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ČECH SYSTEMS AND APPROXIMATE INVERSE SYSTEMS

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Abstract. We generalize a result of the first author who proved that the Čech system of open covers of a Hausdorff arc-like space cannot induce an approximate system of the nerves of these covers under any choices of the meshes and the projections.

1. Introduction

We generalize a result of the first author in [3] where it was proved that the Čech system of open covers of a Hausdorff arc-like space cannot induce an approximate system of the nerves of these covers under any choices of the meshes and the projections because it cannot satisfy axiom (A2) (see below for the definitions of these terms).

Before we state the main result of [3], let us review the definition of an approximate (inverse) system according to [2]. Recall that if f and $g : X \to Y$ are functions and $\mathcal{U}$ is a cover of Y, then one writes $(f, g) < \mathcal{U}$ to mean that for each $x \in X$, there exists $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. For each space X, $\mathrm{Cov}(X)$ is the collection of normal covers of X, and map means continuous function. An *approximate system* is a quadruple $(X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, (\Lambda, \preceq))$ where the pair $(\Lambda, \preceq)$ is a pre-ordered set (antisymmetry not required) which is directed and unbounded, for each $\lambda \in \Lambda$, a space X_λ and $\mathcal{U}_\lambda \in \mathrm{Cov}(X_\lambda)$ (called the *mesh* of X_λ) and for $\lambda \preceq \lambda'$, a map $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ such that $p_{\lambda\lambda} = \mathrm{id}_{X_\lambda}$. We require in addition the following three conditions.

(A1) $(p_{\lambda\lambda'} p_{\lambda'\lambda''}, p_{\lambda\lambda''}) < \mathcal{U}_\lambda$, $\lambda \preceq \lambda' \preceq \lambda''$.

(A2) For each $\lambda \in \Lambda$ and each $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$, there exists $\lambda_0 \in \Lambda$, $\lambda \preceq \lambda_0$, such that $(p_{\lambda\lambda_1} p_{\lambda_1\lambda_2}, p_{\lambda\lambda_2}) < \mathcal{U}$, whenever $\lambda_0 \preceq \lambda_1 \preceq \lambda_2$.

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(A3) For each $\lambda \in \Lambda$ and each $\mathcal{U} \in \mathrm{Cov}(X_\lambda)$, there exists $\lambda' \in \Lambda$, $\lambda \preceq \lambda'$, such that $p_{\lambda\lambda''}^{-1}(\mathcal{U})$ is refined by $\mathcal{U}_{\lambda''}$ whenever $\lambda' \preceq \lambda''$.

In [1], Mardešić studied such systems where the axioms (A1) and (A3) were not required. He was able to show, nevertheless, that these systems exhibited many of the features that are true for those with (A1) and (A3). Delving further into this, in [4] the authors demonstrated that in order to have mappings between systems, that is to have a category, it was necessary to require those two axioms. In this paper it will not matter which of the two perspectives on approximate systems one wishes to deploy because only (A2) will come into play.

There are a few matters of notation and some definitions to put into place. For any collection $\mathcal{U}$ of sets, $N(\mathcal{U})$ will denote the nerve of $\mathcal{U}$ and $|N(\mathcal{U})|$ the polyhedron of $N(\mathcal{U})$ with the weak topology. We will use the concept of a *projection* throughout this work. Let $\mathcal{V}$ and $\mathcal{W}$ be collections of sets such that $\mathcal{V} \preceq \mathcal{W}$, that is, $\mathcal{W}$ refines $\mathcal{V}$, and $p : \mathcal{W} \to \mathcal{V}$ be a function such that for each $W \in \mathcal{W}$, $W \subset p(W)$. Then p is called a *projection* from $\mathcal{W}$ to $\mathcal{V}$. Such p induces a simplicial function from $N(\mathcal{W})$ to $N(\mathcal{V})$, and in turn a map from $|N(\mathcal{W})|$ to $|N(\mathcal{V})|$, which we usually denote $p_{\mathcal{V}\mathcal{W}}$. Plainly, the composition of projections is a projection. It will be our convention herein that if $\mathcal{V} = \mathcal{W}$, then the only projection $p_{\mathcal{V}\mathcal{W}}$ in play will be the one induced by the identity function from $\mathcal{W}$ to $\mathcal{V}$.

A space X is called arc-like if for each $\mathcal{U} \in \mathrm{Cov}(X)$, there exists $\mathcal{V} \in \mathrm{Cov}([0,1])$ and a surjective map $f : X \to [0,1]$ such that the open cover $f^{-1}(\mathcal{V})$ of X refines $\mathcal{U}$.

Proposition 1.1 (see Proposition 2.2 of [3]). *Let a* T_1*-space* X *be arc-like. Then* X *is a nontrivial Hausdorff continuum.*

Plainly, any space as in Proposition 1.1 is paracompact, so all its open covers are normal covers. The main result, Theorem 2.13 of [3], can be stated as follows.

Theorem 1.2. *Let* X *be a Hausdorff arc-like space. For each* $\mathcal{U} \in \mathrm{Cov}(X)$*, let* $\mathcal{H}_\mathcal{U} \in \mathrm{Cov}(|N(\mathcal{U})|)$*, and if* $\mathcal{V} \in \mathrm{Cov}(X)$ *with* $\mathcal{U} \preceq \mathcal{V}$*, let* $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$ *be a projection. Then the* Čech *system*

$$\mathbf{U} = (|N(\mathcal{U})|, \mathcal{H}_\mathcal{U}, p_{\mathcal{U}\mathcal{V}}, (\mathrm{Cov}(X), \preceq))$$

does not satisfy (A2) *of the definition of approximate system. Hence it is impossible to make the choices of the elements* $\mathcal{H}_\mathcal{U} \in \mathrm{Cov}(|N(\mathcal{U})|)$ *and the projections* $p_{\mathcal{U}\mathcal{V}}$ *so that* $\mathbf{U}$ *would be an approximate system.*

Our main result, Theorem 4.1, is a substantial generalization of this one although our proof techniques are offshoots of those used in [3].

2. Chains

Definition 2.1. *Let* $(U_1, \ldots, U_m)$ *be a finite sequence of sets,* $a \in U_1$*, and* $b \in U_m$*. Suppose that,*

1. *whenever* $1 \leq i \leq j \leq m$*, then* $U_i \cap U_j \neq \emptyset$*, if and only if* $j - i \leq 1$*,*
2. $a \in U_i$ *if and only if* $i = 1$*, and*
3. $b \in U_i$ *if and only if* $i = m$*.*

Then $(U_1, \ldots, U_m)$ *is called a* **simple chain** *from* a *to* b*.*

Lemma 2.2. *Let* $(U_1, \ldots, U_m)$*,* $m > 1$*, be a simple chain from* a *to* b*. Then for all* $1 \leq i < m$*, both* $U_i \subset U_{i+1}$ *and* $U_{i+1} \subset U_i$ *are false.*

Proof. We cannot have $U_1 \subset U_2$, for this would violate Definition 2.1(2). If $1 < i$ and $U_i \subset U_{i+1}$, then $\emptyset \neq U_{i-1} \cap U_i \subset U_{i+1}$, so $U_{i-1} \cap U_{i+1} \neq \emptyset$, contrary to Definition 2.1(1). Now $U_m \subset U_{m-1}$ is not possible because of Definition 2.1(3). Suppose that $i < m - 1$ and $U_{i+1} \subset U_i$. Then $\emptyset \neq U_{i+2} \cap U_{i+1} \subset U_i$, so $U_i \cap U_{i+2} \neq \emptyset$, which cannot happen because of Definition 2.1(1). □

Definition 2.3. *Let* X *be a space,* $\mathcal{U}$ *an open cover of* X*,* $D \subset X$*, and* $\{a, b\} \subset D$*. A simple chain* $(U_1 \cap D, \ldots, U_m \cap D)$ *from* a *to* b *is called a simple* $\mathcal{U}_D$*-chain from* a *to* b *if for each* $1 \leq i \leq m$*,* $U_i \in \mathcal{U}$*.*

Lemma 2.4. *Let* X *be a space,* $\mathcal{U}$ *an open cover of* X*,* D *a component of* X*, and* $\{a, b\} \subset D$*. Then there exists a simple* $\mathcal{U}_D$*-chain from* a *to* b*.*

Proof. Let C_a be the set of points x in D having the property that there exists a simple $\mathcal{U}_D$-chain from a to x. There exists $U_1 \in \mathcal{U}$ with $a \in U_1$, so the simple $\mathcal{U}_D$-chain $(U_1 \cap D)$ witnesses the fact that $a \in C_a$. Thus $C_a \neq \emptyset$. We claim that C_a is open and closed in the connected space D, which will show that $C_a = D$, and will complete our proof.

Let $x \in C_a$ and select a simple $\mathcal{U}_D$-chain $(U_1 \cap D, \ldots, U_m \cap D)$ from a to x. For each $y \in U_m \cap D$, either $(U_1 \cap D, \ldots, U_m \cap D)$ or $(U_1 \cap D, \ldots, U_{m-1} \cap D)$ is a simple $\mathcal{U}_D$-chain from a to y, so $U_m \cap D \subset C_a$. Since $U_m \cap D$ is an open set in the subspace D, it follows that $U_m \cap D$ is contained in the interior of C_a in D. In particular, $x \in U_m \cap D$; so x lies in the interior of C_a in D, which shows that C_a is open in D.

To prove that C_a is also closed in D, let y be a limit point of C_a in D. Select an element $U \in \mathcal{U}$ with $y \in U$; hence $y \in U \cap D$. If $a \in U \cap D$, then $(U \cap D)$ is a simple $\mathcal{U}_D$-chain from a to y, so $y \in C_a$. Assume that $a \notin U \cap D$. There exists $x \in C_a \cap U$, so we may choose a simple $\mathcal{U}_D$-chain $(U_1 \cap D, \ldots, U_k \cap D)$ from a to x. Note that $x \in U_k \cap U \cap D$, so $U_k \cap U \cap D \neq \emptyset$. Let $m = \min\{i \in \{1, \ldots, k\} \,|\, U_i \cap U \cap D \neq \emptyset\}$. We claim that $U_m \cap D$ is not contained in $U \cap D$. This is obvious if $m = 1$, since $a \notin U \cap D$. If $m > 1$, $U_m \cap D$ is not contained in $U \cap D$, for otherwise we would get $\emptyset \neq U_{m-1} \cap U_m \cap D \subset U_{m-1} \cap U \cap D$, which contradicts the definition of m. Now we conclude that either $(U_1 \cap D, \ldots, U_m \cap D)$ or $(U_1 \cap D, \ldots, U_m \cap D, U \cap D)$

is a simple $\mathcal{U}_D$-chain from a to y. This implies that $y \in C_a$ and shows that C_a is closed in D. ∎

3. Nerves of Covers

Since all polyhedra P are paracompact, then every open cover of a polyhedron is a normal cover.

LEMMA 3.1. *Let P be a polyhedron and L a triangulation of P. Then $\{\mathrm{st}(v, L) \mid v \in L^{(0)}\} \in \mathrm{Cov}(P)$; if $\{v, w\} \subset L^{(0)}$ and $\mathrm{card}(\{v, w\}) = 2$, then $w \notin \mathrm{st}(v, L)$.* ∎

DEFINITION 3.2. *For each space X, let $\mathcal{O}(X)$ denote the set of nonempty collections of open subsets of X.*

Our Corollary 3.4 can be extracted from the first part of the proof of Theorem 2.12 of [3]. However, using the technique of that proof, we have a somewhat more general lemma that implies Corollary 3.4.

LEMMA 3.3. *Let X be a space and $\mathcal{C}$ be a nonempty subset of $\mathcal{O}(X)$. For each $\{\mathcal{U}, \mathcal{V}\} \subset \mathcal{C}$ such that $\mathcal{U} \preceq \mathcal{V}$, select a projection $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$, and suppose that the obtained system $(|N(\mathcal{U})|, p_{\mathcal{U}\mathcal{V}}, (\mathcal{C}, \preceq))$ satisfies (A2) in the sense that for each $\mathcal{U} \in \mathcal{C}$ and $\mathcal{A} \in \mathrm{Cov}(|N(\mathcal{U})|)$, there exists $\mathcal{U}_0 \in \mathcal{C}$ such that $\mathcal{U} \preceq \mathcal{U}_0$ and for any $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ with $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, $(p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}, p_{\mathcal{U}\mathcal{W}}) < \mathcal{A}$. Then for any $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ with $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}} = p_{\mathcal{U}\mathcal{W}}$.*

PROOF. Let $\mathcal{U} \in \mathcal{C}$ and $\mathcal{A} = \{\mathrm{st}(U, N(\mathcal{U})) \mid U \in N(\mathcal{U})^{(0)}\} \in \mathrm{Cov}(|N(\mathcal{U})|)$. Since the postulated system satisfies (A2), then there exists $\mathcal{U}_0 \in \mathcal{C}$ such that $\mathcal{U} \preceq \mathcal{U}_0$ and for any $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ with $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, $(p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}, p_{\mathcal{U}\mathcal{W}}) < \mathcal{A}$.

Suppose that $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ and $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$. Let $W \in N(\mathcal{W})^{(0)}$. Since the projections are simplicial, then $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}(W)$ and $p_{\mathcal{U}\mathcal{W}}(W)$ lie in $N(\mathcal{U})^{(0)}$. However, $(p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}, p_{\mathcal{U}\mathcal{W}}) < \mathcal{A}$, so there exists $U \in N(\mathcal{U})^{(0)}$ with both $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}(W)$ and $p_{\mathcal{U}\mathcal{W}}(W)$ in $\mathrm{st}(U, N(\mathcal{U}))$. By Lemma 3.1, $\mathrm{st}(U, N(\mathcal{U}))$ contains at most one element of $N(\mathcal{U})^{(0)}$, so $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}(W) = p_{\mathcal{U}\mathcal{W}}(W)$, and it follows that $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}} = p_{\mathcal{U}\mathcal{W}}$. ∎

COROLLARY 3.4. *Let X be a space and $\mathcal{C}$ be a nonempty subset of $\mathrm{Cov}(X)$. For each $\{\mathcal{U}, \mathcal{V}\} \subset \mathcal{C}$ such that $\mathcal{U} \preceq \mathcal{V}$, select a projection $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$, and suppose that the obtained system$(|N(\mathcal{U})|, p_{\mathcal{U}\mathcal{V}}, (\mathcal{C}, \preceq))$ satisfies (A2) in the sense that for each $\mathcal{U} \in \mathcal{C}$ and $\mathcal{A} \in \mathrm{Cov}(|N(\mathcal{U})|)$, there exists $\mathcal{U}_0 \in \mathcal{C}$ such that $\mathcal{U} \preceq \mathcal{U}_0$ and for any $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ with $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, $(p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}, p_{\mathcal{U}\mathcal{W}}) < \mathcal{A}$. Then for any $\{\mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$ with $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, $p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}} = p_{\mathcal{U}\mathcal{W}}$.* ∎

4. Main Theorem

THEOREM 4.1. *Let X be a Hausdorff paracompactum that has a nontrivial component. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathcal{H}_{\mathcal{U}} \in \mathrm{Cov}(|N(\mathcal{U})|)$, and if $\mathcal{V} \in \mathrm{Cov}(X)$*

with $\mathcal{U} \preceq \mathcal{V}$*, let* $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$ *be a projection. Then the* Čech *system*

$$\mathbf{U} = (|N(\mathcal{U})|, \mathcal{H}_{\mathcal{U}}, p_{\mathcal{U}\mathcal{V}}, (\mathrm{Cov}(X), \preceq))$$

does not satisfy (A2) *of the definition of approximate system. Hence it is impossible to make the choices of the elements* $\mathcal{H}_{\mathcal{U}} \in \mathrm{Cov}(|N(\mathcal{U})|)$ *and the projections* $p_{\mathcal{U}\mathcal{V}}$ *so that* $\mathbf{U}$ *would be an approximate system.*

Proof. Let us assume the contrary, i.e., that (A2) of the definition of an approximate system holds true for $\mathbf{U}$. To shorten the notation, let $\mathcal{C} = \mathrm{Cov}(X)$. Select a nontrivial component D of X and let $\{a, b\} \subset D$ be chosen so that $\mathrm{card}(\{a, b\}) = 2$. Let $U_a = X \setminus \{b\}$ and $U_b = X \setminus \{a\}$. Then $\mathcal{U} = \{U_a, U_b\} \in \mathcal{C}$, U_a is the only element of $\mathcal{U}$ that contains a, and U_b is the only element of $\mathcal{U}$ that contains b. Select $\mathcal{U}_0 \in \mathcal{C}$ with $\mathcal{U} \preceq \mathcal{U}_0$ in accordance with Corollary 3.4 as applied to $(|N(\mathcal{U})|, p_{\mathcal{U}\mathcal{V}}, (\mathcal{C}, \preceq))$. So,

(1) if $\{\mathcal{U}^*, \mathcal{V}, \mathcal{W}\} \subset \mathcal{C}$, and $\mathcal{U}_0 \preceq \mathcal{U}^* \preceq \mathcal{V} \preceq \mathcal{W}$, then each of $p_{\mathcal{U}\mathcal{V}}p_{\mathcal{V}\mathcal{W}} = p_{\mathcal{U}\mathcal{W}}$, $p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{V}} = p_{\mathcal{U}\mathcal{V}}$, and $p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{W}} = p_{\mathcal{U}\mathcal{W}}$ holds true.

Since D is a component of X, then Lemma 2.4 gives us a simple $(\mathcal{U}_0)_D$ chain $(U_1^0 \cap D, \dots, U_m^0 \cap D)$ from a to b. Plainly $a \in U_1^0 \subset U_a$, $a \notin U_i^0$ for $i > 1$, $b \in U_m^0 \subset U_b$, $b \notin U_i^0$ for $i < m$, and obviously $1 < m$. Making use of the fact that the nontrivial connected T_1-space D is perfect, we may select for each $1 \le i < m$, two elements x_i, $y_i \in D \cap U_i^0 \cap U_{i+1}^0$ in such a manner that no x_i or y_i equals a or b, $\{x_1, \dots, x_{m-1}\} \cap \{y_1, \dots, y_{m-1}\} = \emptyset$, and if $1 \le i < j < m$, then $x_i \ne x_j$ and $y_i \ne y_j$. Let

$$E = \{a, b\} \cup \{x_i \,|\, 1 \le i < m\} \cup \{y_i \,|\, 1 \le i < m\}.$$

Of course E is a finite and hence closed subset of X.

We "inscribe" a refinement $\mathcal{U}^*$ inside the open cover $\mathcal{U}_0$ as follows. For each $U \in \mathcal{U}_0 \setminus \{U_1^0, \dots, U_m^0\}$, let $U^* = U \setminus E$. If $U \in \{U_1^0, \dots, U_m^0\}$, put $U^* = U$. Then define $\mathcal{U}^* = \{U^* \,|\, U \in \mathcal{U}_0\}$. Our construction of $\mathcal{U}^*$ shows that:

(2) if $U \in \mathcal{U}^*$ and $U \cap E \ne \emptyset$, then $U \in \{U_1^0, \dots, U_m^0\}$.

Plainly,

(3) $p_{\mathcal{U}\mathcal{U}^*}(U_1^0) = U_a$,

(4) $p_{\mathcal{U}\mathcal{U}^*}(U_m^0) = U_b$, and

(5) for all $1 < i < m$, either $p_{\mathcal{U}\mathcal{U}^*}(U_i^0) = U_a$ or $p_{\mathcal{U}\mathcal{U}^*}(U_i^0) = U_b$.

It follows from (3)–(5) that,

(6) there exists $k \in \{1, \dots, m-1\}$ such that $p_{\mathcal{U}\mathcal{U}^*}(U_k^0) = U_a$ and $p_{\mathcal{U}\mathcal{U}^*}(U_{k+1}^0) = U_b$.

Since $(U_1^0 \cap D, \dots, U_m^0 \cap D)$ is a simple $(\mathcal{U}_0)_D$-chain from a to b, and (2) is true, then,

(7) the only elements of $\mathcal{U}^*$ that contain x_k or y_k are U_k^0 and U_{k+1}^0.

We want to form two "adjustments," $\mathcal{V}_1$ and $\mathcal{V}_2$, to $\mathcal{U}^*$. First, let $\mathcal{M} = \mathcal{U}^* \setminus \{U_k^0, U_{k+1}^0\}$. Then put,

(8) $\mathcal{V}_1 = \mathcal{M} \cup \{U_k^0 \setminus \{x_k\}, U_{k+1}^0 \setminus \{y_k\}\}$ and

(9) $\mathcal{V}_2 = \mathcal{M} \cup \{U_k^0 \setminus \{y_k\}, U_{k+1}^0 \setminus \{x_k\}\}$.

One then checks that,

(10) $\{\mathcal{V}_1, \mathcal{V}_2\} \subset \mathcal{C}$,

(11) both $\mathcal{U}^* \preceq \mathcal{V}_1$ and $\mathcal{U}^* \preceq \mathcal{V}_2$,

(12) the only element of $\mathcal{V}_1$ that contains x_k is $U_{k+1}^0 \setminus \{y_k\}$, and

(13) the only element of $\mathcal{V}_2$ that contains x_k is $U_k^0 \setminus \{y_k\}$.

Now $y_k \in U_k^0$. Hence it is not possible that $U_{k+1}^0 \setminus \{y_k\} \subset U_k^0$ since in that case (see (7)) one would have $U_{k+1}^0 \cap D \subset U_k^0 \cap D$, and according to Lemma 2.2, $(U_1^0 \cap D, \ldots, U_m^0 \cap D)$ would not be a simple $(\mathcal{U}_0)_D$-chain. So, taking into account (7), we get that,

(14) $p_{\mathcal{U}^*\mathcal{V}_1}(U_{k+1}^0 \setminus \{y_k\}) = U_{k+1}^0$.

It is not possible that $U_k^0 \setminus \{y_k\} \subset U_{k+1}^0$, for in that case (see (7)) one would have $U_k^0 \cap D \subset U_{k+1}^0 \cap D$, and according to Lemma 2.2, $(U_1^0 \cap D, \ldots, U_m^0 \cap D)$ would not be a simple $(\mathcal{U}_0)_D$-chain. So, taking into account (7), we get that,

(15) $p_{\mathcal{U}^*\mathcal{V}_2}(U_k^0 \setminus \{y_k\}) = U_k^0$.

Select $\mathcal{W} \in \mathcal{C}$ in such a manner that $\mathcal{V}_1 \preceq \mathcal{W}$ and $\mathcal{V}_2 \preceq \mathcal{W}$. There exists an element $W_{x_k} \in \mathcal{W}$ with

(16) $x_k \in W_{x_k}$.

Making use of (12) and (16), one sees that,

(17) $p_{\mathcal{V}_1\mathcal{W}}(W_{x_k}) = U_{k+1}^0 \setminus \{y_k\}$.

Similarly, this time using (13) and (16), we get,

(18) $p_{\mathcal{V}_2\mathcal{W}}(W_{x_k}) = U_k^0 \setminus \{y_k\}$.

Since $\mathcal{V}_1 \preceq \mathcal{W}$ and $\mathcal{V}_2 \preceq \mathcal{W}$, then (11) shows that,

(19) both $\mathcal{U}_0 \preceq \mathcal{U}^* \preceq \mathcal{V}_1 \preceq \mathcal{W}$ and $\mathcal{U}_0 \preceq \mathcal{U}^* \preceq \mathcal{V}_2 \preceq \mathcal{W}$.

The relations in (19) along with (1), imply that,

(20) $p_{\mathcal{U}\mathcal{V}_1}p_{\mathcal{V}_1\mathcal{W}} = p_{\mathcal{U}\mathcal{W}} = p_{\mathcal{U}\mathcal{V}_2}p_{\mathcal{V}_2\mathcal{W}}$, $p_{\mathcal{U}\mathcal{V}_1} = p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{V}_1}$, and $p_{\mathcal{U}\mathcal{V}_2} = p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{V}_2}$.

This allows us to evaluate $p_{\mathcal{U}\mathcal{W}}(W_{x_k})$ in two ways. First, $p_{\mathcal{U}\mathcal{W}}(W_{x_k}) = p_{\mathcal{U}\mathcal{V}_1}p_{\mathcal{V}_1\mathcal{W}}(W_{x_k}) = p_{\mathcal{U}\mathcal{V}_1}(U_{k+1}^0 \setminus \{y_k\}) = p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{V}_1}(U_{k+1}^0 \setminus \{y_k\}) = p_{\mathcal{U}\mathcal{U}^*}(U_{k+1}^0)$ $= U_b$. Second, $p_{\mathcal{U}\mathcal{W}}(W_{x_k}) = p_{\mathcal{U}\mathcal{V}_2}p_{\mathcal{V}_2\mathcal{W}}(W_{x_k}) = p_{\mathcal{U}\mathcal{V}_2}(U_k^0 \setminus \{y_k\}) = p_{\mathcal{U}\mathcal{U}^*}p_{\mathcal{U}^*\mathcal{V}_2}$ $(U_k^0 \setminus \{y_k\}) = p_{\mathcal{U}\mathcal{U}^*}(U_k^0) = U_a$. Since $U_a \neq U_b$, we have arrived at a contradiction. Our proof is complete. □

In Theorem 4.1, the requirement that X be a Hausdorff paracompactum was needed only to insure that all open covers of X are normal open covers. Lemma 3.3 has no separation requirements on X at all. Hence if we replace the requirement for normal covers simply by open covers, drop paracompactness, replace Hausdorff by T_1, and examine our proof of Theorem 4.1 (using Lemma 3.3 where we had Corollary 3.4), we obtain a different theorem. Before stating it, we give one definition.

Definition 4.2. *For each space X,* OCov(X) *will denote the set of open covers of X.*

THEOREM 4.3. *Let X be a T_1-space that has a nontrivial component. For each $\mathcal{U} \in \mathrm{OCov}(X)$, let $\mathcal{H}_{\mathcal{U}} \in \mathrm{Cov}(|N(\mathcal{U})|)$, and if $\mathcal{V} \in \mathrm{OCov}(X)$ with $\mathcal{U} \preceq \mathcal{V}$, let $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$ be a projection. Then the system*

$$\mathbf{U} = (|N(\mathcal{U})|, \mathcal{H}_{\mathcal{U}}, p_{\mathcal{U}\mathcal{V}}, (\mathrm{OCov}(X), \preceq))$$

does not satisfy (A2) *of the definition of approximate system. Hence it is impossible to make the choices of the elements $\mathcal{H}_{\mathcal{U}} \in \mathrm{Cov}(|N(\mathcal{U})|)$ and the projections $p_{\mathcal{U}\mathcal{V}}$ so that $\mathbf{U}$ would be an approximate system.*

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