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# Limit theorems for a jump-diffusion model with Hawkes jumps

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**Abstract.** We consider a jump-diffusion process with Hawkes jumps, which has been widely applied in insurance, finance, queueing theory, statistics, and many other fields. This model can be compared with the Poissonian jump-diffusion model familiar to financial economists since Merton [24]. We study the limit theorems for a jump-diffusion process with Hawkes jumps. In particular, we obtain the law of large numbers, central limit theorems, and the large deviations principle. In addition, we provide some examples with i.i.d. random variable $Y_i$ that represent the jumps to illustrate the quantities of the limit behaviors.

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**Key words**: jump-diffusion, the Hawkes process, self-exciting point processes, the law of large numbers, the central limit theorems, the large deviations principles

## 1. Introduction

Jump-diffusion models are particular cases of exponential Lévy models in which the frequency of jumps is finite. They can be considered as prototypes for a large class of more complex models such as the stochastic volatility plus jump model of Bates [3]. Starting with Merton's seminal paper [24] and up to the present date, various aspects of jump-diffusion models have been studied in the academic finance community. In the last decade, the research departments of major banks started to accept jump-diffusions as a valuable tool in their day-to-day modeling. An interest to jump models in finance is increasing because first, in a model with continuous paths like a diffusion model, the price process behaves locally like Brownian motion and the probability that the stock moves by a large amount over a short period of time is very small, unless one fixes an unrealistically high value of volatility. Therefore, in such models the prices of short term out of the money options should be much lower than what one observes in real markets, and second, from the point of view of hedging, continuous models of stock price behavior generally lead to a complete market or to a market, which can be made complete by adding one or two additional instruments, like in stochastic volatility models. Combining Brownian motion with drift and a compound Poisson process, we obtain the simplest case of

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jump-diffusion: a process which sometimes jumps and has a continuous but random evolution between the jump times:

$$X_t = \alpha t + \beta B_t + \sum_{i=1}^{N_t} Y_i. \tag{1}$$

The best known model of this type in finance is the Merton model [24], where the stock price is $S_t = S_0 e^{X_t}$ with $X_t$ as above and the jumps $Y_i$ having a Gaussian distribution. The process in (1) is again a Lévy process and its characteristic function can be computed by multiplying the characteristic function of Brownian motion and that of the compound Poisson process:

$$\mathbb{E}[e^{iuX_t}] = \exp\left\{t\Big(i\mu u - \frac{\sigma^2 u^2}{2} + \lambda \int_{\mathbb{R}} (e^{iux} - 1) f(dx)\Big)\right\}.$$

In the recent paper [1], the authors proposed a model that is capable of reproducing both time and space propagation in a crisis, and developed appropriate estimation and testing methods for that purpose. For this, they needed to leave the widely applied class of Lévy jumps, such as the compound Poisson process that is the usual driving jump process employed in the literature. Lévy processes have independent increments; as a result, they do not allow for any type of serial dependence, whence propagation of jumps over time as well as propagation of jumps across markets are key components we wish to capture. So they employed a different model, i.e., Hawkes process. In a Hawkes process, a jump in one market raises the probability of future jumps both in the same market and elsewhere, thereby generating episodes.

A Hawkes process is a self-exciting simple point process with the clustering effect whose jump rate depends on its entire past history and was introduced by Hawkes [13]. We start with a general description of the Hawkes process.

Let N be a simple point process on $\mathbb{R}$ and let $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ be an increasing family of $\sigma-$algebras for all $t \in \mathbb{R}$ and let $\mathbb{F} := (\mathcal{F}_t^{-\infty})_{t\in\mathbb{R}}$ be a filtration. Any nonnegative $\mathcal{F}_t^{-\infty}$-progressively measurable process $\lambda_t$ with

$$E\left[N(a,b]|\mathcal{F}_a^{-\infty}\right] = E\left[\int_a^b \lambda_s ds|\mathcal{F}_a^{-\infty}\right]$$

a. s. for all intervals or for every interval $(a, b]$, is called an $\mathcal{F}_t^{-\infty}$-intensity of N. We use the notation $N_t := N(0, t]$ to denote the number of points in the interval $(0, t]$.

A general Hawkes process is a simple point process N admitting an $\mathcal{F}_t^{-\infty}$-intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)N(ds)\right),$$

where $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and left continuous, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and we always assume that $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. Here $\int_{-\infty}^t h(t-s)N(ds)$ stands for $\int_{(-\infty,t)} h(t-s)N(ds)$. We always assume that $N(-\infty,0]=0$, i.e. the Hawkes process has empty history. In the literature [13], $h(\cdot)$ and $\lambda(\cdot)$ are usually referred to as an exciting function and a rate function, respectively. The Hawkes process is linear if $\lambda(\cdot)$ is linear and it is nonlinear otherwise. In general, the model described above is non-Markovian since the future evolution of a self-exciting simple point process is controlled by the timing of past events, but it is Markovian for a special case which means that one special case of the Hawkes process is when the exciting function $h(\cdot)$ is exponential. The Hawkes process has a wide range of applications in neuroscience [6, 19], seismology [14, 25], DNA modeling [12, 26], finance [17, 18, 34], and many other fields. It has both self-exciting and clustering properties, which is very appealing to some financial applications. In particular, self-exciting and clustering properties of the Hawkes process make it a viable candidate for modeling correlated defaults and evaluating credit derivatives in finance, for example, see Errais et al. [10] and Dassios and Zhao [8].

Hawkes [13] introduced the linear case, and the linear Hawkes process can be studied via immigration-birth representation, see e.g. Hawkes and Oakes [15]. The stability [7], the law of large numbers [7], the central limit theorem [2], large deviations [4], the Bartlett spectrum [13, 15], etc. have all been studied and are understood very well. Almost all of the applications of the Hawkes process in the literature consider exclusively the linear case. Because of the lack of immigration-birth representation and computational tractability, the nonlinear Hawkes process is much less studied. However, some efforts have already been made in this direction. A nonlinear case was first introduced by Brémaud and Massoulié [5]. Recently, Zhu [37, 35, 32, 33, 34] investigated several results for both a linear and a nonlinear model. The central limit theorem was investigated in Zhu [32] and the large deviation principles have been obtained in Zhu [35]. Limit theorems and rough fractional diffusions as scaling limits for nearly unstable Hawkes processes are obtained in Jaisson and Rosenbaum [17, 18] and the Bartlett spectrum of randomized Hawkes processes is obtained in Kelbert et al. [21]. Zhu [34] has also studied applications to financial mathematics. Some variations and extensions of the Hawkes process have been studied in Dassios and Zhao [8], Zhu [36], Karabash and Zhu [20], Mehrdad and Zhu [23] and Ferro, Leiva and Møller [11]. In the recent paper [28], Seol considers the arrival time $\tau_n$, i.e., the inverse process of the Hawkes process, and studies the limit theorems (the law of large numbers, the central limit theorem and large deviations) for $\tau_n$. Recently, Seol [27] studied the law of large numbers, central limit theorem and invariance principles for discrete Hawkes processes starting from empty history. A moderate deviation principle for marked Hawkes processes was investigated in Seol [29], and limit theorems for the compensator of Hawkes processes were studied by Seol [30].

In this paper, we consider the log-stock prices process $X_t$ defined by

$$X_t = \alpha t + \beta W_t + \sum_{i=1}^{N_t} Y_i, \tag{2}$$

where $\alpha$ is the instantaneous expected return on the stock, $\beta$ is the instantaneous volatility of the return, under the condition that the Hawkes event does not occur,

$W_t$ is standard Brownian motion, $Y_i$ are i.i.d. $\mathbb{R}-$valued random variables, $\sum_{i=1}^{N_t} Y_i$ is compound Hawkes process, where $N_t$ is a linear Hawkes process with intensity

$$\lambda_t := \nu + \int_0^{t-} h(t-s)dN_s.$$

$W_t, N_t$ and $Y_i$ are independent, while $W_t$ and $\sum_{i=1}^{N_t} Y_i$ are independent. Then the stock price process $S_t$ is defined by

$$S_t = S_0 e^{X_t}.$$

The main goal of this article is to establish the several limit behaviors for the process $X_t$.

The paper is structured as follows. Some auxiliary results to prove the main results are stated in Section 2, and the main results are given in Section 3. The proofs for the main theorems are contained in Section 4. In Section 5, we study some examples according to the random variable $Y_1$, that is, we give some examples when $Y_1$ follows a normal distribution or a double exponential distribution, and then we can illustrate the quantities of the limit behaviors with respect to two random distributions.

## 2. Preliminaries

In this section, we introduce some classical results to set up the main goal. We start with some reviews for the results of Hawkes processes.

### 2.1. Limit theorems for Hawkes processes

The limit theorems for both linear and nonlinear models are well known and studied by many authors.

**Linear model:** Since $\lambda(\cdot)$ is linear, say $\lambda(z) = \nu + z$ for some $\nu > 0$, and $\|h\|_{L^1} < 1$, we can use a very nice immigration-birth representation and the limit theorems are well understood and more explicitly represented. Daley and Vere-Jones [7] proved the law of large numbers for a linear Hawkes process. The functional central limit theorem for a linear multivariate Hawkes process under certain assumptions has been obtained by Bacry et al [2]. Bordenave and Torrisi [4] proved that if $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, then $(\frac{N_t}{t} \in \cdot)$ satisfies the large deviation principle. A moderate deviation principle for linear continuous time Hawkes processes is obtained by Zhu [33], while the limit theorems for linear marked Hawkes processes are obtained in Zhu [23].

**Nonlinear model:** Since $\lambda(\cdot)$ is nonlinear, the usual immigration-birth representation no longer works and so a nonlinear model is much harder to study. Brémaud and Massoulié [5] proved that there exists a unique stationary version of nonlinear Hawkes processes under certain conditions and the convergence to equilibrium of a non-stationary version. The central limit theorem is obtained in Zhu [32], and Zhu [37] proved a large deviation for a special case of a nonlinear case when $h(\cdot)$ is exponential or sums of exponentials. Zhu [35] proved a process-level, i.e., the level-3 large deviation principle for nonlinear Hawkes processes for general $h(\cdot)$, and hence by the principle of contradiction, the level-1 large deviation principle for $(\frac{N_t}{t} \in \cdot)$.

Before we proceed, let us review some limit theorem results for the linear Hawkes process in the literature. Daley and Vere-Jones [7] proved the law of large numbers for a linear Hawkes process as follows:

$$\frac{N_t}{t} \to \frac{\nu}{1 - \|h\|_{L^1}} \text{ as } t \to \infty \quad a.s.$$

The functional central limit theorem for a linear multivariate Hawkes process under certain assumptions has been obtained by Bacry et al. [2] and they proved that

$$\frac{N_{\cdot t} - \cdot \mu t}{\sqrt{t}} \to \sigma B(\cdot), \text{ as } t \to \infty,$$

where $B(\cdot)$ is standard Brownian motion and

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}$$

The convergence used in the above theorem is weak convergence on $D[0,1]$, the space of a càdlàg function on $[0,1]$, equipped with the Skorokhod topology. Bordenave and Torrisi [4] proved that if $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, then $(\frac{N_t}{t} \in \cdot)$ satisfies the large deviation principle with the good rate function $I(\cdot)$, which means that for any closed set $C \subset \mathbb{R}$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(N_t/t \in C) \leq -\inf_{x \in C} I(x),$$

and for any open set $G \subset \mathbb{R}$,

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}(N_t/t \in G) \geq -\inf_{x \in G} I(x),$$

where

$$I(x) = \begin{cases} x\theta_x + \nu - \frac{\nu x}{\nu + \|h\|_{L^1} x}, & \text{if } x \in (0, \infty) \\ \nu, & \text{if } x = 0 \\ +\infty, & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is a unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$, of

$$\mathbb{E}(e^{\theta S}) = \frac{x}{\nu + x\|h\|_{L^1}}, \quad x > 0,$$

where S in the above equation denotes the total number of descendants of an immigrant, including the immigrant himself.

**Remark 1.** *The rate function described above $I(x)$ can be represented in a more explicit form. Note that (see [16] for details), for all $\theta \in (-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$, $\mathbb{E}(e^{\theta S})$ satisfies*

$$\mathbb{E}(e^{\theta S}) = e^{\theta} e^{\|h\|_{L^1}(\mathbb{E}(e^{\theta S}) - 1)},$$

*which implies that* $\theta_x = \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - \|h\|_{L^1}\left(\frac{x}{\nu + x\|h\|_{L^1}} - 1\right)$. *Substituting into the formula, we have*

$$I(x) = \begin{cases} x\log\left(\frac{x}{\nu+x\|h\|_{L^1}}\right) - x + \|h\|_{L^1}x + \nu, & \text{if } x \in (0,\infty) \\ \nu, & \text{if } x = 0 \\ +\infty, & \text{if } x \in (-\infty, 0). \end{cases}$$

## 3. Statement of the main results

This section states the main results of this paper. It consists of three parts. The first part is devoted to the law of large numbers, while the second one covers the result for the central limit theorem. Large deviation principles will be obtained in the end. We provide an auxiliary result playing the key role in the proof of the large deviation principle.

We start with the assumptions which will be used throughout the paper as follows:

**Assumption 1.**

(*a*) $Var[Y_1] < \infty$,

(*b*) $\lambda(z) = \nu + z$, *for some* $\nu > 0$,

(*c*) $\|h\|_{L^1} < 1$, *where* $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$.

The first assumption of Assumption 1 says that i.i.d. random variable $Y_i$ has finite expectation and variance. The second assumption says that $\lambda$ is a linear and increasing function and so the Hawkes process has a very nice immigration birth representation (see Hawkes and Oakes [15], 1974). The third assumption says that in the immigration-birth representation, the total number of descendants of any given immigrant is finite with probability 1 (see [2], [4], [7] for details).

**Theorem 1** (Law of large number)**.** *Assume that assumption 1 is satisfied and let* $X_t$ *be the log-stock price process defined in* (2)*; then we have*

$$\frac{X_t}{t} \to \alpha + \mu\mathbb{E}[Y_1]$$

*in probability as* $t \to \infty$, *where* $\mu = \frac{\nu}{1-\|h\|_{L^1}}$.

**Theorem 2** (Central limit theorem)**.** *Assume that assumption 1 is satisfied and let* $X_t$ *be the log-stock price process defined in* (2)*; then we have*

$$\frac{X_t - (\alpha + \mu\mathbb{E}[Y_1])t}{\sqrt{t}} \to N(0, \mu Var[Y_1] + (\mathbb{E}[Y_1])^2\sigma^2)$$

*in distribution as* $t \to \infty$, *where*

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \textit{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

**Theorem 3** (Large deviation principle)**.** *Assume that assumption 1 is satisfied and let $X_t$ be the log-stock price process defined in* (2)*; then* $(\frac{X_t}{t})$ *satisfies the LDP on* $\mathbb{R}$ *with the rate function*

$$\hat{I}(x) = \begin{cases} x\theta_x - \alpha\theta_x - \frac{1}{2}\beta^2\theta_x^2 + \nu - \frac{\nu\mathbb{E}[e^{\theta_x Y_1}](x-\alpha-\beta^2\theta_x)}{\nu\mathbb{E}[Y_1 e^{\theta_x Y_1}] + \|h\|_{L^1}\mathbb{E}[e^{\theta_x Y_1}](x-\alpha-\beta^2\theta_x)}, & \text{if } x \in [0,\infty) \\ +\infty, & \text{if } x \in (-\infty, 0), \end{cases}$$

*where* $\theta = \theta_x$ *is a unique solution in* $(-\infty, \|h\|_{L^1} - 1 - \log\|h\|_{L^1})$, *of*

$$\mathbb{E}\Big[S\exp((\log\mathbb{E}[e^{\theta Y_1}])S)\Big] = \frac{x-\alpha-\beta^2\theta}{\nu}, \quad x > 0 \tag{3}$$

*or equivalently of*

$$\mathbb{E}\Big[\exp((\log\mathbb{E}[e^{\theta Y_1}])S)\Big] = \frac{\mathbb{E}[e^{\theta Y_1}](x-\alpha-\beta^2\theta)}{\nu\mathbb{E}[Y_1 e^{\theta Y_1}] + \|h\|_{L^1}\mathbb{E}[e^{\theta Y_1}](x-\alpha-\beta^2\theta)}, \quad x > 0.$$

To prove Theorem 3, we need to prove a key Lemma 1 as follows.

**Lemma 1.** *For* $\theta \le \|h\|_{L^1} - 1 - \log\|h\|_{L^1}$, *we have*

$$\mathbb{E}\Big[\exp(\theta X_t)\Big] = \exp\left[\alpha\theta t + \frac{1}{2}\beta^2\theta^2 t + \nu\int_0^t (\Lambda(s)-1)ds\right], \tag{4}$$

*where* $\Lambda(t) = \mathbb{E}[e^{\theta Y_1}]\exp\Big(\int_0^t h(s)(\Lambda(t-s)-1)ds\Big)$ *for any* $0 \le s \le t$.

**Proof**. By the properties of stock price process with the Hawkes process, we have

$$\mathbb{E}\Big[\exp(\theta X_t)\Big] = \exp\Big(\alpha\theta t + \frac{1}{2}\beta^2\theta^2 t\Big)\mathbb{E}\Big[\exp((\log\mathbb{E}[e^{\theta Y_1}])N_t)\Big]. \tag{5}$$

Let us consider the term $\mathbb{E}\Big[\exp((\log\mathbb{E}[e^{\theta Y_1}])N_t)\Big]$.

A linear Hawkes process can be represented by using a nice immigration-birth representation and the immigration-birth representation is used in the proof of a theorem. There follows a general description of our nice tool. The immigrant arrives according to a standard homogeneous Poisson process with constant intensity $\nu > 0$, and then each immigrant generates children according to a Galton-Waston tree (see [15] for details ). Let $\eta$ be the number of children in the first generation coming from the same immigrant; then $\eta$ has a Poisson distribution with parameter $\|h\|_{L^1}$, i.e.

$$P(\eta = k) = e^{-\|h\|_{L^1}}\frac{\|h\|_{L^1}^k}{k!}. \tag{6}$$

Let $\Lambda^1(t), \Lambda^2(t), \Lambda^3(t), \cdots, \Lambda^\eta(t)$ be the total number of descendants coming from an immigrant's first, second, third, $\cdots$, $\eta-$th chid. Then we know that conditional on the number of children of an immigrant, the time that a child was born has a probability density function

$$\frac{h(\cdot)}{\|h\|_{L^1}}. \tag{7}$$

Using (6) and (7), we have the following:

$$\begin{aligned}
&\mathbb{E}\Big[\exp((\log \mathbb{E}[e^{\theta Y_1}])N_t)\Big] \\
&\quad = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} e^{-\nu t} \frac{1}{t^k} \int \cdots \int_{t_1<t_2<\cdots<t_k} \Lambda(t_1)\cdots\Lambda(t_k) dt_1 dt_2 \cdots dt_k \\
&\quad = e^{\nu \int_0^t (\Lambda(s)-1)ds}. \qquad (8)
\end{aligned}$$

It follows from (6) and conditional probabilities that

$$\begin{aligned}
\Lambda(t) &= \mathbb{E}\Big[\exp\Big((\log \mathbb{E}[e^{\theta Y_1}])S(t)\Big)\Big] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\Big[\exp\Big((\log \mathbb{E}[e^{\theta Y_1}])S(t)\Big)|\eta = k\Big] P\big(\eta = k\big) \\
&= \mathbb{E}\big[e^{\theta Y_1}\big] \sum_{k=0}^{\infty} \prod_{i=1}^{k} \mathbb{E}\Big[\exp\Big((\log \mathbb{E}[e^{\theta Y_1}])\Lambda^i(t)\Big)\Big] P\big(\eta = k\big) \\
&= \mathbb{E}\big[e^{\theta Y_1}\big] \sum_{k=0}^{\infty} \Bigg[\mathbb{E}\Big[\exp\Big((\log \mathbb{E}[e^{\theta Y_1}])\Lambda^i(t)\Big)\Big]\Bigg]^k P\big(\eta = k\big) \\
&= \mathbb{E}\big[e^{\theta Y_1}\big] \sum_{k=0}^{\infty} \Big(\int_0^t \frac{h(s)}{\|h\|_{L^1}} \Lambda(t-s) ds\Big)^k e^{-\|h\|_{L^1}} \frac{\|h\|_{L^1}^k}{k!} \\
&= \mathbb{E}\big[e^{\theta Y_1}\big] \exp\Big(\int_0^t h(s)(\Lambda(t-s)-1) ds\Big). \qquad (9)
\end{aligned}$$

Thus, from (5) and (8), we have

$$\mathbb{E}\Big[\exp(\theta X_t)\Big] = \exp\left[\alpha\theta t + \frac{1}{2}\beta^2\theta^2 t + \nu \int_0^t (\Lambda(s)-1) ds\right].$$

□

## 4. Proofs of the theorems

In this section, we give proofs of the main theorems.

### 4.1. Law of large numbers

First consider

$$\frac{X_t}{t} = \alpha + \beta\frac{W_t}{t} + \frac{1}{t}\sum_{i=1}^{N_t} Y_i. \qquad (10)$$

Then the second term in (10) will be zero since the properties of standard Brownian motion $W_t$ are as follows:

$$\beta\frac{W_t}{t} \to 0$$

in probability as $t \to 0$. The third term in (10) follows Wald's equation by assumption 1 (a) and the fact that $\mathbb{E}[N_t] < \infty$, and so we have

$$\frac{1}{t}\sum_{i=1}^{N_t} Y_i \to \mathbb{E}[N_t]\mathbb{E}[Y_1] = \mu\mathbb{E}[Y_1]$$

in probability as $t \to 0$.

Thus,

$$\frac{X_t}{t} \to \alpha + \mu\mathbb{E}[Y_1]$$

in probability as $t \to \infty$, where $\mu = \frac{\nu}{1-\|h\|_{L^1}}$. The proof of Theorem 1 is completed.

## 4.2. Central limit theorem

Let us recall that, for a one-dimensional case, under the assumption

$$\int_0^\infty h(t)t^{1/2}dt < \infty,$$

we have [2]

$$\frac{N_t - \mu t}{\sqrt{t}} \to \sigma N(0,1), \quad \textit{in distribution as} \quad t \to \infty,$$

where $\mu = \frac{\nu}{1-\|h\|_{L^1}}$ and $\sigma^2 = \frac{\nu}{(1-\|h\|_{L^1})^3}$.

Note that

$$\begin{aligned}\frac{X_t - (\alpha + \mu\mathbb{E}[Y_1])t}{\sqrt{t}} &= \frac{\beta W_t + \sum_{i=1}^{N_t} Y_i - \mu\mathbb{E}[Y_1]t}{\sqrt{t}} \\ &= \frac{\beta W_t}{\sqrt{t}} + \frac{\sum_{i=1}^{N_t} Y_i - \mu\mathbb{E}[Y_1]t}{\sqrt{t}} \end{aligned} \tag{11}$$

Then the first term in (11) has the following property:

$$\frac{\beta W_t}{\sqrt{t}} \to \beta Z \tag{12}$$

in distribution as $t \to \infty$, where $Z \sim N(0,1)$.

The second term in (11) can be separated as follows:

$$\begin{aligned}\frac{\sum_{i=1}^{N_t} Y_i - \mu\mathbb{E}[Y_1]t}{\sqrt{t}} =& \frac{\sum_{i=1}^{N_t} Y_i - \mathbb{E}[N_t]\mathbb{E}[Y_1]}{\sqrt{t}} \\ &+ \frac{\mathbb{E}[Y_1](\mathbb{E}[N_t] - \mu t)}{\sqrt{t}}, \end{aligned} \tag{13}$$

and thus by following Theorem 2.3 in [11], the first term in (13) has

$$\frac{\sum_{i=1}^{N_t} Y_i - \mathbb{E}[N_t]\mathbb{E}[Y_1]}{\sqrt{t}} \to N(0, \mu Var[Y_1] + (\mathbb{E}[Y_1])^2\sigma^2) \tag{14}$$

in distribution as $t \to \infty$, where

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

By assumption (1), the second term in (13) has

$$\frac{\mathbb{E}[Y_1](\mathbb{E}[N_t] - \mu t)}{\sqrt{t}} \to 0 \tag{15}$$

in probability as $t \to \infty$.

Hence, together with (12), (14) and (15) give us

$$\frac{X_t - (\alpha + \mu\mathbb{E}[Y_1])t}{\sqrt{t}} \to N(0, \mu Var[Y_1] + (\mathbb{E}[Y_1])^2\sigma^2)$$

in distribution as $t \to \infty$, where

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

## 4.3. Large deviations principle

We start with the basic definitions in large deviations theory (e.g., see Dembo and Zeitouni [9] or Varadhan [31] for details). Recall that a sequence $(\phi_n)_{n\in\mathbb{N}}$ of probability measures on a topological space X satisfies the large deviations principle with rate function $I : X \to \mathbb{R}$ if $I$ is non-negative, lower semi-continuous, and for any measurable set $B$, we have

$$-\inf_{x\in B^\circ} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log\phi_n(B) \le \limsup_{n\to\infty} \frac{1}{n}\log\phi_n(B) \le -\inf_{x\in\bar{B}} I(x),$$

where $B^\circ$ is the interior of $B$ and $\bar{B}$ is its closure. A rate function $I(x)$ is said to be good if the level sets $\{x : I(x) \le \beta\}$ are compact for any $\beta$.

**Proof of Theorem 3.** We note that $\Lambda(s)$ in (9) is strictly increasing in s and if s is sufficiently large, we have the minimal solution $\hat{x}$ of the following equation:

$$x = \mathbb{E}[e^{\theta Y_1}]e^{\|h\|_{L^1}(x-1)} \tag{16}$$

Thus, using (4) we have the following limit:

$$\begin{aligned} I(\theta) := \lim_{t\to\infty} \frac{1}{t}\log\mathbb{E}[e^{\theta X_t}] &= \lim_{t\to\infty} \frac{1}{t}\Big[\alpha\theta t + \frac{1}{2}\beta^2\theta^2 t + \nu\int_0^t (\Lambda(s) - 1)ds\Big] \\ &= \alpha\theta + \frac{1}{2}\beta^2\theta^2 + \nu(\hat{x} - 1) \end{aligned}$$

If there is no solution to the equation (16), then $\hat{x} = +\infty$. Therefore, we conclude that $(\frac{X_t}{t})$ satisfies the LDP on $\mathbb{R}$ with speed $t$ and a good rate function

$$\hat{I}(x) = \sup_{\theta\in\mathbb{R}}(\theta x - I(\theta)) = \sup_{\theta \le \|h\|_{L^1} - 1 - \log\|h\|_{L^1}} (\theta x - I(\theta)).$$

Now $\hat{I}(x) = \infty$, and in this such case $\lim_{\theta\to-\infty}(\theta x - I(\theta)) = \infty$. If $x > 0$, letting $\theta_x \in (-\infty, \mu - 1 - \log\mu)$ denote a unique solution to equation (3) it easily follows that

$$\hat{I}(x) = x\theta_x - I(\theta_x), \tag{17}$$

where

$$I(\theta) = \alpha\theta + \frac{1}{2}\beta^2\theta^2 + \nu(\Gamma(\theta) - 1) \quad \text{and} \quad \Gamma(\theta) = \mathbb{E}[e^{\theta Y_1}]e^{\|h\|_{L^1}(\Gamma(\theta)-1)}.$$

Hence, we know that to get optimal $\theta_x$, we have

$$x = I'(\theta_x) = \alpha + \beta^2\theta_x + \nu\Gamma'(\theta_x)$$

and thus $\Gamma'(\theta_x) = \frac{x-\alpha-\beta^2\theta_x}{\nu}$. If we differentiate $\Gamma(\theta_x)$ with respect to $\theta_x$, we have

$$\begin{aligned}
\Gamma'(\theta_x) &= \mathbb{E}[Y_1 e^{\theta_x Y_1}]e^{\|h\|_{L^1}(\Gamma(\theta_x)-1)} + \|h\|_{L^1}\Gamma'(\theta_x)\Gamma(\theta_x) \\
&= \frac{\mathbb{E}[Y_1 e^{\theta_x Y_1}]}{\mathbb{E}[e^{\theta_x Y_1}]}\Gamma(\theta_x) + \|h\|_{L^1}\Gamma'(\theta_x)\Gamma(\theta_x) \\
&= \Gamma(\theta_x)\left[\frac{\mathbb{E}[Y_1 e^{\theta_x Y_1}]}{\mathbb{E}[e^{\theta_x Y_1}]} + \|h\|_{L^1}\Gamma'(\theta_x)\right],
\end{aligned}$$

which yields

$$\Gamma(\theta_x) = \frac{\mathbb{E}[e^{\theta_x Y_1}](x - \alpha - \beta^2\theta_x)}{\nu\mathbb{E}[Y_1 e^{\theta_x Y_1}] + \|h\|_{L^1}\mathbb{E}[e^{\theta_x Y_1}](x - \alpha - \beta^2\theta_x)}.$$

Thus, by (17), for $x > 0$ we have

$$\hat{I}(x) = x\theta_x - \alpha\theta_x - \frac{1}{2}\beta^2\theta_x^2 + \nu - \frac{\nu\mathbb{E}[e^{\theta_x Y_1}](x - \alpha - \beta^2\theta_x)}{\nu\mathbb{E}[Y_1 e^{\theta_x Y_1}] + \|h\|_{L^1}\mathbb{E}[e^{\theta_x Y_1}](x - \alpha - \beta^2\theta_x)}.$$

By the Gärtner-Ellis theorem (see [9] for details), we conclude that $\frac{X_t}{t}$ satisfies the LDP on $\mathbb{R}$ with a large deviation rate function

$$\hat{I}(x) = \begin{cases} x\theta_x - \alpha\theta_x - \frac{1}{2}\beta^2\theta_x^2 + \nu - \frac{\nu\mathbb{E}[e^{\theta_x Y_1}](x-\alpha-\beta^2\theta_x)}{\nu\mathbb{E}[Y_1 e^{\theta_x Y_1}] + \|h\|_{L^1}\mathbb{E}[e^{\theta_x Y_1}](x-\alpha-\beta^2\theta_x)}, & \text{if } x \in [0,\infty) \\ +\infty, & \text{if } x \in (-\infty, 0). \end{cases}$$

The proof of the large deviation principle is completed. □

## 5. Examples

In this section, we give two examples for random variable $Y_1$ to show quantities of several limit behaviors. In particular, we will assume that random variable $Y_1$ will be a normal distribution or a double exponential distribution.

### 5.1. Normal distribution

Let us assume that $Y_1$ follows a normal distribution $N(\hat{\mu}, \hat{\sigma}^2)$. That is,

$$\mathbb{E}[Y_1] = \hat{\mu}, \quad Var[Y_1] = \hat{\sigma}^2, \quad \mathbb{E}[Y_1^2] = \hat{\mu}^2 + \hat{\sigma}^2.$$

Moreover, the moment generating function for $Y_1$ is

$$\mathbb{E}[e^{\theta Y_1}] = \exp\left(\theta\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\theta^2\right).$$

Then we can find quantities of limit behaviors (LLN, CLT, LDP) in detail. The law of large numbers for $Y_1$ is

$$\frac{X_t}{t} \to \alpha + \mu\hat{\mu}$$

in probability as $t \to \infty$, where $\mu = \frac{\nu}{1-\|h\|_{L^1}}$. The central limit theorem for $Y_1$ is

$$\frac{X_t - (\alpha + \mu\hat{\mu})t}{\sqrt{t}} \to N(0, \mu\hat{\sigma}^2 + \hat{\mu}^2\sigma^2)$$

in distribution as $t \to \infty$, where

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

For the large deviation principle, we can say that $(\frac{X_t}{t})$ satisfies the LDP on $\mathbb{R}$ with the rate function

$$\hat{I}(x) = \begin{cases} x\theta_x - \alpha\theta_x - \frac{1}{2}\beta^2\theta_x^2 + \nu - \frac{\nu(x-\alpha-\beta^2\theta_x)}{\nu(\hat{\mu}+\hat{\sigma}^2\theta_x)+\|h\|_{L^1}(x-\alpha-\beta^2\theta_x)}, & \text{if } x \in [0, \infty) \\ +\infty, & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is a unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log\|h\|_{L^1})$, of

$$\mathbb{E}\Big[S\exp[(\theta\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\theta^2)S]\Big] = \frac{x - \alpha - \beta^2\theta}{\nu}, \quad x > 0$$

or equivalently of

$$\mathbb{E}\Big[\exp[(\theta\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\theta^2)S]\Big] = \frac{(x - \alpha - \beta^2\theta)}{\nu(\hat{\mu} + \hat{\sigma}^2\theta) + \|h\|_{L^1}(x - \alpha - \beta^2\theta)}, \quad x > 0.$$

## 5.2. Double exponential distributions

The double exponential jump model, initiated by Steven KOU (see [22]), is an exponential Levy model, which is a compromise between reality and tractability. It gives an explanation of the two empirical phenomena which received much attention in financial markets: the asymmetric leptokurtic feature and the volatility smile. It enables us to obtain analytical solutions to the prices of many derivatives: European call and put options, interest rate derivatives, such as swaptions, caps, floors, and bond options, as well as path-dependant options, such as perpetual American options, barrier, and lookback options.

Let us assume that $Y_1$ follows a double exponential distribution with $f(x|\hat{\mu},\hat{\sigma}) = \frac{1}{2\hat{\sigma}}e^{-|\frac{x-\hat{\mu}}{\hat{\sigma}}|}$. That is,

$$\mathbb{E}[Y_1] = \hat{\mu}, \quad Var[Y_1] = 2\hat{\sigma}^2, \quad \mathbb{E}[Y_1^2] = \hat{\mu}^2 + 2\hat{\sigma}^2.$$

Moreover, the moment generating function for $Y_1$ is

$$\mathbb{E}[e^{\theta Y_1}] = \frac{e^{\theta\hat{\mu}}}{1-\hat{\sigma}^2\theta^2}.$$

Then we can find quantities of limit behaviors (LLN, CLT, LDP) in detail. The law of large numbers for $Y_1$ is

$$\frac{X_t}{t} \to \alpha + \mu\hat{\mu}$$

in probability as $t \to \infty$, where $\mu = \frac{\nu}{1-\|h\|_{L^1}}$. The central limit theorem for $Y_1$ is

$$\frac{X_t - (\eta + \mu\hat{\mu})t}{\sqrt{t}} \to N(0, 2\mu\hat{\sigma}^2 + 2\hat{\mu}^2\sigma^2)$$

in distribution as $t \to \infty$, where

$$\mu = \frac{\nu}{1-\|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1-\|h\|_{L^1})^3}.$$

For the large deviation principle, we can say that $(\frac{X_t}{t})$ satisfies the LDP on $\mathbb{R}$ with the rate function

$$\hat{I}(x) = \begin{cases} x\theta_x - \alpha\theta_x - \frac{1}{2}\beta^2\theta_x^2 + \nu \\ -\frac{\nu(1-\hat{\sigma}^2\theta_x^2)(x-\alpha-\beta^2\theta_x)}{\nu(\hat{\mu}+\hat{\sigma}^2\theta_x)(\hat{\mu}-\hat{\mu}\hat{\sigma}^2\theta_x^2+2\hat{\sigma}^2\theta_x)+\|h\|_{L^1}(1-\hat{\sigma}^2\theta_x^2)(x-\alpha-\beta^2\theta_x)}, & \text{if } x \in [0,\infty) \\ +\infty, & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is a unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log\|h\|_{L^1})$, of

$$\mathbb{E}\Big[S\exp[(\theta\hat{\mu} - \log(1-\hat{\sigma}^2\theta^2))S]\Big] = \frac{x-\alpha-\beta^2\theta}{\nu}, \quad x > 0$$

or equivalently of

$$\begin{aligned}&\mathbb{E}\Big[\exp[(\theta\hat{\mu} - \log(1-\hat{\sigma}^2\theta^2))S]\Big] \\ &\quad = \frac{(1-\hat{\sigma}^2\theta^2)(x-\alpha-\sigma^2\theta)}{\nu(\hat{\mu}+\hat{\sigma}^2\theta)(\hat{\mu}-\hat{\mu}\hat{\sigma}^2\theta^2+2\hat{\sigma}^2\theta)+\|h\|_{L^1}(1-\hat{\sigma}^2\theta^2)(x-\alpha-\beta^2\theta)}, \quad x > 0.\end{aligned}$$

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