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TOTALLY REAL THUE INEQUALITIES OVER IMAGINARY QUADRATIC FIELDS: AN IMPROVEMENT

István Gaál, Borka Jadrijević and László Remete
University of Debrecen, Hungary and University of Split, Croatia

Abstract. In this paper we significantly improve our previous results of reducing relative Thue inequalities to absolute ones.

1. Results

Let $F(x,y)$ be a binary form of degree $n \geq 3$ with rational integer coefficients. Assume that $f(x) = F(x,1)$ has leading coefficient 1 and distinct real roots $\alpha_1, \ldots, \alpha_n$. Let $0 < \varepsilon < 1$ and let $K \geq 1$. Let

$$A = \min_{i \neq j} |\alpha_i - \alpha_j|, \quad B = \min_i \prod_{j \neq i} |\alpha_j - \alpha_i|, \quad C = \frac{K}{(1-\varepsilon)^{n-1} B}, \quad G = \frac{K^{1/n}}{\varepsilon A}.$$

Let $m \geq 1$ be a square-free positive integer, and set $M = \mathbb{Q}(i\sqrt{m})$. Consider the relative inequality

$$|F(x,y)| \leq K \text{ in } x, y \in \mathbb{Z}_M. \tag{1.1}$$

If F is irreducible, then (1.1) is called a Thue inequality. We emphasize that our statements are valid also if F is reducible.

If $m \equiv 3 \pmod 4$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = x_1 + x_2 \frac{1 + i\sqrt{m}}{2} = \frac{(2x_1 + x_2) + x_2 i\sqrt{m}}{2},$$

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$$y = y_1 + y_2 \frac{1 + i\sqrt{m}}{2} = \frac{(2y_1 + y_2) + y_2 i\sqrt{m}}{2},$$

and if $m \equiv 1, 2 \pmod 4$, then

$$x = x_1 + x_2 i\sqrt{m}, \quad y = y_1 + y_2 i\sqrt{m},$$

in both cases with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. Set $s = 2$ if $m \equiv 3 \pmod 4$ and $s = 1$ if $m \equiv 1, 2 \pmod 4$. In the following theorem we formulate our statements parallelly in the two cases.

THEOREM 1.1. *Let $(x, y) \in \mathbb{Z}_M^2$ be a solution of (1.1). Then*

$$|F(sx_1 + (s-1)x_2, sy_1 + (s-1)y_2)| \leq s^n K, \quad |F(x_2, y_2)| \leq \frac{s^n K}{(\sqrt{m})^n}, \tag{1.2}$$

and

$$|F(sx_1 + (s-1)x_2, sy_1 + (s-1)y_2)| \cdot |F(x_2, y_2)| \leq \frac{s^{2n} K^2}{2^n \cdot (\sqrt{m})^n}. \tag{1.3}$$

If $|y| > \max\left\{G, \left(\frac{s \cdot C}{\sqrt{m}}\right)^{\frac{1}{n-2}}\right\}$, then $x_2 y_1 = x_1 y_2$.
If $|y| > \max\left\{G, (s \cdot C)^{\frac{1}{n-1}}\right\}$ and $sy_1 + (s-1)y_2 = 0$, then $sx_1 + (s-1)x_2 = 0$.
If $|y| > \max\left\{G, \left(\frac{s \cdot C}{\sqrt{m}}\right)^{\frac{1}{n-1}}\right\}$ and $y_2 = 0$, then $x_2 = 0$.

REMARK 1.2. The present inequality (1.2) is much sharper than the corresponding inequalities of Theorem 2.1 of [1]. Moreover we obtain these inequalities without any conditions on the variables. This makes the applications much easier. If the values of F are non-zero, then (1.3) yields further new restrictions for the possible solutions of (1.1).

PROOF OF THEOREM 1.1. Let $(x, y) \in \mathbb{Z}_M^2$ be an arbitrary solution of (1.1). Let $\beta_j = x - \alpha_j y, \ j = 1, \ldots, n$, then inequality (1.1) can be written as

$$|\beta_1 \cdots \beta_n| \leq K. \tag{1.4}$$

We have

$$\beta_j = \frac{1}{s}((sx_1 + (s-1)\,x_2) - \alpha_j(sy_1 + (s-1)\,y_2)) + \frac{i\sqrt{m}}{s}(x_2 - \alpha_j y_2).$$

Obviously,

$$|\mathrm{Re}(\beta_j)| \leq |\beta_j|, \quad |\mathrm{Im}(\beta_j)| \leq |\beta_j|, \ 1 \leq j \leq n.$$

Further,

$$\prod_{j=1}^{n} |\mathrm{Re}(\beta_j)| \leq \prod_{j=1}^{n} |\beta_j| \leq K, \quad \text{and} \quad \prod_{j=1}^{n} |\mathrm{Im}(\beta_j)| \leq \prod_{j=1}^{n} |\beta_j| \leq K,$$

which imply (1.2). Moreover,

$$\prod_{j=1}^{n}|\mathrm{Re}(\beta_j)|\cdot\prod_{j=1}^{n}|\mathrm{Im}(\beta_j)| = \prod_{j=1}^{n}\left(|\mathrm{Re}(\beta_j)|\cdot|\mathrm{Im}(\beta_j)|\right)$$
$$\leq \prod_{j=1}^{n}\frac{|\mathrm{Re}(\beta_j)|^2+|\mathrm{Im}(\beta_j)|^2}{2} = \prod_{j=1}^{n}\frac{|\beta_j|^2}{2} \leq \frac{K^2}{2^n},$$

whence we obtain (1.3).

Assume now

$$|y|\geq G. \tag{1.5}$$

Let i_0 be the index with $|\beta_{i_0}|=\min_j|\beta_j|$. Then $|\beta_{i_0}|\leq K^{\frac{1}{n}}$ and for $j\neq i_0$

$$|\beta_j|\geq|\beta_j-\beta_{i_0}|-|\beta_{i_0}|\geq|\alpha_j-\alpha_{i_0}|\cdot|y|-K^{\frac{1}{n}}\geq(1-\varepsilon)\cdot|\alpha_j-\alpha_{i_0}|\cdot|y|. \tag{1.6}$$

From (1.4) and (1.6) we have

$$|\beta_{i_0}|\leq\frac{K}{\prod_{j\neq i_0}|\beta_j|}\leq\frac{C}{|y|^{n-1}}. \tag{1.7}$$

Using that $\alpha_{i_0}|y|^2$ is real, by (1.7) we obtain

$$|\mathrm{Im}(x\overline{y})| = |\mathrm{Im}(\alpha_{i_0}|y|^2-x\overline{y})|\leq\left|\alpha_{i_0}|y|^2-x\overline{y}\right|$$
$$=|y|^2\cdot\left|\alpha_{i_0}-\frac{x\overline{y}}{y\overline{y}}\right| = |y|^2\cdot\left|\alpha_{i_0}-\frac{x}{y}\right|\leq\frac{C}{|y|^{n-2}}.$$

If

$$|y|>\left(\frac{s\cdot C}{\sqrt{m}}\right)^{\frac{1}{n-2}},$$

then this implies $x_2y_1=x_1y_2$.

Inequality (1.7) indicates that $|\beta_{i_0}|$ is small for sufficiently large $|y|$ and so are its real and imaginary parts that can equal zero if we impose some extra assumptions.

– If $|y|>(sC)^{\frac{1}{n-1}}$, then $|(sx_1+(s-1)\,x_2)-\alpha_{i_0}(sy_1+(s-1)\,y_2)|<1$. So, $sy_1+(s-1)\,y_2=0$ implies $sx_1+(s-1)\,x_2=0$.

– If $|y|>\left(\frac{sC}{\sqrt{m}}\right)^{\frac{1}{n-1}}$, then $|x_2-\alpha_{i_0}y_2|<1$. So, $y_2=0$ implies $x_2=0$. □

2. How to apply Theorem 1.1?

Finally, we give useful hints for a practical application of Theorem 1.1. Using the same notation let us consider again the relative inequality (1.1). We describe our algorithm in case $m\equiv 3 \pmod 4$, since the other case is completely similar.

First, we solve $F(x_2, y_2) = k_1$ for all $k_1 \in \mathbb{Z}$ with $|k_1| \leq 2^n K/(\sqrt{m})^n$. Since the equation $F(x_2, y_2) = 0$ can also have non-trivial solutions if F is reducible, we split our arguments into two cases.

A. First suppose $F(x_2, y_2) = 0$. This makes possible to determine x_2, y_2. If F is irreducible, then $x_2 = y_2 = 0$, if F is reducible, then x_2, y_2 can be determined easily (if there are any). We then determine the solutions $(a, b) \in \mathbb{Z}^2$ of $|F(a, b)| = k_2$ for all k_2 with $|k_2| \leq 2^n K$. Using all possible values of x_2, y_2 for each solution (a, b) we determine $x_1 = (a - x_2)/2, y_1 = (b - y_2)/2$ and check if these are integers. (Note that if F is irreducible, then $x_2 = y_2 = 0$ implies $|F(x_1, y_1)| \leq K$ and the procedure can be simplified.) Having all possible x_1, x_2, y_1, y_2 we test if $(x, y) \in \mathbb{Z}_M^2$ is a solution of (1.1).

B. Assume now $F(x_2, y_2) = k_1 \neq 0$ for some $(x_2, y_2) \in \mathbb{Z}^2$. Then we solve $F(a, b) = k_2$ in $(a, b) \in \mathbb{Z}^2$ for all $k_2 \in \mathbb{Z}$ with $|k_1 k_2| \leq 2^n K^2/(\sqrt{m})^n$ (a part of this calculation was already performed by solving $F(x_2, y_2) = k_1$). Having a, b, x_2, y_2 we calculate $x_1 = (a - x_2)/2, y_1 = (b - y_2)/2$. For x_2, y_2 and integer values x_1, y_1 we test if $(x, y)^2 \in \mathbb{Z}_M$ is indeed a solution of (1.1).

Remark 2.1. If m is sufficiently large, then by (1.2) we have $|F(x_2, y_2)| < 1$. In case F is irreducible, this implies $x_2 = y_2 = 0$, whence (1.1) reduces to an inequality in x_1, y_1 over $\mathbb{Z}$.

Remark 2.2. Solving Thue equations over $\mathbb{Z}$ is no problem any more by using well-known computer algebra packages. If F is reducible, this task is even easier.

References

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I. Gaál
Mathematical Institute, University of Debrecen
H–4002 Debrecen Pf.400.
Hungary
E-mail: gaal.istvan@unideb.hu

B. Jadrijević
Faculty of Science, University of Split
Rudera Boškovića 33, 21000 Split
Croatia
E-mail: borka@pmfst.hr

L. Remete
Mathematical Institute, University of Debrecen
H–4002 Debrecen Pf.400
Hungary
E-mail: remete.laszlo@science.unideb.hu

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