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Smooth cohomology of C^*-algebras*

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Abstract. We define a notion of smooth cohomology for C^*-algebras which admit a faithful trace. We show that if $\mathcal{A} \subseteq B(H)$ is a C^*-algebra with a faithful normal trace τ on the ultra-weak closure $\bar{\mathcal{A}}$ of $\mathcal{A}$, and X is a normal dual operatorial $\bar{\mathcal{A}}$-bimodule, then the first smooth cohomology $\mathcal{H}^1_s(\mathcal{A}, X)$ of $\mathcal{A}$ is equal to $\mathcal{H}^1(\mathcal{A}, X_\tau)$, where X_τ is a closed submodule of X consisting of smooth elements.
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1. Introduction

Hochschild cohomology is an important invariant for Banach and operator algebras. A study of cohomology in the algebraic setting was initiated by Hochschild (1945-47) [14, 15, 16]. After Kaplansky (1953), we know that various questions about the properties of derivations on C^*-algebras and von Neumann algebras could be translated into certain cohomology groups equal to each other (or to zero). George Elliott used this along with K-theory groups in the classification of separable AF C^*-algebras [11]. Also, Alain Connes and Uffe Haagerup characterized injectivity and hyperfiniteness of von Neumann algebras by the vanishing of its cohomology group over all dual normal modules [7, 8, 9, 13]. Another example is the proof of equivalence of amenability and nuclearity for C^*-algebras by Alain Connes (1978) (amenable $\Rightarrow$ nuclear) and Uffe Haagerup (1983) (nuclear $\Rightarrow$ amenable).

The study of Hochschild cohomology theory for von Neumann algebras was initiated in the early 1970s in the pioneering work of Johnson, Kadison, and Ringrose [19, 20, 17]. Since then, the theory has seen significant progress and, while nowhere near completion, it is reasonable to say that it has reached maturity. In the case of a von Neumann algebra $\mathcal{M}$, since $\mathcal{M}$ is the dual of $\mathcal{M}_*$, the wealth of topological and measure theoretical properties of $\mathcal{M}$ has led, from the beginning, to the additional assumption that cocycles and coboundaries are normal (i.e. separately ultraweakly continuous in each variable). This, in turn, required that cohomology has coefficients in a dual $\mathcal{M}$-module X. The cohomology groups under these circumstances

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are denoted by $\mathcal{H}^n_w(\mathcal{M}, X)$, and it was proved that $\mathcal{H}^n_w(\mathcal{M}, X) = \mathcal{H}^n(\mathcal{M}, X)$, for all von Neumann algebras $\mathcal{M}$ and all dual $\mathcal{M}$-modules X [17]. The most relevant cases turned out to be $X = \mathcal{M}$ and $X = B(H)$, where in the latter case, $\mathcal{M}$ is faithfully represented on H, and so $B(H)$ is canonically an $\mathcal{M}$-bimodule and the cohomology groups $\mathcal{H}^n(\mathcal{M}, B(H))$ are defined. The Johnson-Kadison-Ringrose conjecture (see [19, 20, 17]), stating that $\mathcal{H}^n(\mathcal{M}, \mathcal{M}) = 0$ for $n > 0$, has been verified for large classes of von Neumann algebras. We mention here the types I [19], II_∞ and III [4], as well as several classes of type II_1 factors, such as those with property Γ [5] and those with Cartan subalgebras [22, 5, 26]. With the exception of the Cartan case, $\mathcal{H}^n(\mathcal{M}, B(H)) = 0$ is also known for the same classes in the corresponding papers just cited. Historically, the vanishing of the first cohomology for $X = \mathcal{M}$ was the first to be settled [18, 24], and now it has fairly short proofs. By contrast, for $X = B(H)$, this is still open and, as proved by Kirchberg [21], it is equivalent to Kadison's similarity problem, asking if every bounded representation of a C^*-algebra is similar to a $*$-representation.

Following R. V. Kadison [18], S. Sakai (1966) showed that every derivation $\delta : \mathcal{M} \to \mathcal{M}$ on a von Neumann algebra $\mathcal{M}$ is inner, which is equivalent to the vanishing of the first continuous cohomology group $\mathcal{H}^1(\mathcal{M}, \mathcal{M})$ [24]. B. E. Johnson, R.V. Kadison and J. R. Ringrose (1972) showed that if $\mathcal{M}$ is hyperfinite and X is an arbitrary dual normal $\mathcal{M}$-bimodule, then $\mathcal{H}^n(\mathcal{M}, X) = 0$ for all $n > 0$ [17]. Later, E. Christensen, E. G. Effros and A. M. Sinclair (1987) used the notion of complete bounded maps and applied operator space techniques to cohomology of operator algebras. This method worked perfectly for von Neumann algebras of types $\mathrm{I}, \mathrm{II}_\infty$ and III. Type I can be handled by hyperfiniteness results, while types II_∞ and III are stable under tensoring with $B(H)$, which is enough to obtain complete boundedness of cocycles. However, not all type II_1 factors have this property. Some partial results for II_1 algebras were obtained by F. Pop and R. R. Smith (1994) [22]. For example, if $\mathcal{M}$ is a separably acting type II_1 von Neumann algebra with a Cartan subalgebra, then $\mathcal{H}^n(\mathcal{M}, \mathcal{M}) = 0$ for all $n > 0$. The case of a II_1 factor was studied by S. Popa and S. Vaes (2014) (for the continuous L^2-cohomology) [23], and A. Galatan and Popa (2017) (for factors with some additional conditions) [12].

In the latter paper, the authors related the so-called smooth cohomology of a von Neumann algebra with coefficients in a Banach module X and the ordinary cohomology with coefficients in the smooth part of X (which is a closed submodule of X), and showed that for factors, each derivation with values in the smooth part is inner. The main objective of this paper is to handle the same correspondence for C^*-algebras. Following [12], we define a notion of smooth cohomology for a C^*-algebra $\mathcal{A}$ with a faithful trace. The main result of the paper asserts that smooth cohomology of $\mathcal{A}$ with coefficients in X and Hochschild cohomology of $\mathcal{A}$ with coefficients in the smooth part of X are the same. In order to do this, we show that the smooth weak continuous cocycles on $\mathcal{A}$ can be extended to its ultra-weak closure $\bar{\mathcal{A}}$, without changing the cohomology groups. The precise statement is as follows:

Theorem 1. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on the ultra-weak closure $\bar{\mathcal{A}}$ of $\mathcal{A}$, and let X be a normal dual $\bar{\mathcal{A}}$-bimodule. Then, for every $n \in \mathbb{N}$ we have*

$$\mathcal{H}^n_{sw}(\mathcal{A}, X) = \mathcal{H}^n_{sw}(\bar{\mathcal{A}}, X).$$

The key point here is that every smooth map on $\mathcal{A}$ can be extended to $\bar{\mathcal{A}}$. This will be checked in Lemma 2. Then, using Proposition 1 and Lemma 3, we show that smooth normal cohomology of $\mathcal{A}$ coincides with smooth cohomology of $\mathcal{A}$:

Theorem 2. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on $\bar{\mathcal{A}}$ and let X be a normal dual $\bar{\mathcal{A}}$-bimodule. Then, for every $n \in \mathbb{N}$ we have*

$$\mathcal{H}^n_{sw}(\mathcal{A}, X) = \mathcal{H}^n_s(\mathcal{A}, X).$$

This is done by the averaging techniques described in [12, Section 3]. This technique, effectively used here, amounts to an integration over the compact unitary group of a finite dimensional C^*-algebra. Taking suitable weak limits of an increasing sequence of finite dimensional algebras, the averaging in all levels leads to certain averages over infinite dimensional algebras. This method is described in an abstract setting by Johnson, Kadison and Ringrose in [17].

Combining Theorem 1 with Theorem 2, we deduce the following equality (Corollary 1):

$$\mathcal{H}^n_s(\mathcal{A}, X) = \mathcal{H}^n_{sw}(\bar{\mathcal{A}}, X).$$

In the case when X is a normal dual operatorial $\bar{\mathcal{A}}$-bimodule (in the sense of [12]), we get the main result of the paper:

Theorem 3. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace τ on $\bar{\mathcal{A}}$ and the ultra-weak closure of $\mathcal{A}$, and let X be a normal dual operatorial $\bar{\mathcal{A}}$-bimodule. Then, $\mathcal{H}^1_s(\mathcal{A}, X) = \mathcal{H}^1(\mathcal{A}, X_\tau)$.*

An example of a normal dual operatorial $\bar{\mathcal{A}}$-bimodule is $B(H)$, the space of all bounded linear operators on a Hilbert space H on which $\mathcal{A}$ is represented. The smooth part of this module is a hereditary C^*-subalgebra of $B(H)$ that contains the space of compact operators $K(H)$ and a large variety of non-compact smooth elements in general [12].

2. Preliminaries

Throughout the paper, $\mathcal{A}_1$ denotes the closed unit ball of a C^*-algebra $\mathcal{A}$. Also, the weak, strong and ultra-weak operator topology on $B(H)$ are denoted by WOT, SOT and UWOT, respectively. .

Let $\mathcal{A}$ be a unital C^*-algebra. A positive linear functional τ on $\mathcal{A}$ is called tracial (or a finite trace) if $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. A trace on $\mathcal{A}$ is called faithful if $a = 0$, whenever $\tau(a^*a) = 0$ for every $a \in \mathcal{A}$. Each faithful trace on $\mathcal{A}$ induces a norm $\|.\|_\tau$ on $\mathcal{A}$ defined by $\|a\|^2_\tau = \tau(a^*a)$, $(a \in \mathcal{A})$.

Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ, and let B be a Banach space. An linear map $T : \mathcal{A} \to B$ is called *smooth* if it is continuous relative to the $\|.\|_\tau$-topology on $\mathcal{A}_1$ and the norm topology on B. A multi-linear map is smooth if it is smooth, separately in each argument.

Let X be a Banach $\mathcal{A}$-bimodule. An element $x \in X$ is called *smooth* if the module maps $\mathcal{A} \to X; a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are smooth. We denoted by X_τ the closed submodule of all smooth elements in X. If $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, then we have $X^{\mathcal{A}}_\tau \subseteq X^{\mathcal{B}}_\tau$ [12].

The Banach $\mathcal{A}$-bimodule X is said to be *dual* if it is the dual of a Banach space and for each $a \in \mathcal{A}$, the maps $X \to X; x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are weak* continuous. If in addition, $\mathcal{A}$ admits a weak* topology (for example, whenever $\mathcal{A}$ is a von Neumann algebra), and for every $x \in X$ the maps $\mathcal{A} \to X; a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weak* continuous, then X is said to be *normal*.

We put $BL^0(\mathcal{A}, X) = X$, and for each $n \in \mathbb{N}$, we denote by $BL^n(\mathcal{A}, X)$ the space of all bounded n-linear maps from $\mathcal{A}^n$ into X. The subscripts "s" and "sw" mean that the maps are smooth and separately UWOT-continuous, respectively. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. An element T of $BL^n(\mathcal{A}, X)$ is called $\mathcal{B}$-*modular* if for each $a_1, \ldots, a_n \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$\begin{aligned} b \cdot T(a_1, \ldots, a_n) &= T(ba_1, \ldots, a_n), \\ T(a_1, \ldots, a_j b, a_{j+1}, \ldots, a_n) &= T(a_1, \ldots, a_j, ba_{j+1}, \ldots, a_n), \\ T(a_1, \ldots, a_n b) &= T(ba_1, \ldots, a_n)b. \end{aligned}$$

The space of all $\mathcal{B}$-modular maps is denoted by $BL^n(\mathcal{A}, X : \mathcal{B})$.

For each $n > 0$, the coboundary operators

$$\delta^n : BL^n(\mathcal{A}, X) \to BL^{n+1}(\mathcal{A}, X)$$

are defined by

$$\begin{aligned} (\delta^n T)(a_1, \ldots, a_{n+1}) := {} & a_1 \cdot T(a_2, \ldots, a_{n+1}) \\ & + \sum_{k=1}^{n} (-1)^k T(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}, \quad (a_1, \ldots, a_{n+1} \in \mathcal{A}), \end{aligned}$$

and $\delta^0 : X \to BL(\mathcal{A}, X)$ is defined by $\delta^0(x)(a) = a \cdot x - x \cdot a$. We have the cochain complex

$$\{0\} \to X \xrightarrow{\delta^0} BL(\mathcal{A}, X) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} BL^n(\mathcal{A}, X) \xrightarrow{\delta^n} BL^{n+1}(\mathcal{A}, X) \xrightarrow{\delta^{n+1}} \cdots$$

called the Hochschild cochain complex.

Letting $\mathcal{Z}^n(\mathcal{A}, X) = \ker \delta^n$ and $\mathcal{B}^n(\mathcal{A}, X) = \operatorname{ran} \delta^n$, we have the quotient linear space

$$\mathcal{H}^n(\mathcal{A}, X) := \mathcal{Z}^n(\mathcal{A}, X)/\mathcal{B}^n(\mathcal{A}, X), \ \mathcal{H}^0(\mathcal{A}, X) = \{x \in X : a \cdot x = x \cdot a (a \in \mathcal{A})\}$$

called the n-th Hochschild cohomology of $\mathcal{A}$ with coefficients in X.

Following [12] and [25], we may use the subscripts "s" and "sw" in $BL^n_s(\mathcal{A}, X)$ and $BL^n_{sw}(\mathcal{A}, X))$. For example, $\mathcal{Z}^1_s(\mathcal{A}, X)$ is the space of smooth derivations on $\mathcal{A}$ to X and $\mathcal{B}^1_s(\mathcal{A}, X)$ is the space of inner derivations that is implemented by a smooth element of X.

3. Smooth cohomology

In this section, we explore the relation between $\mathcal{H}^1_s(\mathcal{A}, X)$ and $\mathcal{H}^1(\mathcal{A}, X_\tau)$. Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on $\mathcal{A}''$. By [2, Theorem

1.2.4], on a bounded ball of $\mathcal{A}$, the WOT, SOT and UWOT agree. Also, the $\|.\|_\tau$-topology agrees with SOT (and also with UWOT) on any bounded subset of $\mathcal{A}$ by [3, III. 2.2.17]. In particular, a bounded net $(a_i) \subseteq \mathcal{A}$ converges to zero strongly if and only if $\|a_i\|_\tau \to 0$. We use these facts several times. The results of this section adapt ideas and techniques from [12].

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-subalgebras of $B(H)$ and let τ be a faithful normal trace on the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$. Let $\varphi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded bilinear smooth form, which is separately UWOT-continuous. Then φ extends uniquely to a separately UWOT-continuous, smooth bilinear form $\bar{\varphi} : \bar{\mathcal{A}} \times \mathcal{B} \to \mathbb{C}$, where $\bar{\mathcal{A}}$ is the UWOT-closure of $\mathcal{A}$.*

Proof. For a fixed $b \in \mathcal{B}$, the bounded linear functional $\varphi_b(a) := \varphi(a, b)$ is smooth and UWOT-continuous, so it extends to a UWOT-continuous linear functional $\widetilde{\varphi_b} : \bar{\mathcal{A}} \to \mathbb{C}$. The Kaplansky density theorem implies that $\|\widetilde{\varphi_b}\| = \|\varphi_b\|$. Hence, the map $\widetilde{\varphi} : \mathcal{B} \to (\bar{\mathcal{A}})_*;\ b \mapsto \widetilde{\varphi_b}$ is linear and bounded with $\|\widetilde{\varphi}\| \le \|\varphi\|$. Since φ is UWOT-continuous in the second argument, $\widetilde{\varphi}$ is continuous in UWOT on $\mathcal{B}$ and in $\sigma((\bar{\mathcal{A}})_*, \mathcal{A})$ on $(\bar{\mathcal{A}})_*$. By [27, Theorem 5.4] or [1, Corollary II.9], $\widetilde{\varphi}(\mathcal{B}_1)$ is relatively $\sigma((\bar{\mathcal{A}})_*, \bar{\mathcal{A}})$-compact in $(\bar{\mathcal{A}})_*$, hence $\sigma((\bar{\mathcal{A}})_*, \bar{\mathcal{A}})$ coincides with the coarser topology $\sigma((\bar{\mathcal{A}})_*, \mathcal{A})$. Combining this with the continuity of $\widetilde{\varphi}$ yields that $\widetilde{\varphi}$ is continuous on $\mathcal{B}_1$ in UWOT into $(\bar{\mathcal{A}})_*$ in $\sigma((\bar{\mathcal{A}})_*, \bar{\mathcal{A}})$. Thus, for each fixed $a \in \bar{\mathcal{A}}$, the linear functional $b \mapsto \widetilde{\varphi_b}(a)$ is UWOT-continuous on $\mathcal{B}_1$ and hence on $\mathcal{B}$. Now the bounded bilinear form $\bar{\varphi} : \bar{\mathcal{A}} \times \mathcal{B} \to \mathbb{C}$ defined by $\bar{\varphi}(a, b) = \widetilde{\varphi_b}(a)$ is separately UWOT-continuous. It remains to show that $\bar{\varphi}$ is smooth. The $\|.\|_\tau$-continuity of $\bar{\varphi}$ on $\mathcal{B}_1$ follows from the continuity of φ. For the first argument of $\bar{\varphi}$, it is enough to show that $\widetilde{\varphi_b} : \bar{\mathcal{A}} \to \mathbb{C}$ is smooth. Since $\widetilde{\varphi_b}$ is UWOT-continuous on $(\bar{\mathcal{A}})_1$, it is also $\|.\|_\tau$-continuous. □

Lemma 2. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on $\mathcal{A}''$ and let X be a dual module with predual X_*. If $\varphi : \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A} \to X$ is a bounded n-linear smooth map which is separately UWOT-weak*-continuous, then it extends uniquely (without changing a norm) to a separately UWOT-continuous, smooth n-linear map $\bar{\varphi} : \bar{\mathcal{A}} \times \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}} \to X$.*

Proof. We give the proof in two cases:

Case 1. Let $X = \mathbb{C}$. We will construct a finite sequence $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n$ of bounded n-linear functionals with the following properties:

(i) $\varphi_k : \underbrace{\bar{\mathcal{A}} \times \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}}}_{k-times} \times \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$,

(ii) φ_k extends φ_{k-1} without changing a norm,

(iii) φ_k is separately UWOT-continuous,

(iv) φ_k is a smooth map.

This proves the existence of $\bar{\varphi} = \varphi_n$. The uniqueness of $\bar{\varphi}$ follows from the fact that φ is separately UWOT-continuous and $\mathcal{A}$ is UWOT-dense in $\bar{\mathcal{A}}$.

For $1 \leq k \leq n$, suppose that $\varphi_0, \varphi_1, \dots, \varphi_{k-1}$ have been constructed. For $j \neq k$, let $a_j \in \mathcal{A}$ be fixed. The linear functional

$$f_k : \mathcal{A} \to \mathbb{C}; \qquad a \mapsto \varphi_{k-1}(a_1, \dots, a_{k-1}, a, a_{k+1}, \dots, a_n)$$

is UWOT-continuous (and so $\|.\|_\tau$-continuous on $\mathcal{A}_1$) and

$$\|f_k\| \leq \max\{\|\varphi_{k-1}\|, \|a_1\|, \dots, \|a_{k-1}\|, \|a_{k+1}\|, \dots, \|a_n\|\}.$$

By the Kaplansky density theorem, f_k extends without changing a norm to a UWOT-continuous, smooth functional $\bar{f}_k$ on $\bar{\mathcal{A}}$.

Now we define $\varphi_k(a_1, \dots, a_k, \dots, a_n) = \bar{f}_k(a_k)$. Clearly, φ_k is a bounded n-linear form on $\underbrace{\bar{\mathcal{A}} \times \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}}}_{k-times} \times \mathcal{A} \times \cdots \times \mathcal{A}$ that extends φ_{k-1} without changing a norm and it is UWOT-continuous and smooth in its first k^{th} argument. We will show that φ_k is UWOT-continuous and smooth in its other arguments for $a_k \in \bar{\mathcal{A}} \setminus \mathcal{A}$. Let $1 \leq j \leq n$ with $j \neq k$ and fix a_i for all $i \neq j, k$ with $a_i \in \bar{\mathcal{A}}$ for $i < k$ or $a_i \in \mathcal{A}$ for $i > k$. Let $\mathcal{B} = \bar{\mathcal{A}}$ if $j < k$ and $\mathcal{B} = \mathcal{A}$ if $j > k$. Let $\psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be the bounded bilinear form defined by $\psi(a_k, a_j) = \varphi_{k-1}(a_1, \dots, a_n) = \varphi_k(a_1, \dots, a_n)$. By assuming φ_{k-1}, ψ is a separately UWOT-continuous, smooth form so, by Lemma 1 it extends uniquely to a bounded bilinear smooth form $\bar{\psi} : \bar{\mathcal{A}} \times \mathcal{B} \to \mathbb{C}$, which is separately UWOT-continuous. Since both $\bar{\psi}(a_k, a_j)$ and $\varphi_k(a_1, \dots, a_n)$ are UWOT-continuous in the variable $a_k \in \bar{\mathcal{A}}$ and they agree on $\mathcal{A}$, it follows that $\bar{\psi}(a_k, a_j) = \varphi_k(a_1, \dots, a_n)$ on $\bar{\mathcal{A}} \times \mathcal{B}$. This shows that for each $a_k \in \bar{\mathcal{A}}$, the map φ_k is UWOT-continuous and smooth in $a_j \in \mathcal{B}$, because $\bar{\psi}$ has these properties.

Case 2. Let X be arbitrary. For each $\xi \in X_*$, the bounded n-linear form

$$\rho_\xi : \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}; \qquad (a_1, \dots, a_n) \mapsto \langle \varphi(a_1, \dots, a_n), \xi \rangle$$

is smooth and separately UWOT-continuous. Thus, by case 1, it extends uniquely (without changing a norm) to a separately UWOT-continuous, smooth n-linear form $\bar{\rho}_\xi$ on $\bar{\mathcal{A}} \times \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}}$. Hence, for every $a_1, \dots, a_n \in \bar{\mathcal{A}}$, the map $\xi \mapsto \bar{\rho}_\xi(a_1, \dots, a_n)$ is a bounded linear functional on X_* and so it belongs to $X = (X_*)^*$. This defines a map $\bar{\varphi}$ satisfying $\|\bar{\varphi}\| = \|\varphi\|$.
The smoothness and UWOT-continuity of $\bar{\varphi}$ follow from the smoothness and UWOT-continuity of $\bar{\rho}_\xi$. □

Proof of Theorem 1. It is immediate by Lemma 2, because the restriction map $\mathcal{H}^n_{sw}(\bar{\mathcal{A}}, X) \to \mathcal{H}^n_{sw}(\mathcal{A}, X)$ is an isomorphism. □

Remark 1. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on the UWOT-closure $\bar{\mathcal{A}}$ of $\mathcal{A}$, and let X be a normal dual $\bar{\mathcal{A}}$-bimodule. If π is the universal representation of $\mathcal{A}$, then it is well known [10] that there is a projection p in the center of the UWOT-closure $\overline{\pi(\mathcal{A})}$ of $\pi(\mathcal{A})$ and an isomorphism $\theta : p\overline{\pi(\mathcal{A})} \to \bar{\mathcal{A}}$ such that*

$$\theta(p\pi(a)) = a \quad \textit{and} \quad \theta(pb) = \pi^{-1}(b) \quad (a \in \mathcal{A}, b \in \overline{\pi(\mathcal{A})}). \tag{1}$$

By [3, III. 2.2.12], θ is a homeomorphism in UWOT. Therefore, X may be regarded as a normal dual $\overline{\pi(\mathcal{A})}$-bimodule with the following actions inherited from the actions of $\bar{\mathcal{A}}$ on X,

$$b \cdot x := \theta(pb) \cdot x \quad \text{and} \quad x \cdot b := x \cdot \theta(pb) \quad (x \in X, b \in \overline{\pi(\mathcal{A})}). \tag{2}$$

In this case, every faithful normal trace τ on $\bar{\mathcal{A}}$ induces a faithful normal trace τ' on $\overline{\pi(\mathcal{A})}$ defined by $\tau'(\pi(a)) = \tau\theta(p\pi(a))$, $a \in \mathcal{A}$, such that for each net $(a_i) \subseteq \mathcal{A}_1$, $\|a_i\|_\tau \to 0$ if and only if $\|\pi(a_i)\|_{\tau'} \to 0$.

Proposition 1. *With the assumptions of Remark 1, there are bounded linear maps*

$$\begin{aligned} T_n &: BL^n_s(\mathcal{A}, X) \to BL^n_{sw}(\overline{\pi(\mathcal{A})}, X), \\ S_n &: BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) \to BL^n_{sw}(\bar{\mathcal{A}}, X), \\ W_n &: BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) \to BL^n_s(\mathcal{A}, X), \end{aligned}$$

such that

(i) $\delta^n_{sw} T_n = T_{n+1}\delta^n_s$ and $\delta^n_{sw} S_n = S_{n+1}\delta^n_{sw}$ such that the following internal diagrams are commutative:

$$\begin{array}{ccc} BL^n_s(\mathcal{A}, X) & \xrightarrow{\delta^n_s} & BL^{n+1}_s(\mathcal{A}, X) \\ {\scriptstyle T_n}\downarrow & & \downarrow{\scriptstyle T_{n+1}} \\ BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) & \xrightarrow{\delta^n_{sw}} & BL^{n+1}_{sw}(\overline{\pi(\mathcal{A})}, X) \\ {\scriptstyle S^n}\downarrow & & \downarrow{\scriptstyle S_{n+1}} \\ BL^n_{sw}(\bar{\mathcal{A}}, X) & \xrightarrow{\delta^n_{sw}} & BL^{n+1}_{sw}(\bar{\mathcal{A}}, X). \end{array}$$

(ii) If $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, then T_n maps $\mathcal{B}$-modular maps to $\overline{\pi(\mathcal{B})}$-modular maps and S_n and W_n map $\overline{\pi(\mathcal{B})}$-modular maps to maps.

(iii) The map S_nT_n is a projection from $BL^n_s(\mathcal{A}, X)$ onto $BL^n_{sw}(\mathcal{A}, X)$.

(iv) If $\mathcal{C}$ is the C^*-algebra generated by 1 and p, the minimal projection in $\overline{\pi(\mathcal{B})}$ with $\overline{\pi(\mathcal{B})} \cdot p = \bar{\mathcal{A}}$ discussed in Remark 1, and if $\psi \in BL^n_{sw}(\overline{\pi(\mathcal{A})}, X : \mathcal{C})$, then

$$W_n\psi = S_n\psi \in BL^n_{sw}(\bar{\mathcal{A}}, X).$$

(v) W_nT_n is the identity map on $BL^n_s(\mathcal{A}, X)$.

Proof. For the projection p as in Remark 1, we have $p \cdot x = x \cdot p = x$, for every $x \in X$. Also, for each $b_1, \ldots, b_n \in \pi(\mathcal{A})$ and $\varphi \in BL^n_s(\mathcal{A}, X)$ the equality

$$\varphi_1(b_1, \ldots, b_n) = \varphi(\theta(b_1), \ldots, \theta(b_n)) \tag{3}$$

defines an element $\varphi_1 \in BL_s^n(\pi(\mathcal{A}), X)$. The map φ_1 is smooth because on bounded sets UWOT agrees with $\|.\|_\tau$-topology and θ is a UWOT-continuous homeomorphism. Since π is the universal representation of $\mathcal{A}$, by [27, Theorem 2.4], each continuous linear functional on $\pi(\mathcal{A})$ is UWOT-continuous. Hence, φ_1 is separately UWOT-weak*-continuous, that is, $\varphi_1 \in BL_{sw}^n(\pi(\mathcal{A}), X)$. Therefore by Lemma 2, φ_1 extends uniquely to some $\tilde{\varphi}_1 \in BL_{sw}^n(\overline{\pi(\mathcal{A})}, X)$ without changing a norm. By Remark 1, the map $\tilde{\varphi}_1$ is smooth. Now we define

$$T_n : BL_s^n(\mathcal{A}, X) \to BL_{sw}^n(\overline{\pi(\mathcal{A})}, X)$$

by $T_n\varphi = \tilde{\varphi}_1$. It is easy to see that T_n is an isometry. If $\varphi \in BL_s^n(\mathcal{A}, X)$ and $b_1, \ldots, b_{n+1} \in \pi(\mathcal{A})$, then the definition of T_n combined with the equations (2) and (3) yields

$$\begin{aligned}
\delta_{sw}^n T_n\varphi(b_1, \ldots, b_{n+1}) =& \theta(pb_1) \cdot \varphi(\theta(pb_2), \ldots, \theta(pb_{n+1})) \\
&+ \sum_{j=1}^{n} (-1)^j \varphi(\ldots, \theta(pb_j)\theta(pb_{j+1}), \ldots) \\
&+ (-1)^{n+1} \varphi(\theta(pb_1), \ldots, \theta(pb_n)) \cdot \theta(pb_{n+1}) \\
=& \theta(pb_1) \cdot \varphi(\theta(pb_2), \ldots, \theta(pb_{n+1})) \\
&+ \sum_{j=1}^{n} (-1)^j \varphi(\ldots, \theta(pb_j b_{j+1}), \ldots) \\
&+ (-1)^{n+1} \varphi(\theta(pb_1), \ldots, \theta(pb_n)) \cdot \theta(pb_{n+1}) \\
=& T_{n+1}\delta_s^n \varphi(b_1, \ldots, b_{n+1}).
\end{aligned}$$

We use the fact that p is a central projection. Both maps $\delta_{sw}^n T_n\varphi$ and $T_{n+1}\delta_s^n\varphi$ are separately UWOT-weak*-continuous, hence

$$\delta_{sw}^n T_n\varphi(b_1, \ldots, b_{n+1}) = T_{n+1}\delta_s^n\varphi(b_1, \ldots, b_{n+1}),$$

for every $b_1, \ldots, b_{n+1} \in \overline{\pi(\mathcal{A})}$. Thus $\delta_{sw}^n T_n = T_{n+1}\delta_s^n$.

If $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ and $\varphi \in BL_s^n(\mathcal{A}, X : \mathcal{B})$, then it follows from the equalities $p \cdot x = x \cdot p = x$. For all $x \in X$, that $T_n\varphi \in BL_{sw}^n(\overline{\pi(\mathcal{A})}, X : \overline{\pi(\mathcal{B})})$: for instance, if $a_1, \ldots, a_n \in \mathcal{A}$ with $b_j = \pi(a_j)$ and $b \in \mathcal{B}$, then

$$\begin{aligned}
T_n\varphi(b_1, \ldots, b_j\pi(b), b_{j+1}, \ldots, b_n) =& \varphi(\theta(pb_1), \ldots, \theta(pb_j\pi(b)), \ldots, \theta(pb_n)) \\
=& \varphi(\theta(pb_1), \ldots, a_j b, a_{j+1}, \ldots, \theta(pb_n)) \\
=& \varphi(\theta(pb_1), \ldots, a_j, ba_{j+1}, \ldots, \theta(pb_n)) \\
=& \varphi(\theta(pb_1), \ldots, \theta(pb_j), \theta(p\pi(b)b_{j+1}), \ldots, \theta(pb_n)) \\
=& T_n\varphi(b_1, \ldots, b_j, \pi(b)b_{j+1}, \ldots, b_n).
\end{aligned}$$

By the UWOT-weak*-continuity of the maps involved, the above calculation holds for each $b_j \in \overline{\pi(\mathcal{A})}$. The calculation of other cases is similar.

Next we define the map S_n. For every $\psi \in BL_{sw}^n(\overline{\pi(\mathcal{A})}, X)$, define

$$S_n(\psi)(a_1, \ldots, a_n) = \psi(\theta^{-1}(a_1), \ldots, \theta^{-1}(a_n)) \quad (a_i \in \bar{\mathcal{A}}).$$

Since ψ and θ^{-1} are UWOT-continuous, $S_n\psi$ is normal and Remark 1 implies that it is a smooth map. Hence, S_n maps $BL^n_{sw}(\overline{\pi(\mathcal{A})}, X)$ into $BL^n_{sw}(\bar{\mathcal{A}}, X)$ and $\|S_n\| \leq 1$. By (1), $\theta(p\theta^{-1}(a)) = a$, $\theta^{-1}(a)\cdot x = \theta(p\theta^{-1}(a))\cdot x = a\cdot x$ and $x\cdot\theta^{-1}(a) = x\cdot a$ for all $a \in \mathcal{A}$ and $x \in X$. Hence,

$$\begin{aligned} S_{n+1}\delta^n_{sw}\psi(a_1,\ldots,a_{n+1}) =&\delta^n_{sw}\psi(\theta^{-1}(a_1),\ldots,\theta^{-1}(a_{n+1}))\\ =&a_1\cdot\psi(\theta^{-1}(a_2),\ldots,\theta^{-1}(a_{n+1}))\\ &+\sum_{j=1}^{n}(-1)^j\psi(\theta^{-1}(a_1),\ldots,\theta^{-1}(a_ja_{j+1}),\ldots,\theta^{-1}(a_{n+1}))\\ &+(-1)^{n+1}\psi(\theta^{-1}(a_1),\ldots,\theta^{-1}(a_n))\cdot a_{n+1}\\ =&\delta^n_{sw}S_n\psi(a_1,\ldots,a_{n+1}), \end{aligned}$$

for every $a_1,\ldots,a_{n+1} \in \mathcal{A}$. By the normality of the maps involved, the equality holds on $\bar{\mathcal{A}}$, that is, $\delta^n_{sw}S_n = S_{n+1}\delta^n_{sw}$. Clearly, $S_n\psi$ is a $\bar{\mathcal{A}}$-module map, whenever ψ is a $\overline{\pi(\mathcal{B})}$-module map.

The map $W_n : BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) \to BL^n_s(\mathcal{A}, X)$, defined by $W_n\psi(a_1,\ldots,a_n) = \psi(\pi(a_1),\ldots,\pi(a_n))$ is a continuous linear map with $\|W_n\| \leq 1$. Note that by Remark 1, the smoothness of $\psi \in BL^n_{sw}(\overline{\pi(\mathcal{A})}, X)$ implies the smoothness of $W_n\psi$.

If $\varphi \in BL^n_s(\mathcal{A}, X)$, then by (1) and (3),

$$\begin{aligned} W_nT_n\varphi(a_1,\ldots,a_n) &= T_n\varphi(\pi(a_1),\ldots,\pi(a_n))\\ &= \varphi(\theta(p\pi(a_1)),\ldots,\theta(p\pi(a_n)))\\ &= \varphi(a_1,\ldots,a_n), \end{aligned}$$

which proves (v).

To prove (iv), let $\psi \in BL^n_{sw}(\overline{\pi(\mathcal{A})}, X : \mathcal{C})$. Since $p^2 = p$ in the center of $\overline{\pi(\mathcal{A})}$ and ψ is a $\mathcal{C}$-module map, we have

$$\begin{aligned} W_n\psi(a_1,\ldots,a_n) &= \psi(\pi(a_1),\ldots,\pi(a_n))\\ &= \psi(\pi(a_1),\ldots,\pi(a_n))\cdot p \qquad \text{(since } p\cdot x = x\cdot p = x)\\ &= \psi(\pi(a_1)p,\ldots,\pi(a_n)p)\\ &= \psi(\theta^{-1}(a_1),\ldots,\theta^{-1}(a_n)) \qquad \text{(by (1), } \theta^{-1}(a_i) = \pi(a_i)p)\\ &= S_n\psi(a_1,\ldots,a_n), \end{aligned}$$

as required. This finishes the proof. □

Lemma 3. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on $\bar{\mathcal{A}}$, and let X be a normal dual $\bar{\mathcal{A}}$-bimodule. Then there is a bounded linear map $J_n : BL^n_s(\mathcal{A}, X) \to BL^{n-1}_s(\mathcal{A}, X)$ with the following properties;*

(i) $\|J_n\| \leq ((n+2)^n - 1)/(n+1)$,

(ii) *if $\varphi \in BL^n_s(\mathcal{A}, X)$ with $\delta^n_s\varphi = 0$, then $\varphi - \delta^{n-1}_sJ_n\varphi \in BL^n_{sw}(\mathcal{A}, X)$,*

(iii) *if $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, then J_n maps $BL^n_s(\mathcal{A}, X : \mathcal{B})$ into $BL^{n-1}_s(\mathcal{A}, X : \mathcal{B})$.*

Proof. Let π be the universal representation of $\mathcal{A}$ and p the central projection in $\overline{\pi(\mathcal{A})}$ as in Remark 1. The unitary subgroup consisting of the two elements $\{1, 2p-1\}$ generates a two-dimensional C^*-subalgebra $\mathcal{C}$ in the center of $\overline{\pi(\mathcal{A})}$. By the averaging techniques similar to [25, Lemma 3.2.4(a)], there is a continuous linear map $K_n : BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) \to BL^{n-1}_{sw}(\overline{\pi(\mathcal{A})}, X)$ such that $(I - \delta^{n-1}_{sw} K_n)\psi$ is a $\mathcal{C}$-module map, for each $\psi \in BL^n_{sw}(\overline{\pi(\mathcal{A})}, X)$ with $\delta^n_{sw}\psi = 0$.

Let T_n, W_n be as in Proposition 1. Define

$$J_n : BL^n_s(\mathcal{A}, X) \to BL^{n-1}_s(\mathcal{A}, X)$$

by $J_n = W_{n-1} K_n T_n$, then the following diagram is commutative:

$$\begin{array}{ccc} BL^n_s(\mathcal{A}, X) & \xrightarrow{\;T_n\;} & BL^n_{sw}(\overline{\pi(\mathcal{A})}, X) \\ {\scriptstyle J_n}\big\downarrow & & \big\downarrow{\scriptstyle K_n} \\ BL^{n-1}_s(\mathcal{A}, X) & \xleftarrow{\;W_{n-1}\;} & BL^{n-1}_{sw}(\overline{\pi(\mathcal{A})}, X). \end{array}$$

By [25, Lemma 3.2.4], $\|J_n\| \leq ((n+2)^n - 1)/(n+1)$, and by Proposition 1, J_n takes $\mathcal{B}$-module maps to $\mathcal{B}$-module maps, and this proves (i) and (iii). Since $W_n T_n$ is the identity on $BL^n_s(\mathcal{A}, X)$, the equation $\delta^{n-1}_s W_{n-1} = W_n \delta^{n-1}_{sw}$ implies that

$$\varphi - \delta^{n-1}_s J_n \varphi = \varphi - \delta^{n-1}_s W_{n-1} K_n T_n \varphi = W_n (T_n \varphi - \delta^{n-1}_{sw} K_n T_n \varphi).$$

Now Proposition 1(i) implies that $\delta^n_{sw} T_n \varphi = T_{n+1} \delta^n_s \varphi = 0$. Hence, $T_n\varphi - \delta^{n-1}_{sw} K_n T_n \varphi$ is a $\mathcal{C}$-module map. Proposition 1(iv) asserts that W_n takes $\mathcal{C}$-module smooth maps to smooth normal maps, so $\varphi - \delta^{n-1}_s J_n \varphi$ is a smooth normal map. This completes the proof. $\square$

Proof of Theorem 2. Consider the natural embedding

$$Q_n : BL^n_{sw}(\mathcal{A}, X) \to BL^n_s(\mathcal{A}, X).$$

If $\varphi \in BL^n_{sw}(\mathcal{A}, X)$ with $\varphi = \delta^{n-1}_s \psi$ for some $\psi \in BL^{n-1}_s(\mathcal{A}, X)$, then by Proposition 1(i) and (iii), $\varphi = S_n T_n \delta^{n-1}_s \psi = \delta^{n-1}_{sw} S_{n-1} T_{n-1} \psi$. Therefore, Q_n induces an injective map $\tilde{Q}_n : \mathcal{H}^n_{sw}(\mathcal{A}, X) \to \mathcal{H}^n_s(\mathcal{A}, X)$, which is surjective by Lemma 3. Hence, $\mathcal{H}^n_{sw}(\mathcal{A}, X) = \mathcal{H}^n_s(\mathcal{A}, X)$. $\square$

Theorems 1 and 2 yield the following result.

Corollary 1. *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra with a faithful normal trace on $\bar{\mathcal{A}}$, and let X be a normal dual $\bar{\mathcal{A}}$-bimodule. Then, for every $n \in \mathbb{N}$ we have*

$$\mathcal{H}^n_s(\mathcal{A}, X) = \mathcal{H}^n_{sw}(\bar{\mathcal{A}}, X).$$

In [12], A. Galatan and S. Popa showed that for a von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ and a normal dual operatorial $\mathcal{M}$-bimodule X we have

$$\mathcal{H}^1_s(\mathcal{M}, X) = \mathcal{H}^1(\mathcal{M}, X_\tau). \tag{4}$$

A Banach $\mathcal{M}$-bimodule X is called operatorial if for every projection $p \in \mathcal{M}$ and $x \in X$,

$$\|p \cdot x \cdot p + (1-p) \cdot x \cdot (1-p)\| = \max\{\|p \cdot x \cdot p\|, \|(1-p) \cdot x \cdot (1-p)\|\}.$$

By [12, Proposition 2.2], every smooth derivation of a von Neumann algebra $\mathcal{M}$ to a dual $\mathcal{M}$-bimodule is normal, that is, UWOT-weak*-continuous. Therefore, $\mathcal{H}^1_s(\mathcal{M}, X) = \mathcal{H}^1_{sw}(\mathcal{M}, X)$. Hence, combining [12, Theorem 3.5] with (4) yields

$$\mathcal{H}^1_{sw}(\mathcal{M}, X) = \mathcal{H}^1_w(\mathcal{M}, X_\tau). \tag{5}$$

We use this fact to prove the main result of this paper.

Proof of Theorem 3.

$$\begin{aligned}
\mathcal{H}^1_s(\mathcal{A}, X) &= \mathcal{H}^1_{sw}(\bar{\mathcal{A}}, X) && \text{(by Corollary 1)}\\
&= \mathcal{H}^1_w(\bar{\mathcal{A}}, X_\tau) && \text{(by (5))}\\
&= \mathcal{H}^1(\mathcal{A}, X_\tau) && \text{(by [25, Theorem 3.3.1]).}
\end{aligned}$$

□

We do not know if the Banach $\mathcal{A}$-bimodule $B(\mathcal{A}, X)$ of bounded $\mathcal{A}$-bimodule maps from $\mathcal{A}$ to an operatorial Banach $\mathcal{A}$-bimodule X is again operatorial. If this is the case, by a standard reduction of order argument for cohomologies, one could conclude that $\mathcal{H}^n_s(\mathcal{A}, X) = \mathcal{H}^n(\mathcal{A}, X_\tau)$, for each $n \geq 1$.

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