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On mappings that preserve Fermat-Torricelli points

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Abstract. Let Δ be the set of all triple points $\{A, B, C\}$ in $\mathbb{R}^n$ such that the largest angle of the triangle ABC is less than $\frac{2\pi}{3}$. In this paper, we proved that if a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Fermat-Torricelli points of the triangles in Δ, then f is an affine transformation.

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1. Introduction

A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is called an affine transformation provided it can be expressed by

$$f(x) = g(x) + a,$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and a is a point in $\mathbb{R}^m$. The affine transformations are well known and fundamental in Euclidean geometry and they have many beautiful properties as follows:

- Any affine transformation is uniquely expressible as $f(x) = g(x) + a$, where $a \in \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.
- Any affine transformation $f(x) = g(x) + a$ is bijective if and only if the linear transformation $g : \mathbb{R}^n \to \mathbb{R}^m$ is bijective.
- A bijective affine transformation is an affine isomorphism.
- If an affine transformation is invertible, then its inverse is also an affine transformation.
- If f_1 and f_2 are two affine transformations, then every linear combination $c_1 f_1 + c_2 f_2$, where $c_1, c_2 \in \mathbb{R}$, is an affine transformation.
- The composition of any two affine transformations is also an affine transformation.

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There are many geometric features that remain invariant under affine transformations. For example, points are mapped to points, lines are mapped to lines (line segments are mapped to line segments), and planes are mapped to planes (hyperplanes are mapped to hyperplanes) under affine transformations. A nice consequence of this fact is that one can calculate the image of a polygon by simply computing the images of its vertices. In addition to these features, parallelism of lines and ratios is preserved under affine transformations. Geometric contraction, expansion, dilation, reflection, rotation, shear, similarity transformations, spiral similarities, and translation are all affine transformations, as are their combinations. In general, an affine transformation is a composition of rotations, translations, dilations, and shears. However, it should be noted that distances and angles may not be preserved under affine transformations. For more details about affine transformations, we refer the readers to [6].

In literature, there are many characterizations of affine transformation as follows:

Theorem 1 (see [1]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is a bijection and that it preserves lines, and suppose that the images of any two parallel lines under f are still parallel lines. Then f is an affine transformation.*

Here, f is said to preserve lines if the image of each line is still a line.

Theorem 2 (see [3]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is a bijection and that it preserves lines. Then f is an affine transformation.*

Theorem 3 (see [2]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is surjective and line-to-line. Then f is an affine transformation.*

Here, f is said to be line-to-line if the image of each line in $\mathbb{R}^n$ is contained in a line of $\mathbb{R}^n$.

Theorem 4 (see [4]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ preserves lines. Then f is an affine transformation if and only if it is non-degenerate, that is, the image of the whole space under f is more than a line or geodesic.*

Theorem 5 (see [5]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is a bijection. Then f is an affine transformation if and only if f is triangle domain preserving.*

Theorem 6 (see [5]). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is a bijection. Then f is an affine transformation if and only if f is triangle preserving.*

In this paper, we try to present a new characterization of affine transformations by use of Fermat-Torricelli points of triangles. In geometry, the Fermat-Torricelli point of a triangle, also called the Fermat point or Torricelli point, is a point such that the total distance from the three vertices of the triangle to the point is as minimal as possible. It is named so because this problem was first raised by Fermat in a private letter to Evangelista Torricelli, who solved it. Let ABC be a triangle in the Euclidean plane $\mathbb{R}^2$. If the measure of the largest angle reaches $\frac{2\pi}{3}$ radians or more, then the vertex at the largest angle is the solution to Fermat's problem. If the measure of the largest angle is less than $\frac{2\pi}{3}$, then the solution to Fermat's problem is an inner point of the triangular domain bounded by the sides of the triangle. To find this point, one approach is to construct equilateral triangles on each side of the triangle (it is enough to draw two of them) and draw the segments connecting the opposite vertices of the original triangle and the newly created equilateral vertices. They meet at a point which is the solution to Fermat's problem. If P is the solution to Fermat's problem for ABC, where ABC is an arbitrary triangle that the measure of the largest angle is less than $\frac{2\pi}{3}$, then $\angle APB = \angle BPC = \angle CPA = \frac{2\pi}{3}$. This feature of the Fermat-Torricelli points will play an important role in our proofs.

2. Main results

Let Δ be the set of all triple points $\{A, B, C\}$ in $\mathbb{R}^2$ such that the largest angle of the triangle ABC is less than $\frac{2\pi}{3}$. Throughout the paper, we denote by A' the image of A under f, by $[A, B]$ the line segment between points A and B, and by AB the line through points A and B, by $|AB|$ the Euclidean distance between A and B.

The assertion $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, meaning that if $\{A, B, C\}$ is an element of Δ with the Fermat-Torricelli point P (i.e. P is the Fermat-Torricelli point of the triangle ABC), then $\{A', B', C'\}$ is an element of Δ with the Fermat-Torricelli point P' (i.e. P' is the Fermat-Torricelli point of the triangle $A'B'C'$).

Lemma 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f is injective.*

Proof. Let A and B be two distinct points in $\mathbb{R}^2$. Now take a point in $\mathbb{R}^2$, say C, such that $ABC \in \Delta$, and denote it's Fermat-Torricelli point by P. By the property of f, we get that P' is the Fermat-Torricelli point of $A'B'C'$, which implies $A' \neq B'$. Hence f is injective. □

Lemma 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserve the Fermat-Torricelli points of all triangles in Δ. If ABC is a triangle with $\angle CAB = \frac{2\pi}{3}$, then $A'B'C'$ is a triangle with $\angle C'A'B' = \frac{2\pi}{3}$.*

Proof. Let g_{AB} be the symmetry function with respect to AB and let us denote the image of C by D under g_{AB}. Clearly, CBD is an isosceles triangle with $\angle DAC = \angle CAB = \angle DAB = \frac{2\pi}{3}$, and this implies $\angle ABC = \angle ABD < \frac{\pi}{3}$, $\angle ACD = \angle ADC < \frac{\pi}{3}$. Moreover, A is the Fermat-Torricelli point of CBD. It is easy to see that BCD is an element of Δ and since f preserves the Fermat-Torricelli points of all triangles in Δ, one can get that A' is the Fermat-Torricelli point of $C'B'D'$. Therefore, $A'B'C'$ is a triangle with $\angle C'A'B' = \frac{2\pi}{3}$. □

Lemma 3. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves collinearity.*

Proof. Let G, C, D be collinear points in $\mathbb{R}^2$ satisfying $|GD| = |GC| + |CD|$, and let A be a point in $\mathbb{R}^2$ such that $\angle AGC = \frac{2\pi}{3}$ and $|AG| = |GC|$. Now construct a triangle ABC, where B is the symmetry of A with respect to GC. Clearly, ABC is an equilateral triangle in Δ. By *Lemma 2*, we have $\angle A'G'C' = \angle A'G'B' = \angle B'G'C' = \frac{2\pi}{3}$ since $\angle AGC = \angle AGB = \angle BGC = \frac{2\pi}{3}$. Observing $\angle AGD = \angle BGD = \frac{2\pi}{3}$, we get $\angle A'G'D' = \angle B'G'D' = \frac{2\pi}{3}$. One can easily see that D' must lie on either $G'C'$ or $G'B'$ since there are two lines ($G'C'$ and $G'B'$) in $\mathbb{R}^2$ such that the measure of the angle between $A'G'$ is $\frac{2\pi}{3}$. Since BGD is a triangle with $\angle BGD = \frac{2\pi}{3}$, by *Lemma 2*, we obtain that $B'G'D'$ is a triangle with $\angle B'G'D' = \frac{2\pi}{3}$. Thus the points D', G', B' are definitely not on the same line, and this implies that D' must lie on $G'C'$. $\square$

Lemma 4. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves betweenness, that is, if A, C, E are three distinct points in $\mathbb{R}^2$ with $|AE| = |AC| + |CE|$, then $|A'E'| = |A'C'| + |C'E'|$ holds.*

Proof. Let A, C, E be three distinct points in $\mathbb{R}^2$ with $|AE| = |AC| + |CE|$. Let us assume $|A'C'| = |A'E'| + |E'C'|$. Now take a point, say D, in $\mathbb{R}^2$ such that $|AC| = |CD|$ and $\angle ACD = \frac{2\pi}{3}$. Let B be the reflection of D with respect to C. By *Lemma 2*, we have $A'C'D' = \angle B'C'E' = \frac{2\pi}{3}$, and by *Lemma 3*, we get that the points B', C', D' are collinear. Now take a point on $[B, E]$, say Y ($B \neq Y \neq E$), such that $\angle ACY = \frac{2\pi}{3}$. Since B, Y, E are collinear points, thus we get that the points B', Y', E' are collinear by *Lemma 3*. Moreover, $\angle A'C'Y' = \frac{2\pi}{3}$ holds by *Lemma 2*. This implies $Y' = B'$, which contradicts injectivity of f. Hence we get $|A'E'| = |A'C'| + |C'E'|$. $\square$

Lemma 5. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the equilateral triangles.*

Proof. Let ABC be an equilateral triangle in $\mathbb{R}^2$ and K a point on AC with $|CK| = |AC| + |AK|$. By *Lemma 2*, *Lemma 3* and *Lemma 4*, we get that K', A', C' are collinear points with $\angle B'A'K' = \frac{2\pi}{3}$ and $|C'K'| = |A'C'| + |A'K'|$. Clearly, we obtain $\angle B'A'C' = \frac{\pi}{3}$. Following the same way, one can easily prove that $\angle A'B'C' = \angle B'C'A' = \frac{\pi}{3}$, which implies that $A'B'C'$ is an equilateral triangle. $\square$

Lemma 6. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the midpoints.*

Proof. Let A and B be two distinct points in $\mathbb{R}^2$ and denote the midpoint of $[A, B]$ by M. Now construct a regular hexagon $A_1 \cdots A_6$, whose centroid is M with $A_1 = A$ and $A_4 = B$. Since f preserves the equilateral triangles by *Lemma 5*, we get that $A_1' \cdots A_6'$ is also a regular hexagon whose centroid is M'. Thus we get that M' is the midpoint of $[A', B']$. $\square$

Lemma 7. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the isosceles triangles of Δ.*

Proof. Let ABC be an isosceles triangle in Δ with $|AB| = |AC|$. Obviously, there exist two points in $\mathbb{R}^2$ that form an equilateral triangle with B, C. Let D be the closest of these two points to A. Let us denote the centroid of BCD by M and the midpoint of $[B, C]$ by E. Since M is the Fermat-Torricelli point of BCD, observing $|AB| = |AC|$, M must be the Fermat-Torricelli point of ABC. Moreover, by *Lemma 3*, the points A', D', E' are collinear since the points A, D, E are collinear. By *Lemma 5*, $B'C'D'$ is an equilateral triangle. Since M' is the Fermat-Torricelli point of $B'C'D'$ and $A'B'C'$, it follows that $A'B'C'$ must be an isosceles triangle in Δ with $|A'B'| = |A'C'|$. $\square$

Lemma 8. *If $f:\mathbb{R}^2\to\mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f is continuous.*

Proof. Let X be a point in $\mathbb{R}^2$ and denote the open ball and the circle centered at X with radius ϵ by $B(X,\epsilon)$ and $C(X,\epsilon)$, respectively. We want to prove that f is continuous at X. It is well known that the sets $B_X=\{B(X,\epsilon):\epsilon\in\mathbb{R}^+\}$ and $B_{X'}=\{B(X',\sigma):\sigma\in\mathbb{R}^+\}$ are local bases of X and X', respectively. Let us consider the open ball $B(X',\rho)$. Let A and B be two distinct points in $\mathbb{R}^2$ such that AXB is an equilateral triangle and assume $|AB|=r$. Since all point pairs that form an equilateral triangle with A such that one side length is equal to r must lie on $C(X,r)$, it follows that A and B are two points in $C(X,r)$. Assume $|A'B'|=r'$. By *Lemma 5* and *Lemma 7*, we get $f(C(X,r))\subset C(X',r')$. Moreover, by *Lemma 6*, we get that $f(C(X,\frac{r}{2^n}))\subset C(X',\frac{r'}{2^n})$ for all $n\in\mathbb{N}$. If $r'\le\rho$, this implies that $f(B(X,r))\subset B(X',\rho)$. If $r'>\rho$ holds, then there exists a positive integer k such that $\frac{r'}{2^k}<\rho$. Hence we get that $f(B(X,\frac{r}{2^k}))\subset f(B(X',\frac{r'}{2^k}))\subset B(X',\rho)$. Therefore, we obtain that f is continuous at X, so it is continuous everywhere. □

Lemma 9. *If $f:\mathbb{R}^2\to\mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the angles of triangles in Δ.*

Proof. Let X be a point in $\mathbb{R}^2$ and construct a sequence, say XA_iA_{i+1}, consisting of isosceles triangles taken from Δ such that $\angle A_iXA_{i+1}=\frac{\pi}{k}$, where k is an integer with $k>1$, and $|XA_i|=|XA_{i+1}|$ for all i with $1\le i\le 2k-1$. Clearly, $A_1A_2\cdots A_{2k}$ is a $2k$-sided regular polygon with centroid X, that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). We claim that $A'_1A'_2\cdots A'_{2k}$ is also a $2k$-sided regular polygon with centroid X'. Firstly, by *Lemma 7*, the triangles $X'A'_iA'_{i+1}$ $(1\le i\le 2k-1)$ and $X'A'_{2k}A'_1$ are isosceles with $|X'A'_i|=|X'A'_{i+1}|$ since the triangles XA_iA_{i+1} $(1\le i\le 2k-1)$ and $XA_{2k}A_1$ are isosceles with $|XA_i|=|XA_{i+1}|$. Using again *Lemma 7*, since $A_iA_{i+1}A_{i+2}$ $(1\le i\le 2k-2)$, $A_{2k-1}A_{2k}A_1$ and $A_{2k}A_1A_2$ are isosceles triangles with $|A_iA_{i+1}|=|A_{i+1}A_{i+2}|$, $|A_{2k-1}A_{2k}|=|A_{2k}A_1|$, $|A_{2k}A_1|=|A_1A_2|$, we get that the image triangles $A'_iA'_{i+1}A'_{i+2}$ $(1\le i\le 2k-2)$, $A'_{2k-1}A'_{2k}A'_1$ and $A'_{2k}A'_1A'_2$ are isosceles with $|A'_iA'_{i+1}|=|A'_{i+1}A'_{i+2}|$, $|A'_{2k-1}A'_{2k}|=|A'_{2k}A'_1|$, $|A'_{2k}A'_1|=|A'_1A'_2|$. Hence $A'_1A'_2\cdots A'_{2k}$ is an $2k$-sided equilateral polygon. Moreover, observing $|X'A'_i|=|X'A'_{i+1}|$ $(1\le i\le 2k-1)$, one can easily see that the triangles $X'A'_iA'_{i+1}$ $(1\le i\le 2k-1)$ and $X'A'_{2k}A'_1$ are congruent by *side-side-side* theorem. Therefore, we get that $A'_1A'_2\cdots A'_{2k}$ is a $2k$-sided equiangular polygon. Thus $A'_1A'_2\cdots A'_{2k}$ is a $2k$-sided regular polygon with centroid X' and it is clear that $\angle A'_iX'A'_{i+1}=\frac{\pi}{k}$ for all i with $1\le i\le 2k-1$. Hence f preserves $\frac{n\pi}{k}$-valued angles at the vertex X, where k,n are integers. As f is continuous by *Lemma 8*, and the set of rational numbers is dense in $\mathbb{R}$, it follows that f preserves all angles at the vertex X, and this finishes the proof. □

Corollary 1. *If $f:\mathbb{R}^2\to\mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the circles.*

Lemma 10. *If $f:\mathbb{R}^2\to\mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ, then f preserves the lines. More precisely, if l is a Euclidean line in $\mathbb{R}^2$, then $f(l)$ is a Euclidean line in $\mathbb{R}^2$.*

Proof. It is clear from *Lemma 3* and *Lemma 4* that, for an arbitrary line AB defined from two arbitrary points A, B in $\mathbb{R}^2$, $f(AB) \subset A'B'$ holds. Let S be a point on $A'B'$, and try to find a point on AB, say X, such that $f(X) = S$. Assume $|A'B'| + |B'S| = |A'S|$. We can find a point, say C', that provides $|A'S| + |SC'| = |A'C'|$ by getting enough symmetries of points A and B relative to each other. Now construct an equilateral triangle whose two vertices are A and C, and denote this triangle by ACD. Then, by *Lemma 5*, we get $A'C'D'$ is an equilateral triangle. Clearly, at least one of the triangles $D'A'S$ and $D'C'S$ must be in Δ. Without loss of generality, we may assume $D'A'S \in \Delta$. If $\angle D'SA' := \alpha$, then by observing ACD and $A'C'D'$ are equilateral triangles, there exists a point on AB, say X, such that $\angle DXA = \alpha$. By *Lemma 9*, we get $f(X) = S$, which finishes the proof. □

From the results we have obtained so far, the function f is non-degenerate and preserves the lines, so we can give our main theorem by *Theorem 4* as follows:

Theorem 7. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the Fermat-Torricelli points of all triangles in Δ. Then f is an affine transformation.*

Naturally, one may wonder whether *Theorem 7* holds for the mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserve the Fermat-Torricelli points of all triangles in Δ? Here Δ is defined in the same sense above. The answer to this question is "yes". Indeed, in Euclidean space $\mathbb{R}^n$, the Fermat-Torricelli point of a triangle ABC in Δ is a point in the plane containing ABC. Let us denote this plane by Ω. For each point X in $\mathbb{R}^n \setminus \Omega$, it is clear that

$$|\psi(X)A| + |\psi(X)B| + |\psi(X)C| < |XA| + |XB| + |XC|$$

holds, where $\psi(X)$ is the orthogonal projection of X on Ω. Therefore, if M is the Fermat-Torricelli point of ABC in Ω, then M is also the Fermat-Torricelli point of ABC in $\mathbb{R}^n$. This ensures that all lemmas we have proved above hold here as well. Thus, we can extend *Theorem 7* to n-dimensional space $\mathbb{R}^n$ without proof.

Theorem 8. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the Fermat-Torricelli points of all triangles in Δ. Then f is an affine transformation.*

Remark 1. *By Theorem 7 and Lemma 9 or Corollary 1, f can be expressed by*

$$f(x) = ag(x) + b,$$

where $g(x)$ is an isometry (or an orthogonal transformation) and $a \in \mathbb{R}$ $(a \neq 0)$, $b \in \mathbb{R}^n$.

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