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Higher order numerical method for a semilinear system of singularly perturbed differential equations*

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Abstract. In this paper, a system of singularly perturbed second order semilinear differential equations with prescribed boundary conditions is considered. To solve this problem, a parameter-uniform numerical method is constructed, which consists of a classical finite difference scheme and a piecewise uniform Shishkin mesh. It is proved that the convergence of the proposed numerical method is essentially second order in the maximum norm. An numerical illustration presented here supports the proved theoretical results.

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Key words: singular perturbation problems, boundary layers, semilinear differential equations, finite difference scheme, Shishkin mesh, parameter-uniform convergence

1. Introduction

A differential equation in which a small positive parameter multiplies the highest derivative term in the equation and/or its lower order derivative terms with some conditions is known as a singular perturbation problem. Most of the singular perturbation problems arising in real life follow a system of nonlinear and semilinear differential equations. For instance, the Navier-Stokes equation for a fluid at a high Reynolds number follows a nonlinear system of second order differential equations [8]. Systems of singularly perturbed semilinear reaction-diffusion equations arise, for example, in catalytic reaction theory [2].

Several numerical methods for different scalar singularly perturbed semilinear differential equations are reported in [12]-[14]. Systems of singularly perturbed semilinear reaction-diffusion equations are solved asymptotically by Jeffries [7], and numerically by Shishkina and Shishkin [13]. In [1], a third order uniformly convergent numerical method consisting of a finite difference scheme of Hermite type with a standard central difference on a piecewise uniform Shishkin mesh is formulated for a semilinear system of singularly perturbed second order differential equations with the same perturbation parameters.

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Gracia et al. [6] have established a first order convergent numerical method for a semilinear system of singularly perturbed differential equations with different perturbation parameters. In the present work, an essentially second order parameter-uniform numerical method for a system of singularly perturbed semilinear reaction-diffusion equations with different perturbation parameters is constructed on a piecewise uniform Shishkin mesh.

Precisely, the following system of singularly perturbed second order semilinear differential equations with prescribed boundary conditions is considered in this paper:

$$\vec{T}\vec{u}(x) := -E\,\vec{u}\,''(x) + \vec{f}(x,\,\vec{u}) = \vec{0} \;\text{ on } \Omega = (0,1), \tag{1}$$

$$\text{with } \vec{u}(0) = \vec{a} \text{ and } \vec{u}(1) = \vec{b}, \tag{2}$$

where $\vec{a} = (a_1, \ldots, a_n)^T$ and $\vec{b} = (b_1, \ldots, b_n)^T$ are constant vectors. For all $x \in \overline{\Omega} = [0,1]$, $\vec{u}(x) = (u_1(x), \ldots, u_n(x))^T$, $\vec{f}(x,\,\vec{u}\,) = (f_1(x,\,\vec{u}\,), \ldots,\, f_n(x,\,\vec{u}\,))^T \in C^4(\overline{\Omega} \times \mathbb{R}^n)$. E is an $n \times n$ diagonal matrix with diagonal elements $\varepsilon_1, \ldots, \varepsilon_n$ such that $0 < \varepsilon_1 < \cdots < \varepsilon_n << 1$. It is assumed that for all $(x, \vec{u}\,) \in \overline{\Omega} \times \mathbb{R}^n$, the nonlinear terms satisfy

$$\frac{\partial f_k(x,\,\vec{u}\,)}{\partial u_j} \leq 0,\; k, j = 1, \ldots, n \text{ and } k \neq j, \tag{3}$$

$$\min_{\substack{x\in\overline{\Omega}\\ i=1,\ldots,n}} \left(\sum_{j=1}^{n} \frac{\partial f_i(x, \vec{u}\,)}{\partial u_j} \right) \geq \alpha > 0, \text{ for some constant } \alpha. \tag{4}$$

The existence of a unique solution $\vec{u}$ to problem (1)-(2) such that $\vec{u} \in (C^4(\overline{\Omega}))^n$ is ensured by the implicit function theorem along with conditions (3) and (4). The reduced problem (obtained by putting $\varepsilon_i = 0, i = 1, \ldots, n$) corresponding to problem (1)-(2) is defined by

$$\vec{f}(x,\,\vec{r}\,) = \vec{0} \;\text{ on }\; \overline{\Omega}. \tag{5}$$

It is not hard to verify that the existence of a unique solution to the reduced problem (5) can be ensured by the implicit function theorem along with conditions (3) and (4) . Further, it is to be noted that the solution $\vec{r}$ has derivatives which are bounded independently of all the perturbation parameters $\varepsilon_i, i = 1, \ldots, n$. Thus we have

$$|r_i^{(k)}(x)| \leq C \quad \text{for} \quad i = 1, \ldots, n, \quad k = 0, 1, 2, 3, 4, \quad \text{and } x \in \overline{\Omega}.$$

Throughout the paper, C indicates a positive constant, which is free from x, $\varepsilon_i, i = 1, \ldots, n$ and the discretization parameter(N).

2. Analytical results

The Shishkin decomposition of the solution $\vec{u}(x)$ of (1)-(2) into a smooth component $\vec{v}(x)$ and a singular component $\vec{w}(x)$ is considered in the following form:

$$\vec{u}(x) = \vec{v}(x) + \vec{w}(x),$$

where

$$-E\,\vec{v}\,''(x)+\vec{f}(x,\,\vec{v})=\vec{0}\text{ on }\Omega,\ \vec{v}(0)=\vec{r}(0),\ \vec{v}(1)=\vec{r}(1)$$

and

$$\begin{gathered}-E\,\vec{w}\,''(x)+\vec{f}(x,\,\vec{v}+\vec{w})-\vec{f}(x,\,\vec{v})=\vec{0}\text{ on }\Omega,\\ \vec{w}(0)=(\vec{u}-\vec{v}\,)(0),\ \vec{w}(1)=(\vec{u}-\vec{v}\,)(1).\end{gathered}\tag{6}$$

The bounds on the smooth component $\vec{v}(x)$ and its derivatives are established in the following lemma:

Lemma 1. *For $i=1,\dots,n$ and for all $x\in\overline{\Omega}$,*

$$|v_i^{(k)}(x)|\le C,\ k=0,1,2\quad\textit{and}\quad|v_i^{(k)}(x)|\le C\left(1+\varepsilon_i^{1-\frac{k}{2}}\right),\ k=3,4.$$

Proof. For convenience, the smooth component $\vec{v}(x)$ of $\vec{u}(x)$ is decomposed into n components $\vec{q}^{\,[1]},\dots,\vec{q}^{\,[n]}$ as follows:

$$\vec{v}(x)=\sum_{i=1}^{n}\vec{q}^{\,[i]}(x),\tag{7}$$

where the n^{th} component $\vec{q}^{\,[n]}$ is the solution to

$$-E_n\,\frac{d^2\vec{q}^{\,[n]}}{dx^2}+\vec{f}\left(x,\,q_1^{[n]},\,q_2^{[n]},\dots,q_n^{[n]}\right)=\vec{0},\vec{q}^{\,[n]}(0)=\vec{v}(0),\,\vec{q}^{\,[n]}(1)=\vec{v}(1),\tag{8}$$

and the other components $\vec{q}^{\,[i]}, i=1,\dots,n-1$ are the solutions to

$$\begin{gathered}-\,E_i\,\frac{d^2\vec{q}^{\,[i]}}{dx^2}+\vec{f}\left(x,\,\sum_{j=i}^{n}q_1^{[j]},\,\sum_{j=i}^{n}q_2^{[j]},\dots,\sum_{j=i}^{n}q_n^{[j]}\right)\\ -\,\vec{f}\left(x,\,\sum_{j=i+1}^{n}q_1^{[j]},\,\sum_{j=i+1}^{n}q_2^{[j]},\dots,\,\sum_{j=i+1}^{n}q_n^{[j]}\right)=\vec{0}\\ \vec{q}^{\,[i]}(0)=\vec{q}^{\,[i]}(1)=\vec{0},\ i=1,\dots,n-1,\end{gathered}\tag{9}$$

where E_i is the diagonal matrix $\operatorname{diag}(0,0,\dots,\varepsilon_i,\varepsilon_{i+1},\dots,\varepsilon_n)$, for $i=1,\dots,n$. It is to be noted that $E_1=E$.

Let $x\in\Omega$. Using (8) and (5),

$$\begin{gathered}-E_n\,\frac{d^2(\vec{q}^{\,[n]}-\vec{r}\,)(x)}{dx^2}+A^{[n]}(x)(\vec{q}^{\,[n]}-\vec{r}\,)(x)=E_n\,\frac{d^2\vec{r}(x)}{dx^2},\\ \vec{q}^{\,[n]}(0)=\vec{v}(0),\ \vec{q}^{\,[n]}(1)=\vec{v}(1),\end{gathered}\tag{10}$$

where $A^{[n]}(x)=(a_{ij}^{[n]}(x))$ with $a_{ij}^{[n]}(x)=\dfrac{\partial f_i}{\partial u_j}\left(x,\,\eta_{1i}^{[n]}(x),\dots,\eta_{ni}^{[n]}(x)\right),i,j=1\dots,n,$ where $\eta_{ki}^{[n]},k=1,\dots,n$ are partial derivatives evaluated at intermediate values.

Note that in system (10), the n^{th} equation is a second order ordinary differential equation, while all other $n-1$ equations are algebraic. Hence, each of the components $q_i^{[n]}-r_i, i=1,\ldots,n-1$ can be expressed in terms of $q_n^{[n]}-r_n$. Now the n^{th} equation in (10) can be written as

$$Lz(x) = -\varepsilon_n \frac{d^2 z(x)}{dx^2} + p(x)z(x) = \varepsilon_n \frac{d^2 r_n}{dx^2}$$

with $z(0) = z(1) = 0$, where $z = q_n^{[n]} - r_n, p(x)$ is a rational function in $a_{ij}^{[n]}(x)$ derived in expressing $q_i^{[n]} - r_i$ in terms of $q_n^{[n]} - r_n$. It is not hard to verify that the operator L satisfies a maximum principle stated in Chapter 6 of [8]. Hence

$$\left|q_n^{[n]} - r_n\right| \le C\,\varepsilon_n, \quad \left|\frac{d^2(q_n^{[n]} - r_n)}{dx^2}\right| \le C.$$

By utilizing the mean value theorem, it is not hard to verify that

$$\left|\frac{d(q_n^{[n]} - r_n)}{dx}\right| \le C\,\varepsilon_n^{1/2}.$$

From the $(n-1)^{th}$ equation of system (10), we find that

$$\left|q_{n-1}^{[n]} - r_{n-1}\right| \le C\,\varepsilon_n.$$

Differentiating the $(n-1)^{th}$ equation of system (10) with respect to x once and twice, we get

$$\left|\frac{d(q_{n-1}^{[n]} - r_{n-1})}{dx}\right| \le C, \quad \left|\frac{d^2(q_{n-1}^{[n]} - r_{n-1})}{dx^2}\right| \le C.$$

Similarly,

$$\left|q_i^{[n]} - r_i\right| \le C\,\varepsilon_n, \quad \left|\frac{d(q_i^{[n]} - r_i)}{dx}\right| \le C, \quad \left|\frac{d^2(q_i^{[n]} - r_i)}{dx^2}\right| \le C, \; i = 1,\ldots,n-2.$$

Differentiating the n^{th} equation of system (10) with respect to x once and twice, we get

$$\left|\frac{d^3(q_n^{[n]} - r_n)}{dx^3}\right| \le C\,\varepsilon_n^{-1/2}, \quad \left|\frac{d^4(q_n^{[n]} - r_n)}{dx^4}\right| \le C\,\varepsilon_n^{-1}.$$

Hence,

$$\left|\frac{d^3(q_i^{[n]} - r_i)}{dx^3}\right| \le C\,\varepsilon_n^{-1/2} \le C\,\varepsilon_i^{-1/2}, \quad \left|\frac{d^4(q_i^{[n]} - r_i)}{dx^4}\right| \le C\,\varepsilon_n^{-1} \le C\,\varepsilon_i^{-1}.$$

By using similar arguments in system (9), it is not hard to prove that

$$\left|q_i^{[k]}(x)\right| \le C\,\varepsilon_i, \quad \left|\frac{dq_i^{[k]}(x)}{dx}\right| \le C, \quad \left|\frac{d^2 q_i^{[k]}(x)}{dx^2}\right| \le C$$

$$\left|\frac{d^3 q_i^{[k]}(x)}{dx^3}\right| \leq C\,\varepsilon_i^{-1/2},\ \left|\frac{d^4 q_i^{[k]}(x)}{dx^4}\right| \leq C\,\varepsilon_i^{-1},\ k=1,\ldots,n, i=1,\ldots,n.$$

Thus the bounds for the smooth component $\vec{v}$ and its derivatives follow from the bounds of the components $\vec{q}^{[1]},\ldots,\vec{q}^{[n]}$ and their derivatives. $\Box$

From (6), for $i=1,\ldots,n$,

$$-\varepsilon_i w_i''(x) + \sum_{j=1}^{n} s_{ij}(x) w_j(x) = 0,$$

where $s_{ij}(x) = \dfrac{\partial f_i}{\partial u_j}(x, \vec{\theta}_{f_i}(x)),\ i,j=1,\ldots,n$, are partial derivatives evaluated at intermediate values.

The singular component $\vec{w}(x)$ is further decomposed as follows:

$$\vec{w}(x) = \vec{w}^{\,l}(x) + \vec{w}^{\,r}(x),$$

where the component $\vec{w}^{\,l}$ is the solution to

$$-E(\vec{w}^{\,l})''(x) + S(x)\vec{w}^{\,l}(x) = \vec{0} \text{ on } \Omega,\ \vec{w}^{\,l}(0) = \vec{w}(0), \vec{w}^{\,l}(1) = \vec{0} \tag{11}$$

and the component $\vec{w}^{\,r}$ is the solution to

$$-E(\vec{w}^{\,r})''(x) + S(x)\vec{w}^{\,r}(x) = \vec{0} \text{ on } \Omega, \vec{w}^{\,r}(0) = \vec{0}, \vec{w}^{\,r}(1) = \vec{w}(1),$$

where $S(x) = (s_{ij}(x))_{n\times n}$.

The layer functions $B_i^l, B_i^r, B_i, i=1,\ldots,n$, related to the solution $\vec{u}$, of the problem (1)-(2) are defined by the following equations:

$$\begin{aligned} B_i^l(x) &= e^{-x\sqrt{\alpha}/\sqrt{\varepsilon_i}},\\ B_i^r(x) &= e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\\ B_i(x) &= B_i^l(x) + B_i^r(x) \text{ on } \overline{\Omega}. \end{aligned}$$

The bounds on the component $\vec{w}^{\,l}(x)$ of the singular component $\vec{w}(x)$ and its derivatives are established in the following lemma. It is not hard to prove that the analogous results hold for the component $\vec{w}^{\,r}(x)$ of the singular component $\vec{w}(x)$ and its derivatives by using the same procedure with $1-x$ instead of x.

Lemma 2. *For $i=1,\ldots,n$ and for any $x\in\overline{\Omega}$,*

$$|w_i^l(x)| \leq C\,B_n^l(x), \qquad |(w_i^l)'(x)| \leq C\sum_{q=i}^{n}\frac{B_q^l(x)}{\sqrt{\varepsilon_q}},$$

$$|(w_i^l)''(x)| \leq C\sum_{q=i}^{n}\frac{B_q^l(x)}{\varepsilon_q}, \qquad |(w_i^l)^{(3)}(x)| \leq C\sum_{q=1}^{n}\frac{B_q^l(x)}{\varepsilon_q^{\frac{3}{2}}},$$

$$|\varepsilon_i (w_i^l)^{(4)}(x)| \leq C\sum_{q=1}^{n}\frac{B_q^l(x)}{\varepsilon_q}.$$

Proof. From (11) we note that the defining equations for $\vec{w}^{\,l}(x)$ are the same as in [10]. Hence the bounds on the component $\vec{w}^{\,l}(x)$ and its derivatives can be derived as in [10]. $\Box$

3. The Shishkin mesh and the discrete problem

On the interval $\overline{\Omega}$ a piecewise uniform Shishkin mesh with N mesh-intervals is now constructed as follows. Let $\Omega^N = \{x_j\}_{j=1}^{N-1}$; then $\overline{\Omega}^N = \{x_j\}_{j=0}^{N}$. The interval $\overline{\Omega}$ is subdivided into $2n+1$ sub-intervals as follows: $[0,\tau_1] \cup (\tau_1,\tau_2] \cup \cdots \cup (\tau_{n-1},\tau_n] \cup (\tau_n, 1-\tau_n] \cup (1-\tau_n, 1-\tau_{n-1}] \cup \cdots \cup (1-\tau_2, 1-\tau_1] \cup (1-\tau_1, 1]$. The transition parameters $\tau_r, r = 1,\dots,n$, separating the uniform meshes, are defined by

$$\tau_n = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\}$$

and for $r = 1,\dots,n-1$,

$$\tau_r = \min\left\{\frac{\tau_{r+1}}{2}, \frac{2\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}.$$

From the total N mesh points, $\frac{N}{2}$ mesh points are placed on the outer domain $(\tau_n, 1-\tau_n]$ uniformly and on each of the inner domains $[0,\tau_1], (1-\tau_1, 1], (\tau_r, \tau_{r+1}]$ and $(1-\tau_{r+1}, 1-\tau_r], r = 1,\dots,n-1$, a uniform fine mesh of $\frac{N}{4n}$ mesh points is placed.

The discrete problem corresponding to problem (1)-(2) is defined as

$$\vec{T}^N \vec{U}(x_j) = -E\,\delta^2 \vec{U}(x_j) + \vec{f}(x_j, \vec{U}(x_j)) = \vec{0}, \text{ for } x_j \in \Omega^N, \tag{12}$$

$$\vec{U}(x_0) = \vec{u}(x_0) \text{ and } \vec{U}(x_N) = \vec{u}(x_N). \tag{13}$$

Here

$$\delta^2 V(x_j) = \frac{(D^+ - D^-)V(x_j)}{\overline{h}_j}, \qquad D^+ V(x_j) = \frac{V(x_{j+1}) - V(x_j)}{h_{j+1}},$$

$$D^- V(x_j) = \frac{V(x_j) - V(x_{j-1})}{h_j}, \qquad h_j = x_j - x_{j-1}, \overline{h}_j = \frac{h_{j+1} + h_j}{2},$$

$$\overline{h}_0 = \frac{h_1}{2} \quad \text{and} \quad \overline{h}_N = \frac{h_N}{2}.$$

4. Error analysis

Let $\vec{Y}$ and $\vec{Z}$ be any two mesh functions defined on $\overline{\Omega}^N$. For $x_j \in \Omega^N$, we have

$$\begin{aligned}
&(\vec{T}^N \vec{Y} - \vec{T}^N \vec{Z})(x_j) \\
&\quad = -E\,\delta^2(\vec{Y} - \vec{Z})(x_j) + \vec{f}(x_j, \vec{Y}(x_j)) - \vec{f}(x_j, \vec{Z}(x_j)) \\
&\quad = -E\,\delta^2(\vec{Y} - \vec{Z})(x_j) + J(\vec{f}, \vec{u})(\vec{Y} - \vec{Z})(x_j) \\
&\quad = (\vec{T}^N)'(\vec{Y} - \vec{Z})(x_j),
\end{aligned} \tag{14}$$

where

$$J(\vec{f},\,\vec{u}) = \left(\frac{\partial f_i}{\partial u_k}(x_j,\,\vec{M}(x_j))\right)_{n\times n}$$

is the Jacobian evaluated at an intermediate value and $(\vec{T}^N)'$ is the Frechet derivative of the nonlinear operator $\vec{T}^N$. Since $(\vec{T}^N)'$ is a linear operator, it satisfies the discrete maximum principle presented in [10]. Hence,

$$\parallel \vec{Y}-\vec{Z} \parallel \leq\ C\ \parallel (\vec{T}^N)'(\vec{Y}-\vec{Z}) \parallel = C\ \parallel \vec{T}^N\vec{Y}-\vec{T}^N\vec{Z} \parallel\ \text{ on } \Omega^N. \tag{15}$$

In the following lemma it is proved that the proposed numerical method is essentially second order parameter-uniform convergent.

Lemma 3. *Let $\vec{u}$ be the solution to problem* (1)-(2) *and $\vec{U}$ the solution to problem* (12)-(13). *Then for $x_j \in \overline{\Omega}^N$,*

$$|(\vec{U}-\vec{u})(x_j)|\ \leq\ C\,N^{-2}(\ln N)^3. \tag{16}$$

Proof. Let $x_j \in \Omega^N$. From (15), we have

$$\parallel \vec{U}-\vec{u} \parallel \leq\ C\ \parallel \vec{T}^N\vec{U}-\vec{T}^N\vec{u} \parallel .$$

Consider

$$\parallel \vec{T}^N\vec{u} \parallel = \parallel \vec{T}^N\vec{u}-\vec{T}^N\vec{U} \parallel .$$

Hence,

$$\begin{aligned}
\parallel \vec{T}^N\vec{u}-\vec{T}^N\vec{U} \parallel &= \parallel \vec{T}^N\vec{u} \parallel \\
&= \parallel \vec{T}^N\vec{u}-\vec{T}\vec{u} \parallel \\
&= E\ \parallel (\delta^2\vec{u}-\vec{u}'')(x_j) \parallel \\
&\leq E\,(\parallel (\delta^2\vec{v}-\vec{v}'')(x_j) \parallel + \parallel (\delta^2\vec{w}-\vec{w}'')(x_j) \parallel).
\end{aligned}$$

Note that the bounds for the smooth component $\vec{v}$ and the singular component $\vec{w}$ are the same as in [10]. Hence for $x_j \in \overline{\Omega}^N$, the required result (16) follows by using the same arguments as in [10] to the linear operator $(\vec{T}^N)'$. □

5. The continuation method

The system of semilinear differential equations in (1)-(2) is modified to an artificial system of semilinear partial differential equations as follows:

$$\begin{aligned}
&\frac{\partial\vec{u}(x,t)}{\partial t} - E\,\frac{\partial^2\vec{u}(x,t)}{\partial x^2} + \vec{f}(x,\,\vec{u}(x,t)) = \vec{0},\ (x,t)\in(0,1)\times(0,T\,],\\
&\vec{u}(0,t)=\vec{u}(0),\ \vec{u}(1,t)=\vec{u}(1),\ t\geq 0\ \text{ and }\ \vec{u}(x,0)=\vec{u}_{ini}(x),\ 0<x<1.
\end{aligned} \tag{17}$$

The continuation method reported for a scalar semilinear differential equation in [3] is modified appropriately for a system of semilinear differential equations as given below, which is used to solve (17). For $j = 1, \ldots, N$ and $k = 1, \ldots K$,

$$D_t^- \vec{U}(x_j, t_k) - E\,\delta_x^2 \vec{U}(x_j, t_k) + \vec{f}(x_j, \vec{U}(x_j, t_{k-1})) = \vec{0},$$

$$\begin{aligned} &\vec{U}(0, t_k) = \vec{u}(0), \quad \vec{U}(1, t_k) = \vec{u}(1) \text{ for all } k \text{ and} \\ &\vec{U}(x_j, 0) = \vec{u}_{ini}(x_j) \text{ for all } x_j \in \overline{\Omega}^N, \end{aligned}$$

where

$$\delta_x^2 V(x_j, t_k) = \frac{(D_x^+ - D_x^-)V(x_j, t_k)}{\bar{h}_j}, \qquad D_x^+ V(x_j, t_k) = \frac{V(x_{j+1}, t_k) - V(x_j, t_k)}{h_{j+1}},$$

$$D_x^- V(x_j, t_k) = \frac{V(x_j, t_k) - V(x_{j-1}, t_k)}{h_j}, \quad D_t^- V(x_j, t_k) = \frac{V(x_j, t_k) - V(x_j, t_{k-1})}{h_t}.$$

The initial guess $\vec{u}_{ini}(x)$ is taken to be $\vec{u}(0) + x(\vec{u}(1) - \vec{u}(0))$. The choices of the step size $h_t = t_k - t_{k-1}$ and the number of iterations K are determined as follows. For each $i = 1, \ldots, n$, define

$$\begin{aligned} e_i(k) &= \max_{1 \le j \le N} \left(\frac{|U_i(x_j, t_k) - U_i(x_j, t_{k-1})|}{h_t} \right) \text{ for } k = 1, \ldots, K, \\ e(k) &= \max_{i=1,..,n} e_i(k). \end{aligned}$$

The step size h_t is chosen sufficiently small so that the error decreases with the increasing k. Precisely, we choose h_t such that

$$e(k) \le e(k-1) \text{ for all } k, 1 < k \le K. \tag{18}$$

The number of iterations K is based on the condition that

$$e(K) \le tol, \tag{19}$$

where *tol* is a prescribed small tolerance. The algorithm given below is used to compute the numerical solution for problem (17).

Begin from t_0 with the starting step size $h_t = 1.0$. Suppose at some value of k, condition (18) is not satisfied, then leave the present step and start from the previous step t_{k-1} with h_t as $h_t/2$ and then continue halving the step size h_t until finding an h_t for which condition (18) is satisfied. If condition (18) is satisfied at each step h_t, then continue the process until either condition (19) is satisfied or $K = 100$. If condition (19) is not satisfied, then it is assumed that the stepping process is stalled because of the choice of a large initial step. In such a case, the entire process is repeated from t_0 by halving the initial step size h_t to $h_t/2$. If condition (19) is satisfied, then the final values of $\vec{U}(x_j, t_K)$ are taken as the numerical approximations to the solution for the corresponding continuous problem.

6. Numerical illustration

An example is presented in this section to illustrate the proposed numerical method for a system of singularly perturbed semilinear differential equations with prescribed with boundary conditions.

Example 1. *Consider the following boundary value problem:*

$$-E\,\vec{u}''(x) + \vec{f}(x,\,\vec{u}\,) = \vec{0}, \quad x \in (0,1)$$
$$\textit{with} \quad \vec{u}(0) = (0.002,\, 0.001)^T \;\; \textit{and} \;\; \vec{u}(1) = (0.002,\, 0.001)^T,$$

$$\textit{where } E = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \textit{ and } \vec{f}(x,\vec{u}\,) = \begin{bmatrix} (u_1(x))^3 + 2u_1(x) - \dfrac{1}{10}u_2(x) \\[2ex] (u_2(x))^3 + 2u_2(x) - u_1(x) \end{bmatrix}.$$

The problem in Example 1 is solved by the continuation method constructed in Section 5 for a system of singularly perturbed semilinear differential equations. The tolerance $'tol'$ is taken to be 0.00001.

Using the general methodology from [3], the $\vec{\varepsilon}$- uniform order of convergence (p^*) and the $\vec{\varepsilon}$- uniform error constant ($C^N_{p^*}$) are calculated. Notations D^N, p^N, C^N_p, p^* and $C^N_{p^*}$ bear the same meaning as in [3].

In Table 1, the maximum pointwise error (D^N) and the rate of convergence (p^N) for the above boundary value problem are presented.

η	Number of mesh points (N)					
	64	128	$\cdots$	1024	2048	4096
2^{-5}	0.048E-6	0.024E-6	$\cdots$	0.003E-6	0.002E-6	0.001E-6
2^{-10}	0.261E-6	0.054E-6	$\cdots$	0.012E-6	0.007E-6	0.004E-6
2^{-15}	2.249E-6	0.696E-6	$\cdots$	0.012E-6	0.007E-6	0.004E-6
2^{-20}	3.118E-6	1.313E-6	$\cdots$	0.038E-6	0.007E-6	0.004E-6
2^{-25}	3.289E-6	1.454E-6	$\cdots$	0.079E-6	0.024E-6	0.006E-6
2^{-30}	3.320E-6	1.480E-6	$\cdots$	0.089E-6	0.030E-6	0.009E-6
2^{-35}	3.325E-6	1.485E-6	$\cdots$	0.091E-6	0.032E-6	0.010E-6
2^{-40}	3.326E-6	1.486E-6	$\cdots$	0.092E-6	0.032E-6	0.010E-6
2^{-45}	3.327E-6	1.486E-6	$\cdots$	0.092E-6	0.032E-6	0.010E-6
2^{-50}	3.327E-6	1.486E-6	$\cdots$	0.092E-6	0.032E-6	0.010E-6
2^{-55}	3.327E-6	1.486E-6	$\cdots$	0.092E-6	0.032E-6	0.010E-6
D^N	3.327E-6	1.486E-6	$\cdots$	0.092E-6	0.032E-6	0.010E-6
p^N	1.1625	1.2477	$\cdots$	1.5205	1.6103	
C^N_p	7.563E-4	7.563E-4	$\cdots$	5.240E-4	4.088E-4	2.997E-4
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^* = 1.1625$						
Computed $\vec{\varepsilon}$-uniform error constant, $C^N_{p^*} = 0.0007563$						

Table 1: $\varepsilon_1 = \frac{\eta}{16}$, $\varepsilon_2 = \frac{\eta}{8}$ *and* $\alpha = 0.9$

It is evident from the table that the maximum pointwise error (D^N) decreases monotonically and the rate of convergence (p^N) increases monotonically when the number of mesh points (N) increases. For $\varepsilon_1 = 2^{-16}$, $\varepsilon_2 = 2^{-15}$ and $N = 256$

the numerical solution for component u_1 is portrayed in Figure 1 and the numerical solution for component u_2 is portrayed in Figure 2. From Figures 1 and 2 we observe that both components of the solution exhibit boundary layers at both boundaries $x = 0$ and $x = 1$. Further, the $Log - log\ plot$ for the error in the suggested numerical method for the above problem is presented in Figure 3. And from Figure 3 we perceive that the maximum pointwise errors are bounded by $30\, N^{-2}(\ln N)^3$, which is proved in Lemma 3.

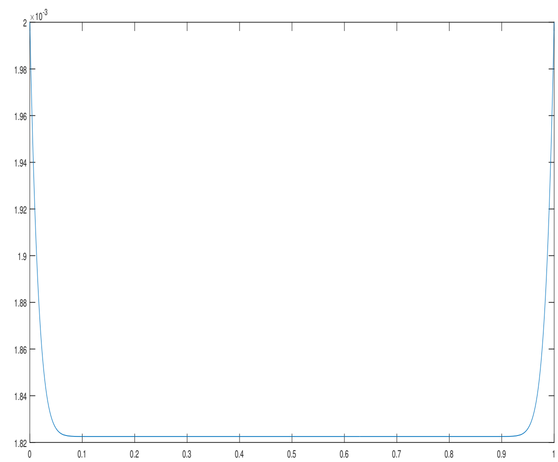

Figure 1: *Solution profile of component* u_1

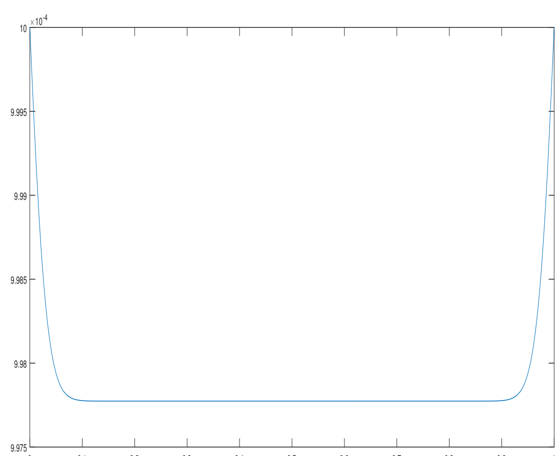

Figure 2: *Solution profile of component* u_2

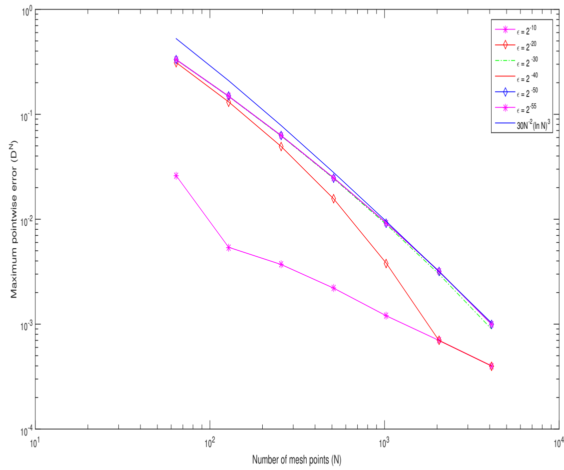

Figure 3: *Log − log plot for the error*

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