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FURTHER RESULTS ON COMMON PROPERTIES OF THE PRODUCTS ac AND bd

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Abstract. In this paper, we continue to investigate common properties of the products ac and bd in various categories under the assumption $acd = dbd$ and $dba = aca$. These properties include generalized strongly Drazin invertibility and generalized Hirano invertibility in rings, abstract index of Fredholm elements and B-Fredholm elements in the Banach algebra context, complementability of kernels and ranges for bounded linear operators on Banach spaces.

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative ring with unit 1. The classical Jacobson's lemma asserts that

$$1 - ab \text{ is invertible if and only if } 1 - ba \text{ is invertible} \tag{1.1}$$

for any $a, b \in \mathcal{R}$. In the last two decades, suitable analogues of Jacobson's lemma for Drazin inverse and generalized Drazin inverse have been found by many researchers around the world (see [6, 8, 14, 16, 17, 24]). Corach et al. in [7] generalized (1.1) and many of its relatives to the case that

$$aba = aca, \tag{1.2}$$

see also [20, 21, 22, 23]. Recently, it has been realized that there are proper counterparts of Jacobson's lemma for Drazin inverse and generalized Drazin inverse under the new condition

$$\begin{cases} acd = dbd, \\ dba = aca, \end{cases} \tag{1.3}$$

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see [15, 18]. Obviously, the case "$a = d$" in (1.3) gives (1.2), the case "$b = c$" in (1.2) results in $aca = aca$.

This paper is a continuation of [15, 18]. In the presence of (1.3), common properties of the products ac and bd are further studied in various categories.

- In section 2, Jacobson's lemma for two new generalized inverses (i.e., generalized strong Drazin inverse and generalized Hirano inverse) are established in rings.
- In section 3, we derive the abstract index equality of Fredholm elements and B-Fredholm elements in the Banach algebra context.
- In section 4, we investigate the common complementability of kernels and ranges for bounded linear operators on Banach spaces.

2. Generalized inverses related to generalized Drazin inverse

For $a \in \mathcal{R}$, the commutant and double commutant of a are defined by $comm(a) = \{x \in \mathcal{R} : ax = xa\}$ and $comm^2(a) = \{x \in \mathcal{R} : xy = yx, \text{for all } y \in comm(a)\}$, respectively. We shall write $\mathcal{R}^{-1}$ and $\mathcal{R}^{nil}$ for the sets of all invertible and nilpotent elements of $\mathcal{R}$, respectively. An element $a \in \mathcal{R}$ is quasinilpotent ([12]) if $1 + ax \in \mathcal{R}^{-1}$ for all $x \in comm(a)$. The set of all quasinilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{qnil}$. Recall that $a \in \mathcal{R}$ is generalized Drazin invertible ([13]) if there exists $b \in \mathcal{R}$ such that

$$b \in comm^2(a), bab = b \text{ and } a - aba \in \mathcal{R}^{qnil}.$$

If such b exists, it is unique, and it is called the generalized Drazin inverse of a, denoted by a^{gD}. The set composed of generalized Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^{gD}$. In [18], the authors obtained the following analogue of Jacobson's lemma for generalized Drazin inverse under the assumption (1.3), which gives an affirmative answer to a conjecture of [15].

Lemma 2.1. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gD}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gD}$. In this case, we have*

$$\beta^{gD} = (1 - d\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gD}bac$$

and

$$\alpha^{gD} = (1 - bac\beta^{\pi}[1 - \beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gD}d,$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gD}, \beta^{\pi} = 1 - \beta\beta^{gD}$.

If we replace the condition $a - aba \in \mathcal{R}^{qnil}$ in the definition of generalized Drazin inverse with $a - ab \in \mathcal{R}^{qnil}$, then a is said to be generalized strongly Drazin invertible and b is called the generalized strong Drazin inverse of a, denoted by a^{gsD} (see [11]). The set composed of generalized strongly Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^{gsD}$. According to [11, Corollary 3.3], $\mathcal{R}^{gsD} \subseteq \mathcal{R}^{gD}$.

THEOREM 2.2. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gsD}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gsD}$. In this case, we have*

$$\beta^{gsD} = (1 - d\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gsD}bac$$

and

$$\alpha^{gsD} = (1 - bac\beta^{\pi}[1 - \beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gsD}d,$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gsD}, \beta^{\pi} = 1 - \beta\beta^{gsD}$.

PROOF. Write $p = \alpha^{\pi}$, $v = [1 - p\alpha(1 + bd)]^{-1}$ and $y = (1 - dpvbac)(1 + ac) + d\alpha^{gsD}bac$. By Lemma 2.1, y is a generalized Drazin inverse of β. To show $y \in \mathcal{R}^{gsD}$, we only need to show that $\beta - \beta y \in \mathcal{R}^{qnil}$. Noting $p = p(bd)^2v = pv(bd)^2$, we deduce that $\alpha - \alpha\alpha^{gsD} = p - bd = (pvbdb - b)d$. From the proof of [18, Theorem 3.3], we get $\beta y = 1 - dpvbac$. Hence $\beta - \beta y = 1 - ac - (1 - dpvbac) = dpvbac - ac = (dpvba - a)c$. Now we put $a' = dpvba - a$ and $b' = pvbdb - b$. Then a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $b'd = \alpha - \alpha\alpha^{gsD} \in \mathcal{R}^{qnil}$, by [19, Lemma 2.6], we conclude that $\beta - \beta y = a'c \in \mathcal{R}^{qnil}$, as required.

Conversely, set $q = \beta^{\pi}$, $u = [1 - q\beta(1 + ac)]^{-1}$ and $x = (1 - bacqud)(1 + bd) + bac\beta^{gsD}d$. By Lemma 2.1, it remains to prove that $\alpha - \alpha x \in \mathcal{R}^{qnil}$. Noting $q = q(ac)^2u = qu(ac)^2$, we get $\beta - \beta\beta^{gsD} = q - ac = (quaca - a)c$. Also, we obtain

$$\begin{aligned}
\alpha x &= (1 - bd)[(1 - bacqud)(1 + bd) + bac\beta^{gsD}d] \\
&= 1 - (bd)^2 - (1 - bd)bacqud(1 + bd) + (1 - bd)bac\beta^{gsD}d \\
&= 1 - [bacd - bac(1 - ac)\beta^{gsD}d] - (1 - bd)bacqud(1 + bd) \\
&= 1 - bacqd - bac(1 - ac)qud(1 + bd) \\
&= 1 - bacqd - bacqu(1 - ac)(1 + ac)d \\
&= 1 - bacqd - bacqu[1 - (ac)^2]d \\
&= 1 - bacqud,
\end{aligned}$$

whence $\alpha - \alpha x = bacqud - bd = (bacqu - b)d$. Now we write $a' = quaca - a$ and $b' = bacqu - b$, a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $a'c = \beta - \beta\beta^{gsD} \in \mathcal{R}^{qnil}$, the desired conclusion $\alpha - \alpha x = b'd \in \mathcal{R}^{qnil}$ then follows by [19, Lemma 2.6]. □

Recently, Abdolyousefi and Chen ([1]) introduced another subclass of generalized Drazin inverse, by replacing $a - aba \in \mathcal{R}^{qnil}$ with $a^2 - ab \in \mathcal{R}^{qnil}$ in the definition of generalized Drazin inverse. In this case, we say that a is generalized Hirano invertible and b is the generalized Hirano inverse of a, denoted by a^{gH}. We use $\mathcal{R}^{gH}$ to denote the set of all generalized Hirono invertible elements in $\mathcal{R}$. By [1, Theorem 2.2], $\mathcal{R}^{gH} \subseteq \mathcal{R}^{gD}$.

THEOREM 2.3. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gH}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gH}$. In this case, we have*

$$\beta^{gH} = (1 - d\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gH}bac$$

and

$$\alpha^{gH} = (1 - bac\beta^{\pi}[1 - \beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gH}d,$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gH}, \beta^{\pi} = 1 - \beta\beta^{gH}$.

PROOF. Write $p = \alpha^{\pi}$, $v = [1 - p\alpha(1 + bd)]^{-1}$ and $y = (1 - dpvbac)(1 + ac) + d\alpha^{gH}bac$. By Lemma 2.1, y is a generalized Drazin inverse of β. To show $y \in \mathcal{R}^{gH}$, we only need to show that $\beta^2 - \beta y \in \mathcal{R}^{qnil}$. Noting $p = p(bd)^2v = pv(bd)^2$, we deduce that $\alpha^2 - \alpha\alpha^{gsD} = p - 2bd + bdbd = (pvbdb - 2b + bdb)d$. From the proof of [18, Theorem 3.3], we get $\beta y = 1 - dpvbac$. Hence $\beta^2 - \beta y = (1 - ac)^2 - (1 - dpvbac) = dpvbac - 2ac + acac = (dpvba - 2a + aca)c$. Now we put $a' = dpvba - 2a + aca$ and $b' = pvbdb - 2b + bdb$. Then a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $b'd = \alpha^2 - \alpha\alpha^{gsD} \in \mathcal{R}^{qnil}$, by [19, Lemma 2.6], we conclude that $\beta^2 - \beta y = a'c \in \mathcal{R}^{qnil}$, as required.

Conversely, put $q = \beta^{\pi}$, $u = [1 - q\beta(1 + ac)]^{-1}$ and $x = (1 - bacqud)(1 + bd) + bac\beta^{gsD}d$. According to Lemma 2.1, it remains to show that $\alpha^2 - \alpha x \in \mathcal{R}^{qnil}$. Since $q = q(ac)^2u = qu(ac)^2$, $\beta^2 - \beta\beta^{gsD} = q - 2ac + acac = (quaca - 2a + aca)c \in \mathcal{R}^{qnil}$. As in the proof of Theorem 2.2, we get $\alpha x = 1 - bacqud$, hence $\alpha^2 - \alpha x = bacqud - 2bd + bdbd = (bacqu - 2b + bdb)d$. Now we set $a' = quaca - 2a + aca$ and $b' = bacqu - 2b + bdb$, it is easy to verify that $a'cd = db'd$ and $db'a' = a'ca'$. Applying [19, Lemma 2.6], again, we get $\alpha^2 - \alpha x = b'd \in \mathcal{R}^{qnil}$ as required. ∎

3. ABSTRACT INDEX OF FREDHOLM AND B-FREDHOLM ELEMENTS

Following [9], an element $a \in \mathcal{R}$ is said to be Drazin invertible if there exist $b \in \mathcal{R}$ and $k \in \mathbb{N}$ such that

$$b \in comm(a), bab = b \text{ and } a^kba = a^k.$$

The element b above is unique if it exists. It is called the Drazin inverse of a and is denoted by a^D. The smallest k for which $a^kba = a^k$ is called the Drazin index of a, and is denoted by $\mathrm{i}(a)$. If $\mathrm{i}(a) \leq 1$, then a is called group invertible. An element $a \in \mathcal{R}$ is invertible precisely when a is Drazin invertible with $\mathrm{i}(a) = 0$. We use $\mathcal{R}^D$ and $\mathcal{R}^{\sharp}$ to denote all Drazin invertible elements and group invertible elements in $\mathcal{R}$, respectively. According to [15, Theorem 2.4], (see also [18, Theorem 3.1]), in the presence of (1.3), we have

$$1 - ac \text{ is Drazin invertible} \Longleftrightarrow 1 - bd \text{ is Drazin invertible}, \tag{3.1}$$

$$1 - ac \text{ is group invertible} \Longleftrightarrow 1 - bd \text{ is group invertible} \tag{3.2}$$

and

$$1 - ac \text{ is invertible} \Longleftrightarrow 1 - bd \text{ is invertible.} \tag{3.3}$$

Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and π the canonical homomorphism from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}$. Following [3] (resp., [4]), an element $r \in \mathcal{R}$ is called a Fredholm element (resp., generalized Fredholm element, B-Fredholm element) relative to $\mathcal{I}$ if $\pi(r) \in (\mathcal{R}/\mathcal{I})^{-1}$ (resp., $\pi(r) \in (\mathcal{R}/\mathcal{I})^{\sharp}$, $\pi(r) \in (\mathcal{R}/\mathcal{I})^{D}$). The set of all Fredholm elements, generalized Fredholm elements and B-Fredholm elements relative to $\mathcal{I}$ will be denoted by $\Phi(\mathcal{R},\mathcal{I})$, $g\Phi(\mathcal{R},\mathcal{I})$ and $B\Phi(\mathcal{R},\mathcal{I})$, respectively. Applying (3.1), (3.2) and (3.3) respectively to $\mathcal{R}/\mathcal{I}$, we get

$$1 - ac \in B\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1 - bd \in B\Phi(\mathcal{R},\mathcal{I}), \tag{3.4}$$

$$1 - ac \in g\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1 - bd \in g\Phi(\mathcal{R},\mathcal{I}) \tag{3.5}$$

and

$$1 - ac \in \Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1 - bd \in \Phi(\mathcal{R},\mathcal{I}), \tag{3.6}$$

provided that (1.3) holds.

Recall that a Banach algebra $\mathcal{A}$ is called semisimple if the radical $\mathrm{Rad}(\mathcal{A})$ of $\mathcal{A}$ is equal to $\{0\}$, and $\mathcal{A}$ is said to be primitive if $\{0\}$ is a primitive ideal of $\mathcal{A}$. Primitive Banach algebras are semisimple. Let $\mathcal{A}$ be a complex semisimple Banach algebra with unit 1 and let $\mathcal{I}$ be a trace ideal (i.e., an ideal on which a trace $\tau : \mathcal{I} \longrightarrow \mathbb{C}$ is defined, see [10, 5] for details) of $\mathcal{A}$. Following [10] (resp., [5]), the index of a Fredholm element (resp., B-Fredholm element) $a \in \mathcal{A}$ relative to trace ideal $\mathcal{I}$ is defined with the aid of the trace as $\iota(a) := \tau(aa_0 - a_0a)$, where $\pi(a_0)$ is an inverse (resp., a Drazin inverse) of $\pi(a)$ in $\mathcal{A}/\mathcal{I}$. The socle $\mathrm{soc}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the sum of minimal ideals, and the set $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ is defined by $\mathrm{kh}(\mathrm{soc}(\mathcal{A})) := \{a \in \mathcal{A} : a + \mathrm{soc}(\mathcal{A}) \in \mathrm{Rad}(\mathcal{A}/\mathrm{soc}(\mathcal{A}))\}$. In the following two results, we obtain the abstract index equality of Fredholm elements and B-Fredholm elements respectively in the Banach algebra context.

THEOREM 3.1. *Let $\mathcal{A}$ be a unital semisimple Banach algebra and let $\mathcal{I}$ be a trace ideal of $\mathcal{A}$ such that $soc(\mathcal{A}) \subseteq \mathcal{I} \subseteq kh(soc(\mathcal{A}))$. If $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$ and $1 - ac$ is a Fredholm element relative to $\mathcal{I}$, then $\iota(1 - ac) = \iota(1 - bd)$.*

PROOF. By [10, Proposition 3.10 and Theorem 3.11], there exist idempotents p, q in $\mathrm{soc}(\mathcal{A})$ and $x \in \mathcal{A}$ such that $p(1 - ac) = 0$, $(1 - ac)q = 0$, $(1 - ac)x = 1 - p$, $x(1 - ac) = 1 - q$ and $\iota(1 - ac) = \tau(q) - \tau(p)$. Now we take $y = 1 + bd + bacxd$. A direct calculation shows that $(1 - bd)y = 1 - bacpd$ and $y(1 - bd) = 1 - bacqd$, which implies that

$$\iota(1 - bd) = \tau((1 - bd)y - y(1 - bd)) = \tau(bacqd - bacpd).$$

Since $p(1-ac)=0$, $\tau(bacpd)=\tau(pdbac)=\tau(pacac)=\tau(pac)=\tau(p)$. Analogously, $\tau(bacqd)=\tau(q)$. Therefore, $\iota(1-bd)=\tau(q)-\tau(p)=\iota(1-ac)$. ∎

THEOREM 3.2. *Let $\mathcal{A}$ be a unital primitive Banach algebra and suppose that $a,b,c,d\in\mathcal{A}$ satisfy $acd=dbd$ and $dba=aca$.*

(1) *If $1-ac$ is a B-Fredholm element relative to $soc(\mathcal{A})$, then $\iota(1-ac)=\iota(1-bd)$.*
(2) *If ac is a B-Fredholm element relative to $soc(\mathcal{A})$, then $\iota(ac)=\iota(bd)$.*

PROOF. (1) By the punctured neighborhood theorem for the index of B-Fredholm element (see [5, Theorem 3.1]), for nonzero λ with $|\lambda|$ small enough, we have

$$1-ac-\lambda\in\Phi(\mathcal{A},\mathrm{soc}(\mathcal{A})),\ \ 1-ba-\lambda\in\Phi(\mathcal{A},\mathrm{soc}(\mathcal{A}))$$

and

$$\iota(1-ac)=\iota(1-ac-\lambda),\ \ \iota(1-ba)=\iota(1-ba-\lambda).$$

Hence, the desired result follows by Theorem 3.1.

(2) The proof is analogous to that above. ∎

4. COMPLEMENTABILITY OF KERNELS AND RANGES

Let $\mathcal{B}(X,Y)$ denote the set of all bounded linear operators from Banach space X to Banach space Y. For $T\in\mathcal{B}(X):=\mathcal{B}(X,X)$, let $\mathcal{N}(T)$ denote its kernel and $\mathcal{R}(T)$ its range. In this section, we discuss the complementability of kernels and ranges of $I-AC$ and $I-BD$ under the assumption $ACD=DBD$ and $DBA=ACA$. Recall that a closed subspace M of a Banach space X is complemented if there exists a (closed) subspace N of X such that $X=M\bigoplus N$. Equivalently, M is complemented in X if and only if there is a bounded projection P such that $\mathcal{R}(P)=M$.

THEOREM 4.1. *Suppose that $A,D\in\mathcal{B}(X,Y)$ and $B,C\in\mathcal{B}(Y,X)$ satisfy $ACD=DBD$ and $DBA=ACA$. Then $\mathcal{N}(I-AC)$ is complemented in Y if and only if $\mathcal{N}(I-BD)$ is complemented in X.*

PROOF. Assume that P is the projection onto $\mathcal{N}(I-AC)$. Then $(I-AC)P=0$, that is, $P=ACP$. Put $Q=BPACD$. From the fact $DBP=DBACP=ACACP=ACP=P$, it follows that

$$Q^2=(BPACD)(BPACD)=BPACPACD=BPACD=Q.$$

Noting that

$$(I-BD)Q=(I-BD)(BPACD)=BPACD-BDBPACD=0,$$

we have $\mathcal{R}(Q)\subseteq\mathcal{N}(I-BD)$. Let $x\in\mathcal{N}(I-BD)$. Then $Dx=DBDx=ACDx$, whence $Dx\in\mathcal{N}(I-AC)=\mathcal{R}(P)$. Thus $PDx=Dx$, and hence

$$Qx=BPACDx=BPACPDx=BPDx=BDx=x,$$

which implies that $\mathcal{N}(I-BD) \subseteq \mathcal{R}(Q)$. Consequently, Q is the projection onto $\mathcal{N}(I-BD)$.

Conversely, assume that U is the projection onto $\mathcal{N}(I-BD)$. Set $V = ACDUBACAC$. Noting that $BDU = U$, it follows

$$\begin{aligned} V^2 &= (ACDUBACAC)(ACDUBACAC) \\ &= ACDUBDBDBDBDUBACAC \\ &= ACDUBACAC \\ &= V. \end{aligned}$$

Since

$$\begin{aligned} (I-AC)V &= (I-AC)(ACDUBACAC) \\ &= ACDUBACAC - ACACDUBACAC \\ &= ACDUBACAC - ACDBDUBACAC \\ &= ACDUBACAC - ACDUBACAC \\ &= 0, \end{aligned}$$

$\mathcal{R}(V) \subseteq \mathcal{N}(I-AC)$. Let $x \in \mathcal{N}(I-AC)$. Then $x = ACx$. Since $BACx = BACACx = BDBACx$, $BACx \in \mathcal{N}(I-BD) = \mathcal{R}(U)$, and hence $UBACx = BACx$. Thus,

$$\begin{aligned} Vx &= ACDUBACACx = ACDUBACx \\ &= ACDBACx = ACACACx = x, \end{aligned}$$

which implies that $\mathcal{N}(I-AC) \subseteq \mathcal{R}(V)$. Consequently, V is the projection onto $\mathcal{N}(I-AC)$. □

THEOREM 4.2. *Suppose that $A, D \in \mathcal{B}(X,Y)$ and $B, C \in \mathcal{B}(Y,X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then $\mathcal{R}(I-AC)$ is complemented in Y if and only if $\mathcal{R}(I-BD)$ is complemented in X.*

PROOF. Assume that P is the projection onto $\mathcal{R}(I-AC)$. Set $Q = I - BAC(I-P)D$. Since $(I-P)(I-AC) = 0$, $(I-P)AC = I-P$. It follows that

$$\begin{aligned} [BAC(I-P)D][BAC(I-P)D] &= BAC(I-P)ACAC(I-P)D \\ &= BAC(I-P)D, \end{aligned}$$

and hence $Q^2 = Q$. Since $\mathcal{R}(P) = \mathcal{R}(I-AC)$,

$$\mathcal{R}(BACPD) \subseteq \mathcal{R}(BAC(I-AC)) = \mathcal{R}((I-BD)BAC) \subseteq \mathcal{R}(I-BD).$$

Noting that

$$\begin{aligned} Q &= I - BAC(I-P)D = I - BACD + BACPD \\ &= I - BDBD + BACPD = (I-BD)(I+BD) + BACPD, \end{aligned}$$

we get $\mathcal{R}(Q) \subseteq \mathcal{R}(I-BD)$. Let $x \in \mathcal{R}(I-BD)$. Then there is an $x_1 \in X$ such that $x = (I-BD)x_1$. Since $Dx = D(I-BD)x_1 = (I-AC)Dx_1 \in \mathcal{R}(P)$,

$$Qx = [I - BAC(I-P)D]x = x,$$

which deduces that $\mathcal{R}(I-BD) \subseteq \mathcal{R}(Q)$. Therefore, $\mathcal{R}(I-BD)$ is complemented in X.

Conversely, suppose that U is the projection onto $\mathcal{R}(I-BD)$ and put

$$V = I - ACD(I-U)BAC.$$

Next we will show that V is the associated projection onto $\mathcal{R}(I-AC)$. Since $(I-U)(I-BD) = 0$, $(I-U)BD = I-U$, and hence

$$\begin{aligned}[ACD(I-U)BAC]^2 &= ACD(I-U)BDBDBD(I-U)BAC \\ &= ACD(I-U)BAC,\end{aligned}$$

which implies that $V^2 = V$. Noting that

$$\begin{aligned}V &= I - ACD(I-U)BAC = I - ACDBAC + ACDUBAC \\ &= I - ACACAC + ACDUBAC,\end{aligned}$$

it follows

$$\begin{aligned}\mathcal{R}(V) &\subseteq \mathcal{R}(I - ACACAC + ACDUBAC) \\ &\subseteq \mathcal{R}[(I-AC)(I+AC+ACAC)] + \mathcal{R}[ACD(I-BD)] \\ &\subseteq \mathcal{R}(I-AC) + \mathcal{R}[(I-AC)ACD] \\ &\subseteq \mathcal{R}(I-AC).\end{aligned}$$

For any $y \in \mathcal{R}(I-AC)$, there exists an element $y_1 \in Y$ such that $y = (I-AC)y_1$. Thus $BACy = BAC(I-AC)y_1 = (I-BD)BACy_1 \in \mathcal{R}(U)$, and so

$$Vy = [I - ACD(I-U)BAC]y = y.$$

Hence $\mathcal{R}(I-AC) \subseteq \mathcal{R}(V)$. Consequently, $\mathcal{R}(I-AC)$ is complemented in Y. ∎

In the following we give an application of Theorem 4.1 and Theorem 4.2. Recall that an operator $T \in \mathcal{B}(X)$ is said to be relatively regular if there exists an operator $S \in \mathcal{B}(X)$ for which $TST = T$ and $STS = S$. Relatively regular operator plays a significant role in operator theory. We refer the reader to [2] for more details. It is known that $T \in \mathcal{B}(X)$ is relatively regular if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are complemented ([2, Theorem 3.88]). Thus it is easy to obtain the following conclusion about relatively regular operators from Theorem 4.1 and Theorem 4.2.

Corollary 4.3. *Suppose that $A, D \in \mathcal{B}(X,Y)$ and $B, C \in \mathcal{B}(Y,X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then $I - AC$ is relatively regular if and only if $I - BD$ is relatively regular.*

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