GLASNIK MATEMATIČKI
Vol. 55(75)(2020), 195 – 201

A NOTE ON THE EXPONENTIAL DIOPHANTINE EQUATION $(A^2n)^x + (B^2n)^y = ((A^2+B^2)n)^z$

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Abstract. Let A, B be positive integers such that $\min\{A,B\} > 1$, $\gcd(A,B) = 1$ and $2|B$. In this paper, using an upper bound for solutions of ternary purely exponential Diophantine equations due to R. Scott and R. Styer, we prove that, for any positive integer n, if $A > B^3/8$, then the equation $(A^2n)^x + (B^2n)^y = ((A^2+B^2)n)^z$ has no positive integer solutions (x,y,z) with $x > z > y$; if $B > A^3/6$, then it has no solutions (x,y,z) with $y > z > x$. Thus, combining the above conclusion with some existing results, we can deduce that, for any positive integer n, if $B \equiv 2 \pmod 4$ and $A > B^3/8$, then this equation has only the positive integer solution $(x,y,z) = (1,1,1)$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let n be a positive integer, and let a, b be positive integers such that $\min\{a,b\} > 1$ and $\gcd(a,b) = 1$. Recently, P.-Z. Yuan and Q. Han ([9]) proposed the following conjecture:

Conjecture 1.1. *For any n, if $\min\{a,b\} \geq 4$, then the equation*

$$(an)^x + (bn)^y = ((a+b)n)^z,\ x,y,z \in \mathbb{N} \tag{1.1}$$

has only the solution $(x,y,z) = (1,1,1)$.

Since Conjecture 1.1 is much broader than Jeśmanowicz' conjecture concerning Pythagorean triples (see [2] and the survey paper on the conjectures of Jeśmanowicz and Terai which was published by G. Soydan, M. Demirci, I. N. Cangül and A. Togbé, ([5])), it is unlikely to be solved in the short term. There are only a few scattered results on Conjecture 1.1 at present (see [6]).

2020 *Mathematics Subject Classification.* 11D61.
Key words and phrases. Ternary purely exponential Diophantine equation.

Let A, B be positive integers such that $\min\{A, B\} > 1$, $\gcd(A, B) = 1$ and $2|B$. In the same paper, P.-Z. Yuan and Q. Han ([9]) deal with the solutions (x, y, z) of (1.1) for the case that $(a, b) = (A^2, B^2)$. Then (1.1) can be rewritten as

$$(A^2n)^x + (B^2n)^y = ((A^2 + B^2)n)^z, x, y, z \in \mathbb{N}. \tag{1.2}$$

For this equation, they proved that, for any $n > 1$, if $B \equiv 2 \pmod 4$, then (1.2) has no solutions (x, y, z) with $y > z > x$; in particular, if $B = 2$, then Conjecture 1.1 is true for any n.

In this paper, using an upper bound for solutions of ternary purely exponential Diophantine equations due to R. Scott and R. Styer ([4]), we prove a general result as follows:

Theorem 1.2. *For any n, if $A > B^3/8$, then (1.2) has no solutions (x, y, z) with $x > z > y$; if $B > A^3/6$, then (1.2) has no solutions (x, y, z) with $y > z > x$.*

Thus, combining Theorem 1.2 with the above mentioned results of [9], we can deduce the following corollary:

Corollary 1.3. *For any n, if $B \equiv 2 \pmod 4$ and $A > B^3/8$, then (1.2) has only the solution $(x, y, z) = (1, 1, 1)$.*

This implies that, for any fixed B with $B \equiv 2 \pmod 4$, then Conjecture 1.1 is true for $(a, b) = (A^2, B^2)$ except for finitely many values of A.

2. Lemmas

For any positive integer m, let $\text{rad}(m)$ denote the product of all distinct prime divisors of m, and let $\text{rad}(1) = 1$. Obviously, $\text{rad}(m)$ is equal to the largest squarefree divisor of m.

Lemma 2.1 ([6, Theorem 1.1], [9, Proposition 3.1]). *Assume $n > 1$ in (1.1) and let (x, y, z) be a solution of (1.1) with $(x, y, z) \neq (1, 1, 1)$. If $\min\{a, b\} > 2$, then either*

$$x > z > y,\ \text{rad}(n) \mid b,\ b = b_1b_2,\ b_1^y = n^{z-y},\ b_1, b_2 \in \mathbb{N},\ b_1 > 1,\ \gcd(b_1, b_2) = 1$$

or

$$y > z > x,\ \text{rad}(n) \mid a,\ a = a_1a_2,\ a_1^x = n^{z-x},$$
$$a_1, a_2 \in \mathbb{N},\ a_1 > 1,\ \gcd(a_1, a_2) = 1.$$

Remark 2.2. Because when $\min\{a, b\} = 2$, there might be a solution (x, y, z) to (1.1) with $y > z = x$ (see [1, 3, 7, 8]), the condition $\min\{a, b\} > 2$ in Lemma 2.1 is necessary.

Lemma 2.3. *If $B \equiv 2 \pmod 4$ and $(x, y, z) \neq (1, 1, 1)$ is a solution to (1.2), then $x > z > y$.*

Proof. When $n > 1$, Lemma 2.1 shows that Lemma 2.3 is equivalent to [9, Theorem 1.3].

So we can assume $n = 1$. Suppose (1.2) has a solution $(x, y, z) \neq (1, 1, 1)$, so that

$$A^{2x} + B^{2y} = (A^2 + B^2)^z. \tag{2.1}$$

Clearly $(1, 1, 1)$ is the only possible solution to (1.2) with $z = 1$, so in (2.1) we have

$$z > 1. \tag{2.2}$$

Since $z \geq 2$, if $\max\{x, y\} \leq z$, then we have

$$(A^2 + B^2)^z = A^{2x} + B^{2y} \leq A^{2z} + B^{2z} < (A^2 + B^2)^z,$$

a contradiction from which we get

$$z < \max\{x, y\}. \tag{2.3}$$

Next, we show that $y < z$ using a straightforward approach which works when $n = 1$ (as well as when $n > 1$ as in [9]).

It is a familiar elementary result (see, for example, [9, Lemma 3.2]) that, if (2.1) holds, there are positive integers u and v such that $2 \mid v$, $u^2 + v^2 = A^2 + B^2$, $(u, v) = 1$, and

$$\pm(u \pm v\sqrt{-1})^z = A^x + B^y\sqrt{-1},$$

with

$$\nu_2(v) + \nu_2(z) = \nu_2(B^y) \tag{2.4}$$

where, for any positive integer m, $2^{\nu_2(m)} \mid\mid m$. $2 \mid\mid B$, so $A^2+B^2 \equiv 5 \pmod 8$, so $2 \mid\mid v$, so that (2.4) becomes

$$1 + \nu_2(z) = y,$$

so that

$$z \geq 2^{y-1} \geq y \tag{2.5}$$

and $z = y$ implies $y \leq 2$. Since $z > 1$ and $y = z = 2$ implies

$$A^{2x} = A^2(A^2 + 2B^2)$$

which contradicts $(A, 2B) = 1$, we must have

$$y < z. \tag{2.6}$$

(2.3) and (2.6) combine to give $y < z < x$. ∎

Lemma 2.4 ([9, Theorem 1.4]). *For any n, if $B = 2$, then* (1.2) *has only the solution* $(x, y, z) = (1, 1, 1)$.

LEMMA 2.5 ([4, Theorem 3]). *Let G, H, K be fixed positive integers with $\min\{G, H, K\} > 1$, $\gcd(G, H) = 1$ and $2 \nmid K$. Further, let PQ be the largest squarefree divisor of GH, with P and Q chosen so that $(GH/P)^{1/2}$ is an integer. If there exists a positive integer Z such that $G + H = K^Z$, then Z satisfies*

$$Z \begin{cases} \leq \frac{1}{2}Q, & \text{if} \quad P = 1, \\ \leq \frac{1}{2}(Q+1), & \text{if} \quad P = 2, \\ < \frac{1}{2}P^{1/2}Q \log P, & \text{if} \quad P \geq 3. \end{cases} \tag{2.7}$$

LEMMA 2.6. *Under the assumptions of Lemma 2.5, we have*

$$Z \leq \frac{1}{2}PQ. \tag{2.8}$$

PROOF. Obviously, by (2.7), (2.8) holds for $P \leq 2$. Let

$$f(t) = \frac{\log t}{t^{1/2}},\ t \geq 3. \tag{2.9}$$

Then we have

$$f'(t) = \frac{2 - \log t}{2t^{3/2}},\ t \geq 3, \tag{2.10}$$

where $f'(t)$ is the derivative of $f(t)$. We see from (2.9) and (2.10) that $f(e^2) = 2/e$ is the maximum value of $f(t)$. Therefore, if $P \geq 3$, then from (2.7) and (2.9) we get

$$\begin{aligned} Z < \frac{1}{2}P^{1/2}Q \log P &= \left(\frac{1}{2}PQ\right)\left(\frac{\log P}{P^{1/2}}\right) \\ &= \left(\frac{1}{2}PQ\right)(f(P)) \leq \left(\frac{1}{2}PQ\right)\left(\frac{2}{e}\right) < \frac{1}{2}PQ. \end{aligned}$$

This implies that (2.8) holds for $P \geq 3$. The lemma is proved. □

LEMMA 2.7. *For any n, the solutions (x, y, z) of* (1.2) *satisfy $z \leq AB/2$.*

PROOF. Since $AB/2 \geq 3$, the lemma holds for $(x, y, z) = (1, 1, 1)$. We now assume that (x, y, z) is a solution of (1.2) with $(x, y, z) \neq (1, 1, 1)$. Then, by Lemma 2.1, we have either $x > z > y$ or $y > z > x$.

Since $\min\{A^2, B^2\} \geq 4$, by Lemma 2.1, if $x > z > y$, then we have

$$B = B_1B_2,\ B_1, B_2 \in \mathbb{N},\ \gcd(B_1, B_2) = 1, \tag{2.11}$$

$$B_1^{2y} = n^{z-y} \tag{2.12}$$

and

$$A^{2x}n^{x-z} + B_2^{2y} = (A^2 + B^2)^z. \tag{2.13}$$

Take $G = A^{2x}n^{x-z}$, $H = B_2^{2y}$, $K = A^2 + B^2$ and $Z = z$. Let PQ be the largest squarefree divisor of GH. Since $\gcd(A, B) = 1$, by (2.11) and (2.12), we have

$$PQ = \operatorname{rad}(GH) = \operatorname{rad}(A^{2x}n^{x-z}) \cdot \operatorname{rad}(B_2^{2y}) = \operatorname{rad}(AB_1) \cdot \operatorname{rad}(B_2) = \operatorname{rad}(AB) \leq AB. \tag{2.14}$$

Therefore, applying Lemma 2.6 to (2.13), we get from (2.14) that

$$z \leq \frac{PQ}{2} \leq \frac{AB}{2}. \tag{2.15}$$

Similarly, if $y > z > x$, then we have

$$A = A_1A_2,\ A_1, A_2 \in \mathbb{N},\ \gcd(A_1, A_2) = 1, \tag{2.16}$$

$$A_1^{2x} = n^{z-x} \tag{2.17}$$

and

$$A_2^{2x} + B^{2y}n^{y-z} = (A^2 + B^2)^z. \tag{2.18}$$

Take $G = A_2^{2x}$, $H = B^{2y}n^{y-z}$, $K = A^2 + B^2$ and $Z = z$. By (2.16) and (2.17), we have

$$PQ = \operatorname{rad}(GH) = \operatorname{rad}(A_2^{2x}) \cdot \operatorname{rad}(B^{2y}n^{y-z}) = \operatorname{rad}(A_2) \cdot \operatorname{rad}(BA_1) = \operatorname{rad}(AB) \leq AB, \tag{2.19}$$

where PQ is the largest squarefree divisor of GH. Therefore, applying Lemma 2.6 to (2.18), we see from (2.19) that z satisfies (2.15). Thus, the lemma is proved. □

3. Proofs

Proof of Theorem 1.2. By Lemma 2.4, the theorem holds for $B = 2$. We may therefore assume that $B \geq 4$.

We now prove the first part of the theorem. Since $2 \nmid A$ and $A > B^3/8$, we have $A \geq 9$. Let (x, y, z) be a solution of (1.2) with $x > z > y$. By (2.13), we have $A^{2x}n^{x-z} < (A^2 + B^2)^z$, whence we get $(A^2n)^{x-z} < (1 + B^2/A^2)^z$ and

$$\log(A^2n) \leq (x - z)\log(A^2n) < z \log\left(1 + \frac{B^2}{A^2}\right). \tag{3.1}$$

Since $\log(1 + t) < t$ for any $t > 0$, by (3.1), we have

$$\frac{A^2}{B^2}\log(A^2n) < z. \tag{3.2}$$

On the other hand, by Lemma 2.7, we have $z \leq AB/2$. Hence, by (3.2), we get

$$\frac{A^2}{B^2}\log(A^2n) < \frac{AB}{2}. \tag{3.3}$$

Further, since $A > B^3/8$, we see from (3.3) that

$$\log(A^2 n) < 4. \tag{3.4}$$

But, since $A \geq 9$ and $n \geq 1$, (3.4) is false. Therefore, the first part of the theorem is proved.

Using the same method as before, we can easily prove the second part of the theorem. Since $2 \nmid A$ and $B > A^3/6$, we have $A \geq 3$ and $B \geq 6$. Let (x, y, z) be a solution of (1.2) with $y > z > x$. By (2.18), we have $B^{2y} n^{y-z} < (A^2 + B^2)^z$, whence we get

$$\frac{B^2}{A^2} \log(B^2 n) \leq \frac{B^2}{A^2}(y - z) \log(B^2 n) < z. \tag{3.5}$$

Further, by Lemma 2.7, we have $z \leq AB/2$. Hence, by (3.5), we get

$$\frac{B^2}{A^2} \log(B^2 n) < \frac{AB}{2}. \tag{3.6}$$

Furthermore, since $B > A^3/6$, we see from (3.6) that

$$\log(B^2 n) < 3. \tag{3.7}$$

But, since $B \geq 6$ and $n \geq 1$, (3.7) is false. Thus, the second part of the theorem is proved. The proof is complete. □

Proof of Corollary 1.3. Combining Theorem 1.2 with Lemma 2.3, we obtain the corollary immediately. □

Acknowledgements.

The authors are grateful for the referees for carefully reading our manuscript and for giving such constructive comments which substantially helped improving the presentation of the paper. We would like to thank Professors Reese Scott and Robert Styer for reading the previous version of this manuscript carefully and giving valuable advice. Especially thanks to Professor Robert Styer for verifying some details of the previous version of this paper and providing other technical assistance. The second author was supported by TÜBİTAK (the Scientific and Technological Research Council of Turkey) under Project No: 117F287.

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Received: 8.12.2019.
Revised: 13.5.2020.