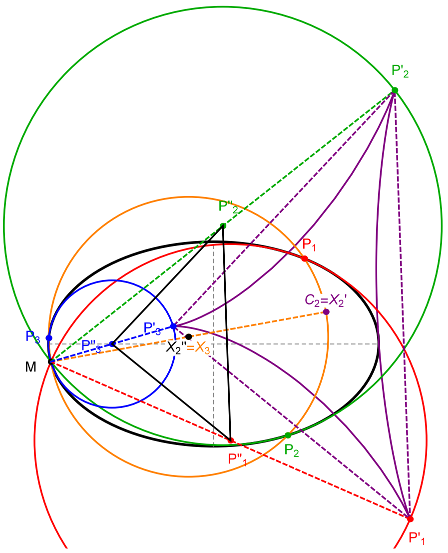

Figure 12: *Lines connecting each cusp P_i' to the center P_i'' of the circle which osculates $\mathcal{E}$ at the pre-image P_i concur at M. Note these lines are diameters of said circles. Therefore M is the perspector of T' and T'', i.e., the ratio of their areas is 4. This perspectivity implies C_2, X_2'', M are collinear. Surprisingly, the X_2'' coincides with the circumcenter X_3 of the pre-image triangle T (not drawn).*

Proposition 15 *Each (extended) side of T' passes through an intersection of two osculating circles. Moreover, those sides are perpendicular to the radical axis of said circle pairs.*

Proof. It suffices to prove it for one of the sides of T' and the others are analogous. Let M_1 be the intersection of $\mathcal{K}_2$ and $\mathcal{K}_3$ different than M and let $M_{1/2}$ be the midpoint of M and M_1. Since MM_1 is the radical axis of $\mathcal{K}_2$ and $\mathcal{K}_3$, the lines $P_2''P_3''$ and MM_1 are perpendicular and their intersection is $M_{1/2}$. Applying an homothety of center M and ratio 2, we get that the lines $P_2'P_3'$ and MM_1 (the radical axis) are perpendicular and their intersection is M_1, as desired. □

Corollary 5 *The Steiner ellipse $\mathcal{E}''$ of triangle T'' is similar to $\mathcal{E}'$. In fact, $\mathcal{E}'' = \frac{1}{2}\mathcal{E}'$.*

Proof. This follows from the homothety of T' and T''. □

8 Relations between T, T', T''

As before we identify Triangle centers as X_k after Kimberling's Encyclopedia [6].

Proposition 16 *The circumcenter X_3 of T coincides with the barycenter X_2'' of T''.*

Proof: Follows from direct calculations using the coordinate expressions of P_i and P_i''. In fact,

$$X_2'' = X_3 = \frac{c^2}{4}\left[\frac{\cos u}{a}, -\frac{\sin u}{b}\right]. \quad \square$$

Corollary 6 *The homothety with center M and factor 2 sends X_2'' to $X_2' = C_2$.*

Proposition 17 *The lines joining a cusp P_i' to its preimage P_i concur at Δ's center of area C_2.*

Proof. From Theorem 2, points M and C_2 both lie on the circumcircle of T and form a diameter of this circle. Thus, for each $i = 1,2,3$, we have $\angle MP_iC_2 = 90°$. By construction, $\angle MP_iP_i' = 90°$, so P_i, P_i', and C_2 are collinear as desired. □

Referring to Figure 14(left):

Corollary 7 *$C_2 = X_2'$ is the perspector of T' and T.*

Lemma 2 *Given a triangle $\mathcal{T}$, and its Steiner Ellipse Σ, the normals at each vertex pass through the Orthocenter of $\mathcal{T}$, i.e., they are the altitudes.*

Proof. This stems from the fact that the tangent to Σ at a vertex of $\mathcal{T}$ is parallel to opposite side of $\mathcal{T}$ [12, Steiner Circumellipse]. □

Referring to Figure 14(right):

Proposition 18 *The orthocenter X_4 is the perspector of T and T''. Equivalently, a line connecting a vertex of T to the respective vertex of T'' is perpendicular to the opposite side of T.*

Proof. Since T has fixed X_2, $\mathcal{E}$ is its Steiner Ellipse. The normals to the latter at P_i pass through centers P_i'' since these are osculating circles. So by Lemma 2 the proof follows. □

Definition 2 *According to J. Steiner [3, p. 55], two triangles ABC and DEF are said to be orthologic if the perpendiculars from A to EF, from B to DF, and from C to DE are concurrent. Furthermore, if this holds, then the perpendiculars from D to BC, from E to AC, and from F to AB are also concurrent. Those two points of concurrence are called the centers of orthology of the two triangles [9].*

Note that orthology is symmetric but not transitive [9, p. 37], see Figure 13 for a non-transitive example involving a reference, pedal, and antipedal triangles.

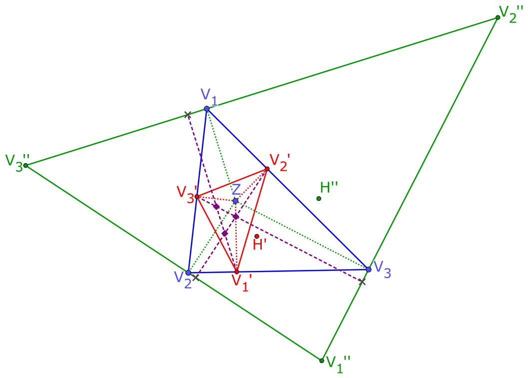

Figure 13: *Consider a reference triangle $\mathcal{T}$ (blue), and its pedal $\mathcal{T}'$ (red) and antipedal $\mathcal{T}''$ (green) triangles with respect to some point Z. Construction lines for both pedal and antipedal (dashed red, dashed green) imply that Z is an orthology center simultaneous for both $\mathcal{T},\mathcal{T}'$ and $\mathcal{T},\mathcal{T}''$, i.e. these pairs are orthologic. Also shown are H' and H'', the 2nd orthology centers of said pairs (construction lines omitted). Non-transitivity arises from the fact that perpendiculars dropped from the vertices of $\mathcal{T}'$ to the sides of $\mathcal{T}''$ (dashed purple, feet are marked X) are non-concurrent (purple diamonds mark the three disjoint intersections), i.e., $\mathcal{T}',\mathcal{T}''$ are not orthologic.*

Lemma 3 *Let $-P_i$ denote the reflection of P_i about $O = X_2$ for $i = 1,2,3$. Then the line from M to $-P_1$ is perpendicular to the line $P_2''P_3''$, and analogously for $-P_2$, and $-P_3$.*

Proof. This follows directly from the coordinate expressions for points M, $P_i = \mathcal{E}(t_i)$ and $P_i'' = \mathcal{E}^*(t_i)$. It follows that $\langle M + P_1, P_2'' - P_3'' \rangle = 0$. □

Referring to Figure 14(right):

Theorem 3 *Triangles T and T' are orthologic and their centers of orthology are the reflections X_{671} of M on X_2 and on X_4.*

Proof. We denote by X_{671} the reflection of $M = X_{99}$ on $O = X_2$. From Lemma 3, the line through M and $-P_1$ is perpendicular to $P_2''P_3''$. Reflecting about $O = X_2$, the line P_1X_{671} is also perpendicular to $P_2''P_3''$. Since $P_2''P_3'' \parallel P_2'P_3'$ from the homothety, we get that $P_1X_{671} \perp P_2'P_3'$. This means the perpendicular from P_1 to $P_2'P_3'$ passes through X_{671}. Analogously, the perpendiculars from P_2 to $P_1'P_3'$ and from P_3 to $P_1'P_2'$ also go through X_{671}. Therefore T and T' are orthologic and X_{671} is one of their two orthology centers.

Let X_h be the reflection of M on X_4. From Proposition 18, the line through P_1 and P_1'' passes through the orthocenter X_4 of T, that is, P_1'' is on the P_1-altitude of T. This means that the perpendicular from P_1'' to the line P_2P_3 passes through X_4. Applying the homothety with center M and ratio 2, the perpendicular from X_1' to X_2X_3 passes through X_h. Analogously, the perpendiculars from X_2' to X_1X_3 and from X_3' to X_1X_2 also pass through X_h. Hence, X_h is the second orthology center of T and T'. □

Theorem 4 *Triangles T and T'' are orthologic and their centers of orthology are X_4 and the reflection X_{671} of M on X_2.*

Proof. From Proposition 18, the perpendiculars from P_1'' to P_2P_3, from P_2'' to P_1P_3, and from P_3'' to P_1P_2 all pass through X_4. Thus, triangles T and T'' are orthologic and X_4 is one of their two orthology centers.

As before, we denote by X_{671} the reflection of $M = X_{99}$ on $O = X_2$. Again, from Lemma 3, the line through M and $-P_1$ is perpendicular to $P_2''P_3''$, so reflecting it at X_2, we get that $P_1X_{671} \perp P_2''P_3''$. Since the triangles T'' and T' have parallel sides, we get $P_1X_{671} \perp P_1'P_3' \parallel P_1''P_3''$. Thus, X_{671} is the second orthology center of T and T''. □

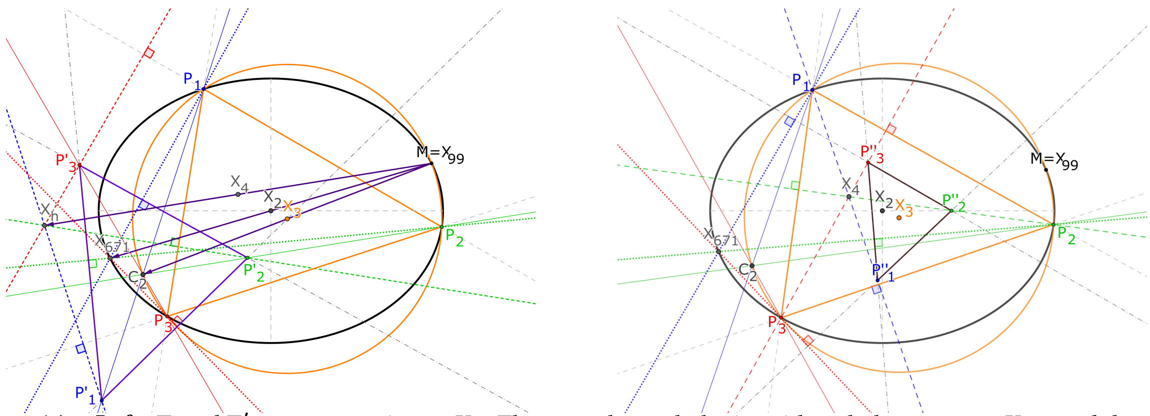

Figure 14: ***Left:*** *T and T' are perspective on X_2. They are also orthologic, with orthology centers X_{671} and the reflection of X_{99} on X_4.* ***Right:*** *T and T'' are perspective on X_4. They are also orthologic, with orthology centers X_4 and X_{671}.*

Theorem 5 (Sondat's Theorem) *If two triangles are both perspective and orthologic, their centers of orthology and perspectivity are collinear. Moreover, the line through these centers is perpendicular to the perspectrix of the two triangles [11, 9].*

Referring to Figure 15:

Theorem 6 *The perspectrix of T and T' is perpendicular to the Euler Line of T.*

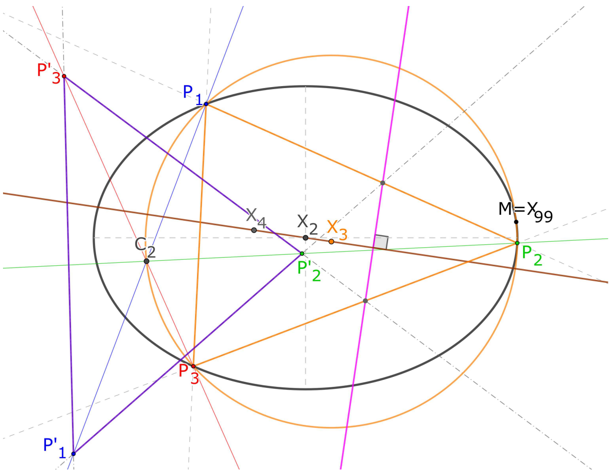

Figure 15: *The perspectrix of T, T'' is perpendicular to the line through X_4 and X_{671}. Compare with Figure 13: the perspectrix of T, T' is perpendicular to the Euler Line of T.*

Proof. Since T and T' are both orthologic and perspective from Corollary 7 and Theorem 3, by Sondat's Theorem, their perspectrix is perpendicular to the line through their orthology centers (reflections of M at X_2 and X_4) and perspector ($X_2' = C_2 = X_{98}$ =reflection of M at X_3). By applying a homothety of center M and ratio 1/2, this last line is parallel to the line through X_2, X_3, and X_4, the Euler line of T. Therefore the perspectrix of T and T' is perpendicular to the Euler line of T. □

Proposition 19 *The perspectrix of T and T'' is perpendicular to the line $X_4 X_{671}$ (which is parallel to the line through M and X_{376}, the reflection of X_2 at X_3).*

Proof. Since T and T'' are both orthologic and perspective from Proposition 18 and Theorem 4, by Sondat's Theorem, their perspectrix is perpendicular to the line through their orthology centers X_4 and X_{671}. Reflecting this last line at X_2, we find that it is parallel to the line through M and the reflection of X_4 at X_2, which is the same as the reflection of X_2 at X_3. □

Table 2 lists a few pairs of triangle centers numerically found to be common over T, T' or T, T''.

T	T'	T''
X_3	–	X_2''
X_4	–	X_{671}''
X_5	–	X_{115}''
X_{20}	–	X_{99}''
X_{76}	–	X_{598}''
X_{98}	X_2'	–
X_{114}	X_{230}'	–
X_{382}	–	X_{148}''
X_{548}	–	X_{620}''
X_{550}	–	X_{2482}''

Table 2: *Triangle Centers which coincide T, T' or T, T''.*

9 Addendum: Rotated Negative Pedal Curve

The Negative Pedal Curve is the envelope of lines $L(t)$ passing through $P(t)$ and perpendicular to $P(t) - M$. Here we consider the envelope Δ_θ^* of the $L(t)$ rotated clockwise a fixed θ about $P(t)$; see Figure 16.

Proposition 20 *Δ_θ^* is the image of the NPC Δ under the similarity which is the product of a rotation about M through θ and a homothety with center M and factor $\cos\theta$.*

Proof. For variable parameter t, the lines $L(t)$ and $L_\theta^*(t)$ perform a motion which sends P along $\mathcal{E}$, while the line through P orthogonal to $L(t)$ slides through the fixed point M. Due to basic results of planar kinematics [1, p. 274], the instantaneous center of rotation I lies on the normal to $\mathcal{E}$ at P and on the normal to MP at M. We obtain a rectangle with vertices P, M and I. The fourth vertex is the enveloping point C of $L(t)$. The enveloping point C^* of L_θ^* is the pedal point of I. Since the circumcircle of the rectangle with diameter MC also passes through C^*, we see that C^* is the image of C under the stated similarity, Figure 17.

This holds for all points on Δ, including the cusps, but also for the center C_2. At poses where C reaches a cusp P_i' of Δ, then for all lines $L_\theta^*(t)$ through P the point C^* is a cusp of the corresponding envelope. Then the point is the so-called return pole, and the circular path of C the return circle or cuspidal circle [1, p. 274]. □

Corollary 8 *The area of Δ_θ^* is independent of M and is given by:*

$$A = \frac{c^4 \cos^2\theta\, \pi}{2ab}.$$

Note this is equal to $\cos^2\theta$ of the area of Δ, see Equation 5.

Remark 11 *For variable* θ *between* -90° *and* 90°, *the said similarity defines an equiform motion where each point in the plane runs along a circle through* M *with the same angular velocity. For each point, the configuration at* $\theta = 0$ *and* M *define a diameter of the trajectory.*

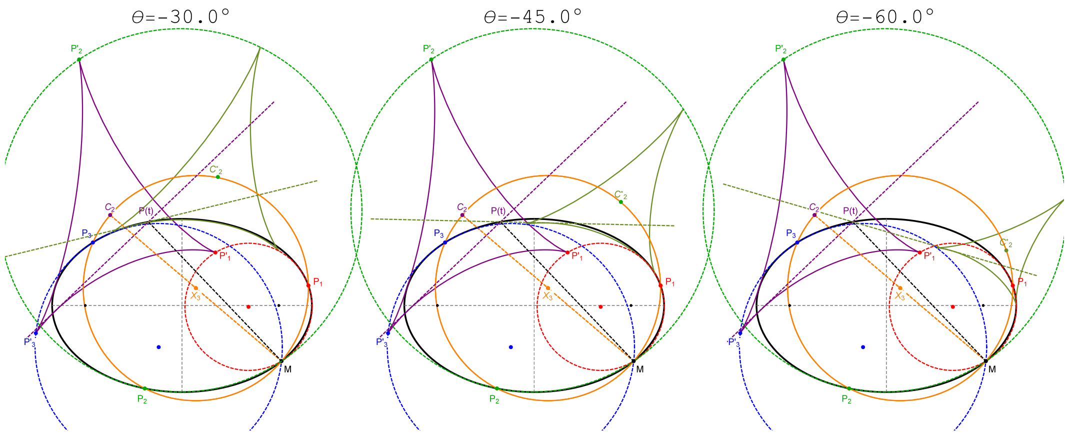

Figure 16: *From left to right: for a fixed* M, *the line passing through* $P(t)$ *and perpendicular to the segment* $P(t) - M$ *(dashed) purple is rotated clockwise by* $\theta = 30, 45, 60$ *degrees, respectively (dashed olive green). For all* $P(t)$ *these envelop new constant-area crooked hats* Δ^* *(olive green) whose areas are* $\cos(\theta)^2$ *that of* Δ. *For the* θ *shown, these amount to* $3/4, 1/2, 1/4$ *of the area of* Δ *(purple). As one varies* θ, *the center of area* C_2^* *of* Δ^* *sweeps the circular arc between* C_2 *and* M *with center at angle* 2θ. *The same holds for the cusps running along the corresponding osculating circles (shown dashed red, green, blue), which are stationary and independent of* θ. ***Video:*** *[10, PL#06]*

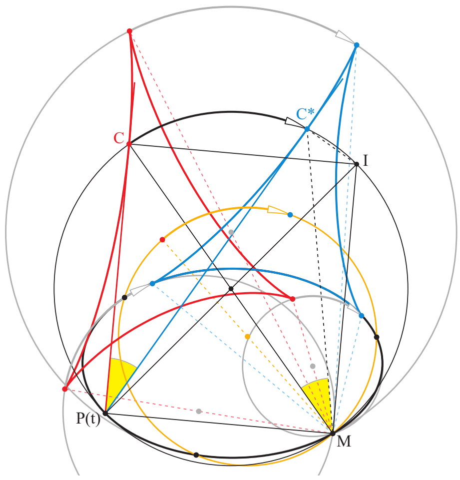

Figure 17: *The construction of points* C, C^* *of the envelopes* Δ *(red) and* Δ^* *(blue) with the help of the instant center of rotation* I *reveals that the rotation about* M *through* θ *and scaling with factor* $\cos\theta$ *sends* Δ *to* Δ^* *(Proposition 20).*

Recall the pre-images P_i of the cusps of Δ have vertices at $\mathcal{E}(t_i)$, where $t_1 = -\frac{u}{3}$, $t_2 = -\frac{u}{3} - \frac{2\pi}{3}$, $t_3 = -\frac{u}{3} - \frac{4\pi}{3}$, see Theorem 2.

Corollary 9 *The cusps* P_i^* *of* Δ_θ^* *have pre-images on* $\mathcal{E}$ *which are invariant over* θ *and are congruent with the* $P_i, i = 1,2,3$.

Corollary 10 *Lines* $P_iP_i^*$ *concur at* C_2^*.

Corollary 11 C_2^* *is a rotation of* C_2 *by* 2θ *about the center* X_3 *of* $\mathcal{K}$. *In particular, When* $\theta = \pi/2$, $C_2^* = M$, *and* Δ_θ^* *degenerates to point* M.

10 Conclusion

Before we part, we would like to pay homage to eminent swiss mathematician Jakob Steiner (1796–1863), discoverer of several concepts appearing herein: the Steiner Ellipse and Inellipse, the Steiner Curve (or hypocycloid), the Steiner Point X_{99}. Also due to him is the concept of orthologic triangles and the theorem of 3 concurrent osculating circles in the ellipse. Hats off and vielen dank, Herr Steiner!

Some of the above phenomena are illustrated dynamically through the videos on Table 3.

PL#	Title	Narrated
01	Constant-Area Deltoid	no
02	Properties of the Deltoid	yes
03	Osculating Circles at the Cusp Pre-Images	yes
04	Loci of Cusps and C_2	no
05	Concyclic pre-images, osculating circles, and 3 area-invariant triangles	no
06	Rotated Negative Pedal Curve	yes

Table 3: *Playlist of videos. Column "PL#" indicates the entry within the playlist.*

Appendix A. Explicit Expressions for the P_i, P_i', P_i''

$$P_1 = \left[a\cos\frac{u}{3}, -b\sin\frac{u}{3},\right]$$
$$P_2 = \left[-a\sin(\frac{u}{3}+\frac{\pi}{6}), -b\cos(\frac{u}{3}+\frac{\pi}{6})\right]$$
$$P_3 = \left[-a\cos(\frac{u}{3}+\frac{\pi}{6}), b\sin(\frac{u}{3}+\frac{\pi}{6})\right]$$
$$P_1' = \left[\frac{3c^2}{2a}\cos\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2a}\cos u, \frac{3c^2}{2b}\sin\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2b}\sin u\right]$$
$$P_2' = \left[-\frac{3c^2}{4a}\cos\frac{u}{3} - \frac{3\sqrt{3}c^2}{4a}\sin\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2a}\cos u, \frac{3\sqrt{3}c^2}{4b}\cos\frac{u}{3} - \frac{3c^2}{4b}\sin\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2b}\sin u\right]$$
$$P_3' = \left[-\frac{3c^2}{4a}\cos\frac{u}{3} + \frac{3\sqrt{3}c^2}{4a}\sin\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2a}\cos u, -\frac{3\sqrt{3}c^2}{4b}\cos\frac{u}{3} - \frac{3c^2}{4b}\sin\frac{u}{3} - \frac{\left(a^2+b^2\right)}{2b}\sin u\right]$$
$$P_1'' = \left[\frac{3c^2}{4a}\cos\frac{u}{3} + \frac{c^2}{4a}\cos u, \frac{3c^2}{4b}\sin\frac{u}{3} + \frac{c^2}{4b}\sin u\right]$$
$$P_2'' = \left[-\frac{3c^2}{8a}\cos\frac{u}{3} - \frac{3\sqrt{3}c^2}{8a}\sin\frac{u}{3} + \frac{c^2}{4a}\cos u, \frac{3\sqrt{3}c^2}{8b}\cos\frac{u}{3} - \frac{3c^2}{8b}\sin\frac{u}{3} - \frac{c^2}{4b}\sin u\right]$$
$$P_3'' = \left[-\frac{3c^2}{8a}\cos\frac{u}{3} + \frac{3\sqrt{3}c^2}{8a}\sin\frac{u}{3} + \frac{c^2}{4a}\cos u, -\frac{3\sqrt{3}c^2}{8b}\cos\frac{u}{3} - \frac{3c^2}{8b}\sin\frac{u}{3} - \frac{c^2}{4b}\sin u\right]$$

Appendix B.Table of Symbols

symbol	meaning	note
$\mathcal{E}$	main ellipse	
a,b	major, minor semi-axes of $\mathcal{E}$	
c	half the focal length of $\mathcal{E}$	$c^2 = a^2 - b^2$
O	center $\mathcal{E}$	
M, M_u	a fixed point on the boundary of $\mathcal{E}$	$[a\cos u, b\sin u]$, perspector of T', T'', $= X_{99}$
$P(t)$	a point which sweeps the boundary of $\mathcal{E}$	$[a\cos t, b\sin t]$
$L(t)$	Line through $P(t)$ perp. to $P(t) - M$	
Δ, Δ_u	Steiner's Hat, negative pedal curve of $\mathcal{E}$ with respect to M	invariant area
Δ^*_θ	envelope of $L(t)$ rotated θ about $P(t)$	invariant area
$\bar{C}$	average coordinates of Δ	$= C_2 = X'_2$
C_2, C^*_2	area center of Δ, Δ^*	$C_2 = X'_2 = X_{98}$
P'_i, P_i, P''_i	The cusps of Δ, their pre-images, and centers of K_i (see below)	
P^*_i	cusps of Δ^*	$P_iP^*_i$ concur at C^*_2
T, T', T''	triangles defined by the P_i, P'_i, P''_i	invariant area over M
A, A', A''	areas of T, T', T''	$A'/A'' = 4$ for any M, a, b
$\mathcal{S}$	Steiner's Curve	aka. Hypocycloid and Triscupoid
$\mathcal{E}'$	Steiner Circumellipse of cusp (P'_i) triangle	centered at C_2
a', b'	major, minor semi-axes of $\mathcal{E}'$	invariant, axis-parallel and similar to 90°-rotated $\mathcal{E}$
$\mathcal{K}$	Circumcircle of T	center X_3, contains M, P_i, C_2, C^*_2
$\mathcal{K}'$	Circumcircle of T'	
$\mathcal{K}_i$	Circles osculating $\mathcal{E}$ at the P_i	contain P_i, P'_i, M
$\mathcal{E}^*$	evolute of $\mathcal{E}$	the K_i lie on it
$\mathcal{H}$	Apollonius Hyperbola of $\mathcal{E}$ wrt M	Δ is tangent to $\mathcal{E}$ at $\mathcal{H} \cap \mathcal{E}$
X_3	circumcenter of T	$= X''_2$
X_4	perspector of T and T''	
X_{99}	Steiner Point of T	$= M$
X_{98}	Tarry Point of T	$= C_2$
X'_2	centroid of T'	$= C_2$, and perspector of T, T'
X'_{99}	Steiner Point of T'	
X''_2	centroid of T''	$= X_3$

Table 4: *All Symbols used.*

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