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PERMUTATION ORBIFOLDS OF $\mathfrak{sl}_2$ VERTEX OPERATOR ALGEBRAS

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Abstract. We analyze two types of permutation orbifolds: (i) S_2-orbifolds of the universal level k vertex operator algebra $V^k(\mathfrak{sl}_2)$ and of its simple quotient $L_k(\mathfrak{sl}_2)$, and (ii) the S_3-orbifold of the level one simple vertex operator algebra $L_1(\mathfrak{sl}_2)$. We determine their structures and discuss related W-algebras.

1. Introduction

Permutation orbifolds, and orbifolds in general, are important sources of new examples of vertex operator algebras. The state of the art result is that any finite solvable orbifold of a rational and C_2-cofinite vertex algebra is also rational and C_2-cofinite ([11], see also [15]). This way we get examples of rational vertex algebra from S_2, S_3 and S_4 permutation orbifolds, whose rationality is sometimes difficult to prove using the standard methods.

Permutations orbifolds were extensively studied in the physics literature on conformal field theory. We mention an important early work by Bantay ([5]) with focus on the structure of characters of representations of the permutation orbifold algebra. They also appear in various contexts in string theory ([9]).

The structure of permutation orbifold vertex algebras and their representations have already been investigated by several authors. For $n = 2$, Abe proved that any S_2-permutation orbifold is C_2-cofinite provided that the underlying vertex algebra is C_2-cofinite ([2]). Dong, Xu and Yu obtained description of low rank cyclic orbifolds in the case of lattice vertex algebras

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([16, 17]). Barron and Vander Werf studied twisted modules of cyclic permutation orbifolds of fermionic vertex superalgebras ([6]) based on an earlier work for vertex algebras ([7], see also [8]). Adamović, Lam, Pedić and Yu considered modules for certain cyclic orbifolds beyond the category of ordinary modules by considering Whittaker modules ([1]).

They are also sources of interesting W-algebras. In a recent work of the authors, jointly with Shao and with Wauchope ([26, 27]), the authors found several interesting examples of W-algebras coming from S_3-orbifolds of the free fermion, symplectic fermion, and Heisenberg vertex algebras. Very recently, with Sadowski, we also determined the structure of S_2, $\mathbb{Z}_3$ and S_3 permutation orbifolds of the Virasoro vertex algebra both for generic and non-generic central charges ([25]). The second author, with Graybill, Linshaw and Quintero, described all cyclic group orbifolds of the rank two Heisenberg algebra and as an application were able to decompose an arbitrary finite abelian group orbifold of any Heisenberg algebra as modules for these cyclic group orbifolds ([19]). Next, the second author, with Quintero, described all dihedral group orbifolds for the rank two Heisenberg algebra, thus completing the problem to describe all finite group orbifolds of this vertex algebra ([29]). Our PhD students Li and Wauchope described the structure of S_2-permutation orbifolds of the Heisenberg-Virasoro, $N = 1$, and $N = 2$ superconformal vertex algebras in [21].

Let us outline the content of the paper and the main results. We consider two types of permutation orbifolds: (i) the S_2-orbifold of the universal vertex algebras $V^k(\mathfrak{sl}_2)$ and of its simple vertex algebras $L_k(\mathfrak{sl}_2)$, for all levels $k \neq -2$, and (ii) S_3-orbifold of $L_1(\mathfrak{sl}_2)$. In Section 2, we first describe the structure of $\left(V^k(\mathfrak{sl}_2) \otimes V^k(\mathfrak{sl}_2)\right)^{S_2}$ for all k. This vertex algebra is of type $(1^3, 2^6, 3^3)$ for $k \neq 8$ and $(1^3, 2^6, 3^3, 4^3)$ if $k = 8$ (see Theorem 2.1). In Section 3, we show that $(L_k(\mathfrak{sl}_2) \otimes L_k(\mathfrak{sl}_2))^{S_2}$ has essentially the same structure except for $k = 1$, $k = -\frac{4}{3}$. We also discuss the $k = 2$ case in more detail as it's connected to an interesting W-algebra of type $(2, 4, 6)$. Then in Section 4 we consider the S_3-permutation orbifold, denoted by $V(3)^{S_3}$, of the simple vertex algebra associated to the level one basic module for $\widehat{\mathfrak{sl}}_2$. The main result here is Theorem 4.3, giving the decomposition of $V(3)^{S_3}$ as a module for the $\mathbb{Z}_2$-orbifold algebra of the Zamolodchikov's algebra of central charge $\frac{6}{5}$. In Section 5, we prove that this W-algebra, denoted by $W_{\frac{6}{5}}(2, 3)^{\sigma}$, is of type $(2, 6, 8, 10)$. Although we did not find a minimal generating set for $V(3)^{S_3}$, we expect it is of type $(1^3, 2, 3^3)$ and we present enough evidence to support the claim. In the appendix we give several explicit formulas of primary vectors used in the paper.

2. Permutation orbifolds $(V^k(\mathfrak{sl}_2)\otimes V^k(\mathfrak{sl}_2))^{S_2}$

Consider two commuting copies of the universal vertex operator algebra associated to the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ at level $k\neq -2$, generated by the vectors h_1, e_1, f_1 and h_2, e_2, f_2 respectively. Here we use the standard Sugawara conformal vector ω_i so the central charge is $\frac{3k}{k+2}$. The associated nontrivial OPEs are given by

$$\begin{aligned} h_i(z)e_i(w) &\sim \frac{2e_i(w)}{z-w},\\ h_i(z)f_i(w) &\sim \frac{-2f_i(w)}{z-w},\\ h_i(z)h_i(w) &\sim \frac{2k}{(z-w)^2},\\ e_i(z)f_i(w) &\sim \frac{k}{(z-w)^2}+\frac{h_i(w)}{z-w}. \end{aligned} \tag{2.1}$$

The total Virasoro vector will be denoted by $\omega=\omega_1+\omega_2$.

There is an obvious $S_2\cong\mathbb{Z}_2$ action on this algebra where the generator permutes the commuting copies of $V^k(\mathfrak{sl}_2)$. We can diagonalize this action by the following change of basis among the generators.

$$\begin{aligned} h &= h_1+h_2, & \alpha &= h_1-h_2,\\ e &= e_1+e_2, & x &= e_1-e_2,\\ f &= f_2+f_2, & y &= f_1-f_2. \end{aligned} \tag{2.2}$$

Now e,f,h generate a diagonal sub-VOA isomorphic to $V^{2k}(\mathfrak{sl}_2)$ and the generator of S_2 acts via

$$\alpha\mapsto-\alpha,\qquad x\mapsto -x,\qquad y\mapsto -y. \tag{2.3}$$

Next, using Lemma 3.1 of [28] (see also [3]) we may take an initial strong generating set for the orbifold $\big(V^k(\mathfrak{sl}_2)\otimes V^k(\mathfrak{sl}_2)\big)^{S_2}$ to be

$$\begin{aligned}&\{h(-1)\mathbb{1}, e(-1)\mathbb{1}, f(-1)\mathbb{1}, \alpha(m_1)\alpha(m_2)\mathbb{1}, \alpha(m_3)x(m_4)\mathbb{1}, \alpha(m_5)y(m_6)\mathbb{1},\\ &\quad x(m_7)y(m_8)\mathbb{1}, x(m_9)x(m_{10})\mathbb{1}, y(m_{11})y(m_{12})\mathbb{1}|m_i\leq -1, 1\leq i\leq 12\}.\end{aligned}$$

Through standard methods, involving the translation operator, we can immediately reduce this set to

$$\begin{aligned}&\{h,e,f\}\cup\\ &\{\alpha(2m_1-1)\alpha(-1)\mathbb{1}, x(2m_2-1)x(-1)\mathbb{1}, y(1m_3-1)y(-1)\mathbb{1}|m_i\leq -1\}\cup\\ &\{\alpha(n_1-1)x(-1)\mathbb{1}, \alpha(n_2-1)y(-1)\mathbb{1}, x(n_3-1)y(-1)\mathbb{1}, |n_i\leq -1\}.\end{aligned} \tag{2.4}$$

For related orbifolds see [3]. We introduce the following notation for these generating vectors

$$
\begin{aligned}
w_m^{\alpha} &= \alpha(-m-1)\alpha(-1)\mathbb{1}, & w_m^{\alpha,x} &= \alpha(-m-1)x(-1)\mathbb{1},\\
w_m^{x} &= x(-m-1)x(-1)\mathbb{1}, & w_m^{\alpha,y} &= \alpha(-m-1)y(-1)\mathbb{1},\\
w_m^{y} &= y(-m-1)y(-1)\mathbb{1}, & w_m^{x,y} &= x(-m-1)y(-1)\mathbb{1}.
\end{aligned} \tag{2.5}
$$

Then the main result of this section is the following Theorem.

THEOREM 2.1. *For $k \neq 8$, the orbifold $\left(V^k(\mathfrak{sl}_2) \otimes V^k(\mathfrak{sl}_2)\right)^{S_2}$ is of type $(1^3, 2^6, 3^3)$ and minimally strongly generated by three weight one vectors h, e, and f, which generate a sub-VOA which is a copy of $V^{2k}(\mathfrak{sl}_2)$, six weight two vectors*

$$w_0^{\alpha}, w_0^{x}, w_0^{y}, w_0^{\alpha,x}, w_0^{\alpha,y}, w_0^{x,y} \tag{2.6}$$

and three weight three vectors

$$w_1^{\alpha,x}, w_1^{\alpha,y}, w_1^{x,y}. \tag{2.7}$$

Further, the orbifold $\left(V^8(\mathfrak{sl}_2) \otimes V^8(\mathfrak{sl}_2)\right)^{S_2}$ is strongly generated by the above vectors with the addition of

$$w_2^{\alpha,x}, w_2^{\alpha,y}, w_2^{x,y} \tag{2.8}$$

and so it is of type $(1^3, 2^6, 3^3, 4^3)$.

PROOF. Throughout our argument we will perform calculations involving the vertex operators as opposed to working directly with the vectors and abuse notation by (for instance) writing $:\alpha\alpha:$ instead of $:\alpha(z)\alpha(z):$. Furthermore, we set

$$
\begin{aligned}
W_m^{\alpha} &= m!\,Y(w_m^{\alpha}, z) = :(\partial^m \alpha)\alpha:, & W_m^{\alpha,x} &= m!\,Y(w^{\alpha,x}, z) = :(\partial^m \alpha)x:,\\
W_m^{x} &= m!\,Y(w_m^{x}, z) = :(\partial^m x)x:, & W_m^{\alpha,y} &= m!\,Y(w^{\alpha,y}, z) = :(\partial^m \alpha)y:,\\
W_m^{y} &= m!\,Y(w_m^{y}, z) = :(\partial^m y)y:, & W_m^{x,y} &= m!\,Y(w^{x,y}, z) = :(\partial^m x)y:.
\end{aligned} \tag{2.9}
$$

We will construct explicit relations at all weights in order to write elements from the set (2.4) in terms of lower weight vectors, thus providing an inductive path to write every element of the orbifold in terms of the vectors h, e, f as well as those listed in (2.6) and (2.7).

We begin by focusing on reducing the need for the higher weight generators $W_m^{x,y}$. We begin with

$$
\begin{aligned}
2(k-8)W_2^{x,y} = &:W_0^{x,y}W_0^{x,y}: - :W_0^{x}W_0^{y}: - 2(k+4)\partial W_1^{x,y}\\
&+ (k+1)\partial^2 W_0^{x,y} - 6:h, W_1^{x,y}: + :h\partial W_0^{x,y}:\\
&- :(\partial h)W_0^{x,y}: - \frac{1}{2}:(\partial^2 h)h: - \frac{k}{3}\partial^3 h,
\end{aligned} \tag{2.10}
$$

which for $k \neq 8$ allows us to write $w_2^{x,y}$ in terms of lower weight vectors in the orbifold. Next, we outline our strategy for constructing decoupling relations. By direct calculation we have

$$\begin{aligned} &{}^\circ_\circ W^{x,y}_{2m} W^{x,y}_0 {}^\circ_\circ - {}^\circ_\circ W^x_{2m} W^y_0 {}^\circ_\circ \\ &\quad = \frac{(-2k+4)m^2 + (-7k+14)m - 3k + 9}{(m+1)(2m+1)} W^{x,y}_{2m+2} \\ &\quad + \frac{2km+k+1}{(2m+1)(m+1)} {}^\circ_\circ x\partial^{2m+2} y {}^\circ_\circ - \frac{4m+4}{2m+1} {}^\circ_\circ (\partial^{2m+1} x)(\partial y) {}^\circ_\circ \\ &\quad - \frac{6m+5}{2m+1} {}^\circ_\circ h W^{x,y}_{2m+1} {}^\circ_\circ + \frac{1}{2m+1} {}^\circ_\circ h x (\partial^{2m+1} y) {}^\circ_\circ \\ &\quad - \frac{2}{2m+1} {}^\circ_\circ (\partial x)(\partial^{2m+1} y) {}^\circ_\circ - {}^\circ_\circ (\partial h) W^{x,y}_{2m} {}^\circ_\circ \\ &\quad - \frac{k}{(2m+3)(m+1)} \partial^{2m+3} h. \end{aligned} \tag{2.11}$$

Next, we can use the combinatorial identity (which follows from the inverse of the Pascal matrix)

$$\begin{aligned} &{}^\circ_\circ (\partial^{m_1} x)(\partial^{m_2} y) {}^\circ_\circ \\ &\quad = \sum_{j=0}^{m_2} (-1)^{j+m_2} \binom{m_2}{j} \partial^j W^{x,y}_{m_1+m_2-j} \\ &\quad = (-1)^{m_2} W^{x,y}_{m_1+m_2} + \sum_{j=1}^{m_2} (-1)^{j+m_2} \binom{m_2}{j} \partial^j W^{x,y}_{m_1+m_2-j} \end{aligned} \tag{2.12}$$

to rewrite all of the terms in (2.11) as combinations of elements from our generating set (2.7) while controlling the coefficient $W^{x,y}_{2m+2}$. This leads to

$$\frac{(m+2)(2km-8m+k-8)}{(2m+1)(m+1)} W^{x,y}_{2m+2} = {}^\circ_\circ W^x_{2m} W^y_0 {}^\circ_\circ - {}^\circ_\circ W^{x,y}_{2m} W^{x,y}_0 {}^\circ_\circ + \Psi^{x,y}_1, \tag{2.13}$$

where $\Psi^{x,y}_1$ is a normally ordered polynomial of fields (and their derivatives) with lower weight. In this case, the exact structure $\Psi^{x,y}_1$ can be inferred from (2.11) and (2.12). Through a similar construction we have

$$\begin{aligned} &\frac{4(m+2)(3k-2)}{6m+3} W^{x,y}_{2m+2} \\ &\quad = {}^\circ_\circ W^{x,y}_{2m-1} W^{x,y}_1 {}^\circ_\circ - {}^\circ_\circ \partial(W^x_{2m-1}) W^y_0 {}^\circ_\circ + {}^\circ_\circ W^x_{2m} W^y_0 {}^\circ_\circ + \Psi^{x,y}_2. \end{aligned} \tag{2.14}$$

Now solving the system of equations

$$\frac{(m+2)(2km-8m+k-8)}{(2m+1)(m+1)} = 0 \text{ and } \frac{4(m+2)(3k-2)}{6m+3} = 0$$

we see that solutions occur at $m = -2$ and $(m,k) = \left(-\frac{11}{10}, \frac{2}{3}\right)$, but since we have $m \in \mathbb{N}$ we see that in all cases $W^{x,y}_{2m+2}$ can be written in terms of lower weight vectors. We also have the following odd analogues of (2.13) and (2.14)

$$
\begin{aligned}
&\frac{2km-8m-3k-16}{2m+3}W^{x,y}_{2m+3} = {}^\circ_\circ W^{x}_{2m+1}W^{y}_{0}{}^\circ_\circ - {}^\circ_\circ W^{x,y}_{2m+1}W^{x,y}_{0}{}^\circ_\circ + \Psi^{x,y}_{3},\\
&\frac{(24k-16)m^3+(60k-24)m^2+(42k+40)m+9k-48}{3(2m+3)(2m+1)(m+1)}W^{x,y}_{2m+3}\\
&\qquad = {}^\circ_\circ W^{x,y}_{2m}W^{x,y}_{1}{}^\circ_\circ - {}^\circ_\circ \partial(W^{x}_{2m})W^{y}_{0}{}^\circ_\circ + {}^\circ_\circ W^{x}_{2m+1}W^{y}_{0}{}^\circ_\circ + \Psi^{x,y}_{4}.
\end{aligned}
\tag{2.15}
$$

The simultaneous zeros of the coefficients of $W^{x,y}_{2m+3}$ in (2.15) are

$$
(m,k)\in\left\{\left(-\frac{5}{2},\frac{1}{2}\right),(0,-2),\left(-\frac{9}{10},-\frac{11}{6}\right),\left(0,-\frac{16}{3}\right)\right\},
$$

none of which are problematic (since $m\in\mathbb{Z}_{\geq 0}$) except for $m=0$ and $k=-\frac{16}{3}$, which we can eliminate with the following

$$
\begin{aligned}
W^{x,y}_{3} = &-\frac{3}{4k}({}^\circ_\circ W^{\alpha,x}_{1}W^{\alpha,y}_{0}{}^\circ_\circ - {}^\circ_\circ W^{\alpha}_{1}W^{x,y}_{0}{}^\circ_\circ) + \frac{3(3k-4)}{8k}\partial W^{\alpha}_{2}\\
&- \frac{3(k-2)}{16k}\partial^2 W^{\alpha}_{0} - \frac{3}{k}\partial W^{x,y}_{2} + \frac{3}{2k}\partial^2 W^{x,y}_{1} - \frac{1}{2k}\partial^3 W^{x,y}_{0} + \psi,
\end{aligned}
\tag{2.16}
$$

where ψ is a normally ordered polynomial in lower weight vectors.

Next, we focus on the generators W^{α}_{m}. The lowest weight relation involving these vectors is

$$
\begin{aligned}
&(k-8)W^{\alpha}_{2} + 2(k-8)W^{x,y}_{2}\\
&\quad = {}^\circ_\circ W^{\alpha}_{0}W^{x}_{0}{}^\circ_\circ - {}^\circ_\circ W^{\alpha,x}_{0}W^{\alpha,x}_{0}{}^\circ_\circ - 2\partial^2 W^{\alpha}_{0} - 12\partial W^{x,y}_{1}\\
&\quad + 4\partial^2 W^{x,y}_{0} + 6{}^\circ_\circ e W^{\alpha,y}_{1}{}^\circ_\circ - 6{}^\circ_\circ f W^{\alpha,x}_{1}{}^\circ_\circ + 2{}^\circ_\circ f(\partial W^{\alpha,x}_{0}){}^\circ_\circ\\
&\quad - \frac{1}{2}{}^\circ_\circ h(\partial W^{\alpha}_{0}){}^\circ_\circ + 2{}^\circ_\circ(\partial e)W^{\alpha,y}_{0}{}^\circ_\circ - 2{}^\circ_\circ(\partial^2 e)f{}^\circ_\circ - \frac{2}{3}\partial^3 h,
\end{aligned}
\tag{2.17}
$$

which for $k\neq 8$, together with (2.10), can be used to write W^{α}_{2} in terms of lower weight vectors. Next, for all $m\geq 1$ we have

$$
\begin{aligned}
&\frac{2k(10m^2+21m+13)}{3(2m+1)(m+1)}W^{\alpha}_{2m+2}\\
&\quad = {}^\circ_\circ(\partial W^{\alpha}_{2m-1})W^{\alpha}_{0}{}^\circ_\circ - {}^\circ_\circ(\partial W^{\alpha}_{2m})W^{\alpha}_{0}{}^\circ_\circ - {}^\circ_\circ W^{\alpha}_{2m-1}W^{\alpha}_{1}{}^\circ_\circ + \Psi^{\alpha}_{1},\\
&\frac{(2k-8)m+k-8}{2m+1}W^{\alpha}_{2m+2} - \frac{16}{2m+1}W^{x,y}_{2m+2}\\
&\quad = {}^\circ_\circ W^{\alpha,x}_{2m}W^{\alpha,y}_{0}{}^\circ_\circ - {}^\circ_\circ W^{\alpha}_{m}W^{x,y}{}^\circ_\circ + \Psi^{\alpha}_{2},
\end{aligned}
\tag{2.18}
$$

where Ψ^{α}_{1} and Ψ^{α}_{2} are normally ordered polynomials in lower weight generators. For all $k\neq 0$ the first equation in (2.18) can be used to write the generators W^{α}_{n} in terms of lower weight vectors. Further, if $k=0$ a combination of the second equation in (2.18), (2.13), and (2.14) will have the same result.

Next, we focus on generators of the for $W^{\alpha,x}_m$. The lowest weight relation involving these vectors is

$$
\begin{aligned}
(k-8)W^{\alpha,x}_2 =& {}^{\circ}_{\circ}W^{x,y}_0 W^{\alpha,x}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{x}_0 W^{\alpha,y}_0{}^{\circ}_{\circ} + 2(k-5)\partial W^{\alpha,x}_1 \\
& - (k-5)\partial^2 W^{\alpha,x}_0 + 6{}^{\circ}_{\circ}eW^{x,y}_1{}^{\circ}_{\circ} - 2{}^{\circ}_{\circ}e(\partial W^{x,y}_0){}^{\circ}_{\circ} \\
& + {}^{\circ}_{\circ}f(\partial W^{x}_0){}^{\circ}_{\circ} + 3{}^{\circ}_{\circ}\alpha W^{\alpha,x}_1{}^{\circ}_{\circ} - 3{}^{\circ}_{\circ}h(\partial W^{\alpha,x}_0){}^{\circ}_{\circ} \\
& + 2{}^{\circ}_{\circ}(\partial e)W^{x,y}_0{}^{\circ}_{\circ} + 2{}^{\circ}_{\circ}(\partial f)W^{x}_0{}^{\circ}_{\circ} - 1{}^{\circ}_{\circ}(\partial h)W^{\alpha,x}_0{}^{\circ}_{\circ} \\
& + 3{}^{\circ}_{\circ}h(\partial^2 e){}^{\circ}_{\circ} + 2{}^{\circ}_{\circ}(\partial h)(\partial e){}^{\circ}_{\circ} + {}^{\circ}_{\circ}(\partial^2 h)e{}^{\circ}_{\circ} + \frac{2}{3}(k-6)\partial^3 e,
\end{aligned}
\tag{2.19}
$$

which can be used to write $W^{\alpha,x}_2$ in terms of lower weight generators for $k \neq 8$. Next, for $m \geq 1$, we have

$$
\begin{aligned}
& {}^{\circ}_{\circ}W^{x}_{2m} W^{\alpha,y}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{x,y}_{2m} W^{\alpha,x}_0{}^{\circ}_{\circ} = \frac{(2k-8)m+k-8}{(m+1)(2m+1)} W^{\alpha,x}_{2m+2} + \Psi^{\alpha,x}_1, \\
& {}^{\circ}_{\circ}W^{\alpha,x}_{2m-1} W^{\alpha}_1{}^{\circ}_{\circ} - {}^{\circ}_{\circ}(\partial W^{x,y}_{2m-2})W^{\alpha,x}_0{}^{\circ}_{\circ} + {}^{\circ}_{\circ}W^{x,y}_{2m-1} W^{\alpha,x}_0{}^{\circ}_{\circ} \\
& \qquad = \frac{-2((2k-8)m-5k-8)}{(2m+1)} W^{\alpha,x}_{2m+2} + \Psi^{\alpha,x}_2,
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
& {}^{\circ}_{\circ}W^{x}_{2m-1} W^{\alpha,y}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{x,y}_{2m-1} W^{\alpha,x}_0{}^{\circ}_{\circ} = \frac{2k}{2m+1} W^{\alpha,x}_{2m+1} + \Psi^{\alpha,x}_3, \\
& {}^{\circ}_{\circ}W^{\alpha}_{2m-1} W^{\alpha,x}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{\alpha,x}_{2m-1} W^{\alpha}_0{}^{\circ}_{\circ} \\
& \qquad = \frac{2((2k-8)m-5k-8)}{2m+1} W^{\alpha,x}_{2m+1} + \Psi^{\alpha,x}_4,
\end{aligned}
\tag{2.21}
$$

which allow us to write $W^{\alpha,x}_n$ for $n \geq 3$ in terms of lower weight generators. .

Next we move onto the generators W^x_n starting with the lowest weight relation

$$
\begin{aligned}
& {}^{\circ}_{\circ}W^{\alpha,x}_0 W^{\alpha,x}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{\alpha}_0 W^{x}_0{}^{\circ}_{\circ} \\
& \qquad = 2(k-8)W^x_2 + 2\partial^2 W^x_0 + 2{}^{\circ}_{\circ}e(\partial W^{\alpha,x}_0){}^{\circ}_{\circ} \\
& \qquad\quad - 12{}^{\circ}_{\circ}eW^{\alpha,x}_1{}^{\circ}_{\circ} - 2{}^{\circ}_{\circ}(\partial e)W^{\alpha,x}_0{}^{\circ}_{\circ} - 2{}^{\circ}_{\circ}(\partial^2 e)e{}^{\circ}_{\circ},
\end{aligned}
\tag{2.22}
$$

which for $k \neq 8$ can be used to write W^x_2 in terms of lower weight generators. Furthermore, the relations

(2.23)

$$
\begin{aligned}
& {}^{\circ}_{\circ}W^{\alpha,x}_{2m} W^{\alpha,x}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{\alpha}_{m} W^{x}_0{}^{\circ}_{\circ} = \frac{2((2k-8)m+k-8)}{(m+1)(2m+1)} W^{x}_{2m+2} + \Psi^{x}_1, \\
& {}^{\circ}_{\circ}W^{x,y}_{m} W^{x}_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}W^{x}_{2m} W^{x,y}_0{}^{\circ}_{\circ} = \frac{(2k-8)m^2+(k-14)m-4}{(m+1)(2m+1)} W^{x}_{2m+2} + \Psi^{x}_2,
\end{aligned}
$$

where Ψ^x_1 and Ψ^x_2 are normally order polynomials in lower weight terms. It is easy to check that the only simultaneous zero for the coefficients of W^x_{2m+2} in

(2.23) occurs at $m = -\frac{2}{3}$ and $k = -8$, but since $m \in \mathbb{N}$ this is not problematic. Thus for all $n \geq 2$, we can write W^x_{2n} in terms of lower weight generators.

In parallel to (2.19)-(2.23) we have relations which allow us to write $W^{\alpha,y}_n$ and W^y_{2n} in terms of lower weight generators for $n \geq 2$ if $k \neq 8$. These along with all of the relations (2.10)-(2.23) allow us to write any element from the generating set in terms of those described by (2.6)-(2.8).

It is easy to see that this set of generators is minimal by explicit computation. □

We can choose generators of the orbifold such that they are primary vectors for all k. Moreover, for $k \neq \frac{1}{2}$ and $k \neq -\frac{2}{3}$ these primaries can be chosen so that they are also highest weight vectors for $\widehat{\mathfrak{sl}_2}$. Their explicit formulas are given in the appendix.

Remark 2.2. In the sequel, for k generic, we will compute the Zhu's algebra and irreducible modules of this orbifold.

3. Permutation orbifolds $(L_k(\mathfrak{sl}_2) \otimes L_k(\mathfrak{sl}_2))^{S_2}$

We now move to describe some simplification that can take place inside the simple quotient at various levels. We denote by $L_k(\mathfrak{sl}_2)$ the simple quotient of $V^k(\mathfrak{sl}_2)$ and consider $(L_k(\mathfrak{sl}_2) \otimes L_k(\mathfrak{sl}_2))^{S_2}$.

We make a few elementary observations here. Let V be a vertex algebra with ideal I and G a finite group of automorphisms preserving I then $(V/I)^G \cong V^G/I^G$. This implies that we only have analyze those vertex algebras for which $L_k(\mathfrak{sl}_2) \neq V^k(\mathfrak{sl}_2)$. These are precisely admissible levels and the critical level $k = -2$ which we do not consider. If the universal vertex algebra has no generators above conformal weight 4 from Theorem 2.1 we immediately have the following result.

Corollary 3.1. *Let k be such that no singular vector in $V^k(\mathfrak{sl}_2)$ occurs in degree ≤ 4, then the S_2-permutation orbifold of $L_k(\mathfrak{sl}_2) \otimes L_k(sl_2)$ is of the type described in Theorem 2.1.*

Next we discuss several cases in more detail where null vectors do occur in degree ≤ 4.

3.1. $L_1(\mathfrak{sl}_2)$. Although this level well-understood we present explicit computation. At this level it is well-known that $e_i(-1)^2\mathbb{1}$ are singular vectors for $i = 1, 2$. Now if we consider the following calculations involving these singular vectors

$$e_1(-1)^2\mathbb{1} + e_2(-1)^2\mathbb{1} = \frac{1}{2}(e(-1)^2\mathbb{1} + w^x_0\mathbb{1}),$$

$$f(0)(e_1(-1)^2\mathbb{1} + e_2(-1)^2) = -w^{\alpha,x}_0 - h(-1)e(-1)\mathbb{1} + 2e(-2)\mathbb{1},$$

$$y(-1)(e_1(-1)^2\mathbb{1} - e_2(-1)^2) = -w^{\alpha,x}_1 + e(-1)w^{x,y}_0 - h(-2)e(-1)\mathbb{1}$$

$$\begin{aligned}
&\qquad + 2e(-3)\mathbb{1},\\
f(0)^2(e_1(-1)^2\mathbb{1} + e_2(-1)^2) &= -2w_0^{x,y} + \alpha(-1)^2\mathbb{1} - 2e(-1)f(-1)\mathbb{1}\\
&\qquad + h(-1)^2\mathbb{1} + 2h(-2)\mathbb{1},\\
e(0)x(-1)(f_1(-1)^2\mathbb{1} - f_2(-1)^2\mathbb{1}) &= -2w_1^{x,y} + f(-1)w_0^{\alpha,x} + h(-1)w_0^{x,y}\\
&\qquad - 2e(-2)f(-1)\mathbb{1} + 2h(-3)\mathbb{1}
\end{aligned}$$

and similar equations involving $w_0^{\alpha,y}$, $w_1^{\alpha,y}$, and w_0^y. All of these together allow us to write w_0^x, $w_0^{\alpha,x}$, $w_1^{\alpha,x}$, $w_0^{x,y}$, $w_1^{x,y}$,$w_0^{\alpha,y}$, $w_1^{\alpha,y}$, and w_0^y in terms of h, f, e and w_0^α making the simple orbifold $(L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2))^{S_2}$ of type $(1, 1, 1, 2)$. If we take the weight 2 generator to be

$$\omega = \frac{1}{48}(2w_0^\alpha + 8w_0^{x,y} - 4e(-1)f(-1)\mathbb{1} - h(-1)^2\mathbb{1} - 2h(-2)\mathbb{1}) \tag{3.1}$$

it is easy to check that this is a conformal vector of central charge 1/2 that commutes with h, e, f, so we have

$$(L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2))^{S_2} \cong L_2(\mathfrak{sl}_2) \otimes L_{\mathrm{Vir}}(1/2, 0). \tag{3.2}$$

This is of course well-known and it follows easily from the coset decomposition of $\frac{sl(2)_1 \times sl(2)_1}{sl(2)_2}$.

3.2. $L_{-\frac{1}{2}}(\mathfrak{sl}_2)$. Since $V^{-\frac{1}{2}}(\mathfrak{sl}_2)$ has the first singular vector in degree 4, by the theorem, this is again of type $(1^3, 2^6, 3^3)$.

We give another description of this orbifold. Recall the $\beta\gamma$-realization of $L_{-\frac{1}{2}}(\mathfrak{sl}_2)$ as

$$V_{\beta\gamma}^+ \cong L_{-\frac{1}{2}}(\mathfrak{sl}_2),$$

where the superscript indicates the fixed point algebra under the automorphism + induced by $(\beta, \gamma) \to (-\beta, -\gamma)$. Therefore

$$(L_{-\frac{1}{2}}(\mathfrak{sl}_2) \otimes L_{-\frac{1}{2}}(\mathfrak{sl}_2))^{S_2} \cong \left(V_{\beta\gamma}^+ \otimes V_{\beta\gamma}^+\right)^{S_2} \cong (V_{\beta\gamma} \otimes V_{\beta\gamma})^{D_4}$$

where D_4 is the dihedral group of 8 elements acting on the tensor product. As a consequence, we determined the type of this non-abelian orbifold.

3.3. $L_2(\mathfrak{sl}_2)$. We first determine the type and then we provide another description of this orbifold.

We only have to analyze degree 3 generators - all generators of degree 2 must be accounted for. Compared to $(V^2(\mathfrak{sl}_2) \otimes V^2(\mathfrak{sl}_2))^{S_2}$ inside $V := (L_2(\mathfrak{sl}_2) \otimes L_2(\mathfrak{sl}_2))^{S_2}$ the only new relations come from the singular vector $e_1(-1)^3\mathbb{1} + e_2(-1)^3\mathbb{1}$. Acting with $f(0) = f_1(0) + f_2(0)$ on it, we create a 7-dimensional subspace which is zero in V. Easy inspection with characters implies that $\mathrm{ch}[V] = 1 + 3q + 15q^2 + 42q^3 + O(q^4)$. However, sl_2 and degree 2 generators can contribute at most with a 39-dimensional degree 3 subspace, therefore 3-dimensional subspace is missing and those are precisely contributed with weight 3 generators. Therefore the type is again $(1^3, 2^6, 3^3)$.

It is interesting to analyze this orbifold from a different perspective using cosets. We first need several results regarding the superconformal algebra $L^{N=1}(1,0)$ algebra and its fixed point algebra under the parity automorphism. Denote by $V^{N=1}(c,0)$ the universal $N=1$ superconformal vertex algebra. Next result is known in the physics literature ([10]).

Proposition 3.2. *The even part $V^{N=1}(c,0)_{\mathrm{even}}$ of the universal $N=1$ superconformal algebra $V^{N=1}(c,0)$ is a W-algebra of type $(2,4,6)$. This algebra is not freely generated. For $c=1$, the corresponding simple quotient is rational.*

Proof. We only sketch the proof - full details will appear in [21, 24]. We first show that for a generating set we can choose $L(-2)\mathbb{1}$, $G_{-n-\frac{1}{2}}G_{-\frac{3}{2}}\mathbb{1}$, $n \geq 2$. Using standard methods we can reduce this further to $L(-2)\mathbb{1}$, $G_{-\frac{5}{2}}G_{-\frac{3}{2}}\mathbb{1}$, $G_{-\frac{9}{2}}G_{-\frac{3}{2}}\mathbb{1}$. Out of the weight 4 and 6 generators we can form two primary fields. Here we give explicit formulas for $c=1$ needed later

$$
\begin{aligned}
w &= G_{-5/2}G_{-3/2}\mathbb{1} - \frac{17}{27}L(-2)^2\mathbb{1} - \frac{2}{9}L(-4)\mathbb{1},\\
z &= 6G_{-9/2}G_{-3/2}\mathbb{1} - \frac{66}{17}G_{-7/2}G_{-5/2}\mathbb{1} - \frac{52}{17}L(-2)G_{-5/2}G_{-3/2}\mathbb{1}\\
&\quad - \frac{8}{17}L(-2)^3\mathbb{1} - \frac{92}{51}L(-3)^2\mathbb{1} + \frac{284}{51}L(-4)L(-2)\mathbb{1} + \frac{960}{153}L(-6)\mathbb{1}.
\end{aligned}
$$

Comparing the character $\mathrm{ch}[V^{N=1}(c,0)_{even}](\tau) = \frac{(-q^{3/2},q)_\infty + (q^{3/2},q)_\infty}{2(q;q)_\infty}$ with the "free" character we see that the orbifold has a first relation among ω, w and z in degree 10. Rationality of the simple quotient follows from a general result for rational vertex superalgebras. $\square$

From now on, we use $W^c(2,4,6)$ to denote the even part of $V^{N=1}(c,0)$ and its simple quotient by $W_c(2,4,6)$. The group of automorphisms of $W^c(2,4,6)$ is mostly trivial.

Lemma 3.3. *For $c \neq 1$, $Aut(W^c(2,4,6))$ is trivial.*

Proof. This can be seen from the structural constants in the vertex algebra (coefficients of the OPE) with respect to the weight 4 and 6 generators. Using OPE we compute

$$
\begin{aligned}
C_{4,4}^4 &= \frac{(6(-82+47c+10c^2))}{(22+5c)},\\
C_{4,4}^6 &= \frac{2(-1+c)(50+c)}{(3(24+c))},\\
C_{4,6}^4 &= \frac{(96(-1+c)(11+c)(22+5c)(11+14c))}{((50+c)(-1+2c)(68+7c))},
\end{aligned}
$$

$$C_{4,6}^6 = \frac{(5(-1+2c)(20+3c)(68+7c))}{((24+c)(22+5c))},$$
$$C_{6,6}^6 = \frac{80(-1+c)(724096+574876c+183931c^2+19106c^3+616c^4))}{((24+c)(50+c)(-1+2c)(68+7c))},$$

where the subscript indicate the weights of the relevant primaries. Any automorphisms of $W^c(2,4,6)$ is uniquely determined with its action of the generators. Easy analysis shows that this action takes form $(\omega, w, z) \to (\omega, \pm w, \pm z)$. Using the formulas above it is clear that the only possible automorphism is $(\omega, w, z) \to (\omega, w, -z)$ and this can only occur for $c = 1$. □

From now on we only consider $W_c(2,4,6)$ with the central charge $c = 1$. Using the character formula for the vertex superalgebra $L^{N=1}(1,0)$, we see that the first singular vector in $V^{N=1}(1,0)$ occurs at degree 8, therefore the even part of $L^{N=1}(1,0)$, $W_1(2,4,6)$, is also of type $(2,4,6)$.

Recall that $F = L(2\Lambda_0) \oplus L(2\Lambda_1)$ has a vertex superalgebra structure with the even part $L(2\Lambda_0)$. Then Goddard-Kent-Olive ([20]) obtained a decomposition

$$\begin{aligned} F \otimes L(2\Lambda_0) \cong L(4\Lambda_0) \otimes L^{N=1}(1,0) \oplus L(2\Lambda_0 + 2\Lambda_0) \\ \otimes L^{N=1}\left(1, \frac{1}{6}\right) \oplus L(4\Lambda_0) \otimes L^{N=1}(1,1). \end{aligned}$$

Since we are interested in the even part we immediately obtain conformal embedding

$$L(4\Lambda_0) \otimes W_1(2,4,6) \hookrightarrow L(2\Lambda_0) \otimes L(2\Lambda_0).$$

Furthermore, taking the S_2-orbifold fixes $L(4\Lambda_0)$ and therefore the automorphism must act non-trivially on $W_1(2,4,6)$. Since the only non-trivial automorphism, σ, of this algebra can be described in Lemma 3.3, we conclude that $(L(2\Lambda_0) \otimes L(2\Lambda_0))^{S_2}$ is an extension of the rational vertex algebra $L(4\Lambda_0) \otimes W_1(2,4,6)^\sigma$. From Lemma 3.3 we have the following statement.

Corollary 3.4. *$Aut(W_1(2,4,6)) = \mathbb{Z}_2$.*

Remark 3.5. It seems that the automorphism σ does not lift to an automorphism of $W^1(2,4,6)$.

There is another description of $W_1(2,4,6)$ using lattice orbifolds. Dong and Jiang characterized rational vertex algebras with $c = 1$ under suitable conditions. Their main result in [13] gives the following result.

Proposition 3.6. *We have $W_1(2,4,6) \cong V^+_{\sqrt{12}\mathbb{Z}}$. Consequently,*

$$W_1(2,4,6)^\sigma \cong \left(V^+_{\sqrt{12}\mathbb{Z}}\right)^\sigma,$$

where + automorphism is induced by $\alpha \to -\alpha$ in the lattice. Therefore $(L_2(\mathfrak{sl}_2) \otimes L_2(\mathfrak{sl}_2))^{S_2}$ is an extension of $L_4(\mathfrak{sl}_2) \otimes \left(V^+_{\sqrt{12}\mathbb{Z}}\right)^\sigma$.

3.4. $L_{-\frac{4}{3}}(\mathfrak{sl}_2)$. As in Section 1.1., straightforward computation with the singular vector of degree 3,

$$\begin{aligned}v_i &= 4e_i(-1)^2 f_i(-1)\mathbb{1} + h_i(-1)^2 e_i(-1)\mathbb{1} - \frac{16}{3} h_i(-1)e_i(-2)\mathbb{1} \\ &\quad - \frac{2}{3} h_i(-2)e_i(-1)\mathbb{1} + \frac{80}{9} e(-3)\mathbb{1},\end{aligned} \tag{3.3}$$

for $i = 1, 2$ in each copy of $V_{-\frac{4}{3}}(\mathfrak{sl}_2)$ and generators shows that all weight 3 vectors can be eliminated. For instance, we have the following

$$\begin{aligned}\alpha(0)(v_1 - v_2) &= \frac{20}{3} w_1^{\alpha,x} - \frac{10}{3} w_0^{\alpha,x}(-2)\mathbb{1} + \frac{1}{2} e(-1)w_0^{\alpha} + 2e(-1)^2 f(-1)\mathbb{1} \\ &\quad + 4e(-1)w_0^{x,y} + 2f(-1)w_0^{x} + h(-1)w_0^{\alpha,x} + \frac{1}{2} h(-1)^2 e(-1)\mathbb{1} \\ &\quad - \frac{2}{3} h(-2)e(-1)\mathbb{1} - \frac{16}{3} h(-1)e(-2)\mathbb{1} + \frac{106}{9} e(-3)\mathbb{1},\end{aligned} \tag{3.4}$$

which shows that $w_1^{\alpha,x}$ can be written in terms of lower weight generators and an element of the maximal ideal. Since no weight 2 generators can be eliminated we get the following result.

Proposition 3.7. *$(L_{-\frac{4}{3}}(\mathfrak{sl}_2) \otimes L_{-\frac{4}{3}}(\mathfrak{sl}_2))^{S_2}$ is of type $(1^3, 2^6)$.*

3.5. $L_3(\mathfrak{sl}_2)$. Here we have a singular vector of weight 4. Since the level is not eight this orbifold is of type $(1^3, 2^6, 3^3)$. Explicit decomposition of the coset W-algebra seems fairly complicated - we leave this for future investigation.

4. The permutation orbifold $(L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2))^{S_3}$

4.1. *Notation and characters.* In this section we discuss the structure of the simplest non-abelian permutation orbifold coming from $\mathfrak{sl}_2$. Let S_3 denote the symmetric group on 3 letters. To simplify notation we let

$$V(3) := L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2).$$

As in section two we denote by $e_i(z)$, $f_i(z)$ and $h_i(z)$, $i = 1, 2, 3$ standard generators of $V(3)$. The well-known formula for the S_3-invariant VOA character gives

$$\mathrm{ch}[V(3)^{S_3}](\tau) = \frac{1}{6}(\mathrm{ch}[V](\tau))^3 + \frac{1}{2}\mathrm{ch}[V](2\tau) \cdot \mathrm{ch}[V](\tau) + \frac{1}{3}\mathrm{ch}[V](3\tau),$$

where

$$\mathrm{ch}[L_1(\mathfrak{sl}_2)] = \frac{\sum_{n\in\mathbb{Z}} q^{n^2}}{\eta(q)},$$

where $\eta(q) = q^{1/24}\prod_{i\geq 1}(1-q^i)$ from the lattice construction. Clearly, $L_3(\mathfrak{sl}_2) \subset V(3)^{S_3}$. By the Weyl-Kac character formula

$$\begin{aligned}\mathrm{ch}[L_3(\mathfrak{sl}_2)] &= \frac{\displaystyle\sum_{m\in\mathbb{Z}}\left(5(m+\frac{1}{10})q^{5(m+\frac{1}{10})^2} - 5(m-\frac{1}{10})q^{5(m-\frac{1}{10})^2}\right)}{\eta(\tau)^3}\\ &= q^{-3/40}(1+3q+9q^2+22q^3+42q^4+81q^5+151q^6+264q^7+\cdots).\end{aligned}$$

4.2. *Full character.* Since $h(0) = h_1(0)+h_2(0)+h_3(0)$ is fixed under S_3, it also defines the charge on $V(3)^{S_3}$. So we can also consider

$$\mathrm{ch}[V(3)](x,\tau) := \mathrm{tr}_{V(3)}x^{H(0)}q^{L(0)}$$

and

$$\mathrm{ch}[V(3)^{S_3}](x,\tau) := \mathrm{tr}_{V(3)^{S_3}}x^{H(0)}q^{L(0)}.$$

We can now improve the above formula by adding the charge variable

Proposition 4.1.

$$\begin{aligned}\mathrm{ch}[V(3)^{S_3}](x,\tau) = \frac{1}{6}(\mathrm{ch}[V](x,\tau))^3 + \frac{1}{2}\mathrm{ch}[V](x^2,2\tau)\cdot\mathrm{ch}[V](x,\tau)\\ + \frac{1}{3}\mathrm{ch}[V](x^3,3\tau).\end{aligned}$$

4.3. *Decomposition of* $V(3)$*.* Using the well-known GKO coset construction of minimal models ([20]) we immediately get

Proposition 4.2. *As an* $L_3(\mathfrak{sl}_2)\times Vir$*-module*

$$\begin{aligned}V(3) = L(3\Lambda_0)\otimes\left(L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{7}{10},\frac{3}{2})\right)\\ \oplus L(\Lambda_0+2\Lambda_1)\otimes\left(L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{7}{10},\frac{1}{10})\oplus L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\right).\end{aligned}$$

For a generalization to any positive level see the recent paper [23].

4.4. *Construction of* $W_{\frac{6}{5}}(2,3)$*.* In this part we recall a lattice construction of the simple (rational) Zamolodchikov's vertex algebra $W(2,3)$ of central charge $c=\frac{6}{5}$. We follow closely Dong et al. ([14]).

Let $L=\sqrt{2}A_2$ be a rescaled $sl(3)$ root lattice of type A_2 and V_L its vertex algebra constructed as in [14]. This algebra contains three orthogonal conformal vectors ω^1,ω^2 and ω^3 of central charge $\frac{1}{2},\frac{7}{10}$ and $\frac{4}{5}$ and thus the lattice vertex algebra contains the triple product $L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},0)$ conformally embedded in it. An explicit decomposition of V_L with respect to this vertex subalgebra was obtained by Lam and Yamada ([18]).

Next we consider the subalgebra

$$M_k^0 := \{v\in V_L : L_{\frac{4}{5}}(0)v = 0\},$$

where $L_{\frac{4}{5}}(0)$ is the degree zero operator of the conformal vector of central charge $\frac{4}{5}$ inside $L(\frac{4}{5},0)$. Then [14] gives

$$M_k^0 \cong L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{7}{10},\frac{3}{2}). \tag{4.1}$$

This vertex algebra has three primary vectors of degree 2 explicitly described in [14]. Moreover, [14, Theorem 2.1] $Aut(M_k^0)\cong S_3=\langle\sigma,\tau\rangle$, where σ and τ are explicitly constructed 2- and 3-cycles, respectively, acting on those three primary vectors of degree 2. In particular, we have ([14])

$$(M_k^0)^{\mathbb{Z}_3}\cong W_{\frac{6}{5}}(2,3),$$

where the right-hand side denotes the simple Zamolodchikov W-algebra $\mathcal{W}(\mathfrak{sl}_3,f_{princ})$ at $c=\frac{6}{5}$. Then the full fixed point vertex subalgebra is

$$(M_k^0)^{S_3}\cong W_{\frac{6}{5}}(2,3)^{\sigma},$$

where the right hand side denotes the fixed point subalgebra under the automorphism induced by $J\to -J$, where J denotes the weight 3 primary generator (this is the only non-trivial automorphism of the $W(2,3)$ algebra).

Appearance of the vertex algebra $M_k^0 := L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{7}{10},\frac{3}{2})$ in Proposition 4.2 and in (4.1) is of course no accident as we briefly explain below.

We first realize $V_{\sqrt{2}A_2}$ inside the triple tensor product $V(3)$. Let $Q=\bigoplus_{i=1}^3\mathbb{Z}\alpha_i$, where $\langle\alpha_i,\alpha_j\rangle=2\delta_{i,j}$. and $L'=\mathbb{Z}(\alpha_1-\alpha_2)\oplus\mathbb{Z}(\alpha_2-\alpha_3)$. Then $L'\cong\sqrt{2}A_2$. and $\langle L',\underbrace{\alpha_1+\alpha_2+\alpha_3}_{:=\gamma}\rangle=0$. Therefore $V(3)$ decomposes as $V_{\mathbb{Z}\gamma}\otimes V_{\sqrt{2}A_2}$-module. Let $S_3=\langle(123),(12)\rangle\subset Aut(V(3))$ acting by permuting tensor factors. Action of (123) on $\beta_0:=-\beta_1-\beta_2$, $\beta_1:=\alpha_1-\alpha_2$ and $\beta_2:=\alpha_2-\alpha_3$ is given by

$$\beta_1\to\beta_2\to\beta_0\to\beta_1.$$

Action of (12) is given by

$$\beta_1\to-\beta_1;\beta_2\to-\beta_0;\beta_0\to-\beta_2.$$

This S_3 action induces an action on M_k^0 which coincides with the S_3 action on M_k^0 defined in [14] via three primary vectors $w(\alpha)$. To see $L(3\Lambda_0)$ from this point of view, we can use decomposition

$$L(3\Lambda_0)=V_{\mathbb{Z}\gamma}\otimes K(\mathfrak{sl}_2,3)\oplus V_{\mathbb{Z}\gamma-\gamma/3}\otimes M_1\oplus V_{\mathbb{Z}\gamma-2\gamma/3}\otimes M_2$$

where $K(\mathfrak{sl}_2,3)=L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ is the parafermionic algebras and M_i are certain irreducible modules thereof.

The above discussion gives almost all arguments needed to prove the following statement.

Theorem 4.3. *As an* $L_3(\mathfrak{sl}_2) \times W_{\frac{6}{5}}(2,3)^\sigma$*-module:*

$$V(3)^{S_3} = L(3\Lambda_0) \otimes W_{\frac{6}{5}}(2,3)^\sigma \oplus L(\Lambda_0 + 2\Lambda_1) \otimes M^\sigma,$$

where M^σ *(resp.* M*) is an irreducible* $W_{\frac{6}{5}}(2,3)^\sigma$*-module (resp.* $W_{\frac{6}{5}}(2,3)$*-module) of lowest conformal weight* $\frac{13}{5}$*.*

Proof. We already argued that $W_{\frac{6}{5}}(2,3)^\sigma$ appears as the coset subalgebra. The automorphism σ of order two acts on the two summands in the decomposition and fixes $L(\Lambda_0 + 2\Lambda_1)$, acting non-trivially on the M. Explicit computation shows that in $V(3)^{S_3}$ there is a 3-dimensional space of weight 3 primary vectors annihilated by $t \cdot \mathfrak{sl}_2[t]$. These vectors are given explicitly in the Appendix. Since the lowest conformal weight of $L(\Lambda_0 + 2\Lambda_1)$ is $\frac{2}{5}$, the lowest conformal weight of M^σ is $\frac{13}{5}$. Irreducibility of M^σ follows from Quantum Galois theory as discussed in [14, Section 4.1]. ∎

5. The structure of $W_{\frac{6}{5}}(2,3)^\sigma$

In this part we describe the orbifold $W_{\frac{6}{5}}(2,3)^\sigma$ in more detail. We prove the following result.

Theorem 5.1. *The rational vertex algebra* $W_{\frac{6}{5}}(2,3)^\sigma$ *is a* W*-algebra of type* $(2,6,8,10)$.

We denote generators of $W_{\frac{6}{5}}(2,3)$ by ω and J of degree 3. As usual we use $L(n)$ to denote the modes of ω and for convenience we let

$$J(n) := J_{n-2},$$

so that $\deg J(n) = n$.

In [14] an explicit formula for the character of $W_{\frac{6}{5}}(2,3)$ was given. In particular, this gives

$$\begin{aligned}\mathrm{ch}[W_{\frac{6}{5}}(2,3)](\tau) = 1 &+ q^2 + 2q^3 + 3q^4 + 4q^5 + 8q^6 + 10q^7 + 17q^8 + 24q^9 \\ &+ 36q^{10} + 50q^{11} + 75q^{12} + 100q^{13} + O(q^{14}).\end{aligned}$$

Since $W_{\frac{6}{5}}(2,3)$ is unitary as a Virasoro algebra module, it decomposes (uniquely) as a direct sum of $L(\frac{6}{5},0)$-modules. We need

Lemma 5.2. *Each module* $L(\frac{6}{5},h)$*,* $h \in \mathbb{N}$ *is generic. Consequently,*

$$q^{1/20}\mathrm{ch}[L(6/5,0)] = \frac{1-q}{(q;q)_\infty},$$

$$q^{1/20}\mathrm{ch}[L(6/5,h)] = \frac{q^h}{(q;q)_\infty}, h \in \mathbb{N}.$$

Using the lemma we get decomposition up to degree 12

$$W_{\frac{6}{5}}(2,3) = L(6/5,0) \oplus L(6/5,3) \oplus L(6/5,6) \oplus L(6/5,8) \oplus L(6/5,9) \\ \oplus L(6/5,10) \oplus 2L(6/5,11) \oplus 2L(6/5,12) \oplus \cdots.$$

In [4], the universal W-algebra $W^c(2,3)$ and its orbifold $W^c(2,3)^{\mathbb{Z}_2}$ was thoroughly studied. In particular, their main result for $c = \frac{6}{5}$ gives the following statement.

Proposition 5.3. *$W^{\frac{6}{5}}(2,3)^{\mathbb{Z}_2}$ is of type* $(2,6,8,10,12)$.

We have to examine what is the structure in the simple case.

Using OPE we can compute Virasoro primaries in the universal algebra $W^{\frac{6}{5}}(2,3)$ of degree 6, 8, 10 and 12 (see Appendix). There are three primaries of degree 12 also given in the appendix. If we set

$$U^3_{12} = 7346581U^1_{12} - 425509U^2_{12}$$

we see that $J(n)U^2_{12} = 0$ for all $n \geq 0$, in other words it is a singular vector for the universal algebra $W^{\frac{6}{5}}(2,3)$ in agreement with the character formula obtained earlier.

It is clear that only primary vectors of degree 6, 8, 10, and two of the three primary vectors of degree 12 are preserved under the automorphism induced by $J \to -J$.

Therefore, as a Virasoro algebra, the fixed point subalgebra $W^{\frac{6}{5}}(2.3)^{\mathbb{Z}_2}$ of the universal W-algebra is isomorphic to

$$L(6/5,0) \oplus L(6/5,6) \oplus L(6/5,8) \oplus L(6/5,10) \oplus 2L(6/5,12) \oplus \cdots.$$

In particular,

$$\begin{aligned} \operatorname{ch}[W_{\frac{6}{5}}(2,3)^{\mathbb{Z}_2}](\tau) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 5q^6 + 5q^7 \\ &\quad + 10q^8 + 12q^9 + 20q^{10} + 25q^{11} + \mathbf{40}q^{12} + O(q^{13}). \end{aligned} \tag{5.1}$$

We can now prove

Theorem 5.4. *$W_{\frac{6}{5}}(2,3)^{\mathbb{Z}_2}$ is of type $(2,6,8,10)$ and the corresponding generators in the Appendix form a minimal set of generators.*

Proof. As already discussed, from the Virasoro decomposition, the primaries of weight $(2,6,8,10)$ must be inside the algebra. It is not too difficult to show using explicit generators $(2,6,8,10)$ in the appendix that there are no algebraic relations among them up to and including conformal weight 12. Therefore these generators must be among a minimal set of generators. Suppose that the weight 12 generator is also a part of the minimal generating set.

Then the character of the orbifold $O(q^{13})$ would be

$$\frac{1}{(q^2;q)_\infty(q^6;q)_\infty(q^8;q)_\infty(q^{10};q)_\infty(1-q^{12})}$$
$$= 1+q^2+q^3+2q^4+2q^5+5q^6+5q^7+10q^8+12q^9$$
$$+20q^{10}+25q^{11}+\mathbf{41}q^{12}+O(q^{13}).$$

On the other hand, by formula (5.1), the dimension of the corresponding graded subspace is 40. However

$$\frac{1}{(q^2;q)_\infty(q^6;q)_\infty(q^8;q)_\infty(q^{10};q)_\infty}$$
$$= 1+q^2+q^3+2q^4+2q^5+5q^6+5q^7$$
$$+10q^8+12q^9+20q^{10}+25q^{11}+\mathbf{40}q^{12}+O(q^{13}),$$

and thus we have an algebra of type $(2,6,8,10)$. $\square$

Corollary 5.5. *The vertex algebra orbifold* $V(3)^{S_3}$ *is generated by the diagonal* $\mathfrak{sl}_2$*, generators of weight* U_i*,* $i\in\{2,6,8,10\}$ *and three primary vectors* W_i*,* $1\le i\le 3$ *of weight* 3 *given in Appendix.*

We have computational evidence that a stronger result holds.

Conjecture 5.6. *The orbifold* $V(3)^{S_3}$ *is of type* $(1^3,2,3^3)$.

In theory it should be straightforward to verify that $(2,6,8,10)$ generators can be eliminated if we include the weight 3 primaries. However, the subspace $V(3)^{S_3}_{10}$ of conformal weight 10 is more than 100,000-dimensional and we were unable to perform this kind of computation.

We finish with another conjecture still much beyond reach.

Conjecture 5.7. *The orbifold* $(V^k(\mathfrak{sl}_2)\otimes V^k(\mathfrak{sl}_2)\otimes V^k(\mathfrak{sl}_2))^{S_3}$ *is generically of type* $(1^3,2^6,3^{13},4^{14},5^6)$.

6. Future work

We plan to extend ideas from this paper to study vertex algebra associated to odd lattice vertex algebra V_L, where $L=\mathbb{Z}\alpha$, $\langle\alpha,\alpha\rangle=3$. The corresponding S_3-permutation orbifold is an infinite extension of the $N=2$ superconformal vertex algebra of central charge 3.

Appendix A.

A.1. *Primary generators of* $(V^k(\mathfrak{sl}_2)\otimes V^k(\mathfrak{sl}_2))^{S_2}$.

$$w_{xy0}=(x(-1)y(-1)-\frac{1}{2}a(-1)^2+\frac{1}{2k-1}e(-1)f(-1)-\frac{1}{2(2k-1)}h(-1)^2$$
$$-\frac{k}{2k-1}h(-2))\mathbb{1},$$

$$
\begin{aligned}
w_{ax0} &= \left(a(-1)x(-1) + \frac{1}{2k-1}h(-1)e(-1) - \frac{2k}{2k-1}e(-2)\right)\mathbb{1},\\
w_{xx0} &= x(-1)^2\mathbb{1} + \frac{1}{2k-1}e(-1)^2\mathbb{1},\\
w_{ay0} &= \left(a(-1)y(-1) + \frac{1}{2k-1}h(-1)f(-1) + \frac{2k}{2k-1}f(-2)\right)\mathbb{1},\\
w_{yy0} &= y(-1)^2\mathbb{1} + \frac{1}{2k-1}f(-1)^2\mathbb{1},\\
w_{xy1} &= \Big(4(3+k)x(-2)y(-1) + e(-1)\alpha(-1)y(-1) + \frac{4(2+k)}{2+3k}e(-1)f(-2)\\
&\quad + f(-1)\alpha(-1)x(-1) - \frac{1}{k}h(-1)\alpha(-1)^2 - \frac{2(2+k)}{k}h(-1)x(-1)y(-1)\\
&\quad + \frac{4(2+k)}{k(2+3k)}h(-1)e(-1)f(-1) - \frac{2+k}{k(2+3k)}h(-1)^3\\
&\quad - \frac{4(2+k)}{2+3k}e(-2)f(-1) + \frac{3(2+k)}{2+3k}h(-2)h(-1)\\
&\quad - 2(3+k)(x(-1)y(-1))_{-2} + \frac{2(k^2+5k+2)}{2+3k}h(-3)\Big)\mathbb{1},\\
w_{ax1} &= \Big(-4(3+k)\alpha(-2)x(-1) + (2+k)e(-1)\alpha(-1)\alpha(-1)\\
&\quad + 2(4+k)e(-1)x(-1)y(-1) - \frac{8(2+k)}{2+3k}e(-1)^2f(-1) - 2kf(-1)x(-1)^2\\
&\quad - kh(-1)\alpha(-1)x(-1) - \frac{2(2+k)}{2+3k}h(-1)^2\alpha(-1)\\
&\quad + \frac{2(8+2k+k^2)}{2+3k}h(-1)e(-2) - \frac{2k(2+k)}{2+3k}h(-2)e(-1)\\
&\quad + 2(3k+k^2)(\alpha(-1)x(-1))_{-2} - \frac{4(4-2k+2k^2+k^3)}{2+3k}e(-3)\Big)\mathbb{1},\\
w_{ay1} &= \Big(2k(3+k)a(-2)y(-1) - ke(-1)y(-1)^2 - \frac{4(2+k)}{2+3k}e(-1)f(-1)^2\\
&\quad + \frac{k+2}{2}f(-1)\alpha(-1)^2 + (4+k)kf(-1)x(-1)y(-1)\\
&\quad - \frac{k}{2}h(-1)\alpha(-1)y(-1) - \frac{2+k}{2+3k}h(-1)^2f(-1)\\
&\quad - \frac{8+2k+k^2}{2+3k}h(-1)f(-2) - \frac{2(-4+2k+k^2)}{2+3k}h(-2)f(-1)\\
&\quad - (3k+k^2)(\alpha(-1)y(-1))_{-2} - \frac{2(4+8k+5k^2+k^3)}{2+3k}f(-3)\Big)\mathbb{1}.
\end{aligned}
$$

A.2. *Generators of* $W_{\frac{6}{5}}(2,3)^{\sigma}$.

$$U_6 = (56154J(-3)^2 - 25120L(-2)^3 + 33144L(-4)L(-2)$$

$$
\begin{aligned}
&- 32565L(-3)^2 + 42432L(-6))\mathbb{1},\\
U_8 =\ &(585900J(-3)^2L(-2) + 641235J(-4)^2 - 1601460J(-5)J(-3)\\
&- 34600L(-2)^4 - 490560L(-4)L(-2)^2 - 34050L(-3)^2L(-2)\\
&- 782280L(-6)L(-2) - 21834L(-4)^2 + 260700L(-8)\\
&- 785970L(-5)L(-3))\mathbb{1},\\
U_{10} =\ &(9181667250J(-3)^2L(-2)^2 + 43808964500J(-4)^2L(-2)\\
&- 75774769000J(-5)J(-3)L(-2) + 26130259050J(-3)^2L(-4)\\
&- 26366049500J(-4)J(-3)L(-3) + 52763594940J(-5)^2\\
&- 129634199520J(-6)J(-4) + 162197568680J(-7)J(-3)\\
&+ 97980000L(-2)^5 - 13597613000L(-4)L(-2)^3\\
&+ 1572749375L(-3)^2L(-2)^2 - 51958104000L(-6)L(-2)^2\\
&- 21154627200L(-4)^2L(-2) - 147488715940L(-8)L(-2)\\
&- 28534352000L(-5)L(-3)L(-2) - 45938374980L(-5)^2\\
&- 943649625L(-4)L(-3)^2 + 37861248852L(-10)\\
&- 63112979280L(-6)L(-4) - 122225317890L(-7)L(-3))\mathbb{1}.
\end{aligned}
$$

A.3. *Weight* 12 *primaries in* $W^{\frac{6}{5}}(2,3)$.

$$
\begin{aligned}
U^1_{12} = (&- 614701126078130938560J(-3)^2L(-2)^3\\
&+ 684779143460211534000J(-4)^2L(-2)^2\\
&+ 391818730508521511328 0J(-5)J(-3)L(-2)^2\\
&- 4396191650253197477760J(-5)^2L(-2)\\
&+ 2353877411882220856 80J(-6)J(-4)L(-2)\\
&- 7499275000234308177600J(-7)J(-3)L(-2)\\
&+ 5515534695729272552352J(-3)^2L(-4)L(-2)\\
&- 6145010146928171826720J(-4)J(-3)L(-3)L(-2)\\
&+ 2588723244855138468420J(-3)^2L(-3)^2\\
&- 8322589801748433335040J(-3)^2L(-6)\\
&+ 10511227468442837375232J(-4)J(-3)L(-5)\\
&- 3625776663088141002000J(-4)^2L(-4)\\
&- 6961122963880684781952J(-5)J(-3)L(-4)\\
&+ 4929939475120655126880J(-5)J(-4)L(-3)\\
&+ 4773532687076653363080J(-6)J(-3)L(-3)
\end{aligned}
$$

$$
\begin{aligned}
&- 171648412285445115468J(-3)^4\\
&- 1066521117142941065550J(-6)^2\\
&+ 12493191821706784715520J(-7)J(-5)\\
&- 5797068994347779833440J(-8)J(-4)\\
&+ 12603643268347174428672J(-9)J(-3)\\
&+ 228266678826758310400L(-2)^6\\
&- 3882692082393713276160L(-4)L(-2)^4\\
&+ 1581754716008505972000L(-3)^2L(-2)^3\\
&+ 2108170613606957312640L(-6)L(-2)^3\\
&+ 8832499573007649131616L(-4)^2L(-2)^2\\
&- 2549342475388773078720L(-8)L(-2)^2\\
&- 9121774545025047556320L(-5)L(-3)L(-2)^2\\
&+ 11219262349279977066432L(-5)^2L(-2)\\
&+ 800472455794442907120L(-4)L(-3)^2L(-2)\\
&+ 1980851526476237525376L(-6)L(-4)L(-2)\\
&- 2545988958876918647040L(-7)L(-3)L(-2)\\
&- 2064428686210240327875L(-3)^4\\
&+ 1062934268596930370784L(-4)^3\\
&- 6375466678009788958464L(-6)^2\\
&+ 14241366043443771963840L(-6)L(-3)^2\\
&+ 11089951838933314485504L(-12)\\
&+ 18738841210216937700672L(-7)L(-5)\\
&+ 13050662062784044674144L(-8)L(-4)\\
&+ 25601205514837174085760L(-9)L(-3)\\
&- 4960605588937809643872L(-5)L(-4)L(-3))\mathbb{1},\\
U_{12}^2 = (&- 6828349711130669400000J(-3)^2L(-2)^3\\
&- 1238547790386007290000J(-4)^2L(-2)^2\\
&+ 45950606112292591752000J(-5)J(-3)L(-2)^2\\
&- 68760390254879388240000J(-5)^2L(-2)\\
&+ 61774703447770919214000J(-6)J(-4)L(-2)\\
&- 157582755702018297336000J(-7)J(-3)L(-2)\\
&+ 25845401839701600324000J(-3)^2L(-4)L(-2)
\end{aligned}
$$

$$
\begin{aligned}
&- 459703436252660013960000J(-4)J(-3)L(-3)L(-2)\\
&+ 171504950209672414545000J(-3)^2L(-3)^2\\
&- 112585948687895047142400J(-3)^2L(-6)\\
&+ 103719931902734081020800J(-4)J(-3)L(-5)\\
&- 495389363491872419220000J(-4)^2L(-4)\\
&- 328793703982930359264000J(-5)J(-3)L(-4)\\
&+ 556173365704131765240000J(-5)J(-4)L(-3)\\
&- 157248221329579724190000J(-6)J(-3)L(-3)\\
&+ 206634866041117790850 0J(-3)^4\\
&- 683370611165183909467 50J(-6)^2\\
&+ 194210340267657714336000J(-7)J(-5)\\
&- 174609762245001030511200J(-8)J(-4)\\
&+ 319573429759307780851200J(-9)J(-3)\\
&+ 199683023958915001600 0L(-2)^6\\
&- 320302366540289311296000L(-4)L(-2)^4\\
&+ 136641204618867341160000L(-3)^2L(-2)^3\\
&+ 319518259107158750496000L(-6)L(-2)^3\\
&+ 861546351615311549419200L(-4)^2L(-2)^2\\
&- 163574416779642731760000L(-8)L(-2)^2\\
&- 810262680271227467928000L(-5)L(-3)L(-2)^2\\
&+ 944458897079867942736000L(-5)^2L(-2)\\
&+ 165243573891337924188000L(-4)L(-3)^2L(-2)\\
&+ 110899518389333144855040L(-10)L(-2)\\
&+ 321820381179607511980800L(-6)L(-4)L(-2)\\
&- 469701032047671228864000L(-7)L(-3)L(-2)\\
&- 189751062371181843048 75L(-3)^4\\
&+ 166630721978441540406 72L(-4)^3\\
&+ 113493996574361727091 20L(-6)^2\\
&+ 117477372846233820052800L(-6)L(-3)^2\\
&+ 321212788014830758616640L(-7)L(-5)\\
&+ 216646683370054699309920L(-8)L(-4)\\
&+ 355551946988511916535040L(-9)L(-3)
\end{aligned}
$$

$$
\begin{aligned}
&- 5638239373432976241264 0L(-5)L(-4)L(-3))\mathbb{1},\\
U_{12}^3 = {}&(103488000J(-4)L(-2)^4 - 1028193600J(-6)L(-2)^3\\
&- 155232000J(-3)L(-3)L(-2)^3 + 3932047200J(-8)L(-2)^2\\
&- 209808000J(-3)L(-5)L(-2)^2 - 734098400J(-4)L(-4)L(-2)^2\\
&+ 1435459200J(-5)L(-3)L(-2)^2 + 452957400J(-4)J(-3)^2L(-2)\\
&- 1178968300J(-4)L(-3)^2L(-2) - 16819505200J(-10)L(-2)\\
&+ 1434952320J(-3)L(-7)L(-2) - 2661520320J(-4)L(-6)L(-2)\\
&+ 656575040J(-5)L(-5)L(-2) + 488209320J(-6)L(-4)L(-2)\\
&- 1870539600J(-7)L(-3)L(-2) + 1415859600J(-3)L(-4)L(-3)L(-2)\\
&+ 1086589650J(-3)L(-3)^3 - 1042286400J(-4)L(-4)^2\\
&+ 1165918275J(-6)L(-3)^2 - 3147857592J(-3)L(-9)\\
&- 1152711040J(-4)L(-8) - 2033897640J(-5)L(-7)\\
&+ 2154938040J(-6)L(-6) - 2099934280J(-7)L(-5)\\
&- 2636649720J(-8)L(-4) + 1042286400J(-3)L(-5)L(-4)\\
&- 679436100J(-3)^3L(-3) - 5683985880J(-9)L(-3)\\
&+ 1122375840J(-3)L(-6)L(-3) - 2246870080J(-4)L(-5)L(-3)\\
&+ 241014000J(-5)L(-4)L(-3) + 1093808240J(-4)^3\\
&+ 2060956170J(-6)J(-3)^2 + 75411426960J(-12)\\
&- 3200898960J(-5)J(-4)J(-3))\mathbb{1}.
\end{aligned}
$$

A.4. *Weight three generators of $V(3)^{S_3}$.*

$$
\begin{aligned}
w_1 = {}& h_1(-1)e_1(-1)f_1(-1) - 2h_1(-1)e_1(-1)f_3(-1) + 2h_1(-1)e_3(-1)f_1(-1)\\
&+ 2h_1(-1)e_3(-1)f_3(-1) - h_1(-1)h_1(-1)h_1(-1) + h_1(-1)h_1(-1)h_3(-1)\\
&+ h_1(-1)h_3(-1)h_3(-1) - 2h_3(-1)e_1(-1)f_1(-1) + 2h_3(-1)e_1(-1)f_3(-1)\\
&+ 2h_3(-1)e_3(-1)f_1(-1) + 2h_3f_1(-1)e_3(-1)f_3(-1) - h_3(-1)h_3(-1)h_3(-1)
\end{aligned}
$$

$$
\begin{aligned}
w_2 = {}& -4e_1(-1)e_1(-1)f_1(-1) - 4e_1(-1)e_2(-1)f_3(-1) - 4e_1(-1)e_3(-1)f_1(-1)\\
&+ 8e_1(-1)e_3(-1)f_3(-1) - 4e_2(-1)e_3(-1)f_1(-1) + 2e_3(-1)e_3(-1)f_3(-1)\\
&- 5h_1(-1)h_1(-1)e_1(-1) + h_1(-1)h_1(-1)e_2(-1) + 4h_1(-1)h_1(-1)e_3(-1)\\
&+ 2h_1(-1)h_3(-1)e_1(-1) - 2h_1(-1)h_3(-1)e_2(-1) - 4h_1(-1)h_3(-1)e_3(-1)\\
&- 5h_3(-1)h_3(-1)e_1(-1) + h_3(-1)h_3(-1)e_2(-1) + 4h_3(-1)h_3(-1)e_3(-1)
\end{aligned}
$$

$$
\begin{aligned}
w_3 = {}& 4e_1(-1)f_1(-1)f_3(-1) - 4e_1(-1)f_2(-1)f_3(-1) - 4e_3(-1)f_1(-1)f_2(-1)\\
&- 8e_3(-1)f_1(-1)f_3(-1) - h_1(-1)h_1(-1)f_1(-1) + h_1(-1)h_1(-1)f_2(-1)
\end{aligned}
$$

$$+ 2h_1(-1)h_3(-1)f_1(-1) - 2h_1(-1)h_3(-1)f_2(-1) - 4h_1(-1)h_3(-1)f_3(-1)$$
$$+ 3h_3(-1)h_3(-1)f_1(-1) + h_3(-1)h_3(-1)f_2(-1) + 2h_3(-1)h_3(-1)f_3(-1)$$
$$+ 4h_1(-2)f_1(-1) - 2h_3(-2)f_3(-1).$$

Then the primary vectors in the orbifold are orbit sums

$$W_i = \sum_{\sigma \in S_3} \sigma(w_i)\mathbb{1}, \quad 1 \le i \le 3,$$

where σ acts by permuting indices.

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