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Negative definite functions on the infinite dimensional special linear group

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Abstract. In this paper, we prove the boundedness of every continuous negative definite function on the infinite dimensional special linear group.
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1. Introduction

Negative definite functions on groups have a long history and arise in various areas of mathematics, such as representation theory and geometric group theory. Let G be a Hausdorff topological group, and K a closed subgroup of G. The pair (G, K) is said to be *spherical* if for every irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$, the subspace $\mathcal{H}^K$ of K-invariant vectors in $\mathcal{H}$ is at most one dimensional. If G is locally compact, and K compact, then the pair (G, K) is said to be a *Gelfand pair* if the convolution algebra of K-biinvariant integrable functions on G is commutative. Furthermore, the pair (G, K) is a Gelfand pair if and only if it is spherical (cf. [10], section 23.1). In the context of Gelfand pairs (G, K), the K-biinvariant continuous negative definite functions are of particular interest since they correspond to equivariant maps of the homogeneous space G/K into affine Hilbert spaces (cf. [6]).

In the 1990s, G. Olshanski introduced a generalization of the notion of a Gelfand pair as follows (cf. [10], section 23.5): let $\left(G_n, K_n\right)_{n\geq 1}$ be a sequence of Gelfand pairs such that G_n is a locally compact topological group which is also a closed subgroup of G_{n+1}, and let K_n be a closed subgroup of K_{n+1} such that $K_n = K_{n+1} \cap G_n$. The family of Gelfand pairs $\left(G_n, K_n\right)_{n\geq 1}$, equipped with the system of continuous canonical embeddings from G_i to G_j with $i \leq j$, constitutes a countable direct system of topological groups. Hence, one can define the following groups:

$$G_\infty = \bigcup_{n\geq 1} G_n \quad \text{and} \quad K_\infty = \bigcup_{n\geq 1} K_n.$$

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The group G_∞ is equipped with the direct limit topology. It is the finest topology in which all canonical embeddings from G_n into G_∞ are continuous. Olshanski proved that the pair (G_∞, K_∞) is spherical (cf. [10, Theorem 23.6]). Hence, one can introduce the following definition: let $\big(G_n, K_n\big)_{n\geq 1}$ be an increasing sequence of Gelfand pairs as above. Then, the direct limit pair (G_∞, K_∞) is called an *Olshanski spherical pair*. It should be noticed that G_∞ is Hausdorff but not locally compact in general (cf. [16, Proposition 2.8]). In this new framework, many results concerning continuous negative definite functions were obtained (cf. [4] and [13, 14, 15]).

In the present paper, we continue the study of negative definite functions in the framework of Olshanski spherical pairs. In fact, we investigate the connection between the boundedness of negative definite functions and Kazhdan's property (T). This property was initially introduced by Kazhdan in [9] for locally compact topological groups in order to prove that a large class of lattices are finitely generated. Then, it was defined in terms of unitary representations using only a limited representation theory background. It is well known that Kazhdan's property (T) implies the boundedness of every negative definite function on Hausdorff topological groups. The converse was stated for σ-compact locally compact topological groups (cf. [2, Theorem 2.12.4]), but the given proof can still be used for countable direct limits of such groups. Our contribution consists in proving that the infinite dimensional special linear group $SL_\infty(\mathbb{F})$ has Kazhdan's property (T), where $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$ or $\mathbb{H}$ is the quaternion field. This enables us to deduce a generalization of the previous result proved in [14, Theorem 4.2]:

Theorem 1. *Let K_∞ be a closed, countable direct limit subgroup of $SL_\infty(\mathbb{F})$ as above. Then, every K_∞-biinvariant, continuous negative definite function on $SL_\infty(\mathbb{F})$ is bounded and has the following integral representation:*

$$\psi(g) = \psi(e_\infty) + \int_\Omega (1 - \omega(g))\, \nu(d\omega),$$

where e_∞ is the neutral element of $SL_\infty(\mathbb{F})$, and ν is a unique bounded positive measure on the spherical dual Ω of $SL_\infty(\mathbb{F})$.

2. Background and preliminary results

We start with some expository material related to Kazhdan's property (T) and negative definite functions. Throughout this section, the topological group G is supposed to be Hausdorff with a neutral element e, K a closed subgroup of G, and π a continuous unitary representation.

Definition 1. *Let $(\pi, \mathcal{H})$ be a unitary representation of a topological group G.*

1. *For a subset Q of G and a real number $\varepsilon > 0$, a unit vector $\xi \in \mathcal{H}$ is said to be (Q, ε)-invariant if $\sup_{g\in Q} ||\pi(g)\xi - \xi|| < \varepsilon$.*

2. *The representation $(\pi, \mathcal{H})$ is said to have almost G-invariant vectors if it has (Q, ε)-invariant vectors, for every compact subset Q of G, and every $\varepsilon > 0$.*

3. *The representation $(\pi, \mathcal{H})$ is said to have non-zero G-invariant vectors if there exists $\xi \neq 0$ in $\mathcal{H}$ such that $\pi(g)\xi = \xi$, for all $g \in G$.*

Remark 1. *Let $(\pi, \mathcal{H})$ be a unitary representation of G which has almost G-invariant vectors. Then there exists a net of unit vectors $(\xi_i)_{i\in I}$ in the Hilbert space $\mathcal{H}$ such that $\lim_i ||\pi(g)\xi_i - \xi_i|| = 0$, uniformly on compact subsets of G.*

A topological group G has Kazhdan's property (T) if there exist a compact subset Q of G and $\varepsilon > 0$ such that, whenever a unitary representation π of G has a (Q, ε)-invariant vector, then π has a non-zero G-invariant vector (cf. [2, Definition 1.1.3]). Using Proposition 1.2.1 in [2], one gets the following equivalent definition in terms of almost G-invariant vectors:

Definition 2. *A topological group G has Kazhdan's property (T) if, whenever a unitary representation π of G has almost G-invariant vectors, π has non-zero G-invariant vectors.*

Definition 3. *A function $\varphi : G \longrightarrow \mathbb{C}$ is said to be positive definite if it holds that $\sum_{i,j=1}^n c_i \overline{c_j} \varphi(g_j^{-1} g_i) \geq 0$, for all $g_1, g_2, \ldots, g_n \in G$ and all $c_1, c_2, \ldots, c_n \in \mathbb{C}$.*

Given a continuous positive definite function φ on a Hausdorff topological group G with a neutral element e, it should be noted that φ satisfies $\varphi(g^{-1}) = \overline{\varphi(g)}$, and $|\varphi(g)| \leq \varphi(e)$, for every $g \in G$. Moreover, if K is a closed subgroup of G, then φ is said to be K-biinvariant if it satisfies $\varphi(k_1 g k_2) = \varphi(g)$, for all $k_1, k_2 \in K$ and all $g \in G$. For a unitary representation $(\pi, \mathcal{H})$ of the group G, we denote by $\mathcal{H}^K$ the subspace of K-invariant vectors in $\mathcal{H}$. Besides, for every vector ξ in $\mathcal{H}$, the complex-valued function defined by $\varphi_{\pi,\xi}(g) = \langle \pi(g)\xi, \xi \rangle$ is continuous and positive definite on G. It is called a positive definite function associated to π and ξ. Remark that if $\xi \in \mathcal{H}^K$, then $\varphi_{\pi,\xi}$ is K-biinvariant. Conversely, every K-biinvariant continuous positive definite function φ on G can be represented by a unique triple (up to a unitary equivalence) $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$, where π_φ is a unitary representation of G in $\mathcal{H}_\varphi$ and ξ_φ is a cyclic vector in $\mathcal{H}_\varphi$ satisfying, $\varphi(g) = \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle$, for every $g \in G$. The previous triple is called the *GNS triple* associated to φ.

Definition 4. *A function ψ, defined on G, with complex values is said to be negative definite if $\psi(e) \geq 0$, $\psi(g^{-1}) = \overline{\psi(g)}$, and it holds that $\sum_{i,j=1}^n c_i \overline{c_j} \psi(g_j^{-1} g_i) \leq 0$, for all $g_1, \ldots, g_n \in G$ and all $c_1, \ldots, c_n \in \mathbb{C}$ such that $c_1 + \cdots + c_n = 0$.*

A continuous negative definite function ψ on G is said to be normalized if $\psi(e) = 0$. If ψ is K-biinvariant, then $\psi - \psi(e)$ is also a K-biinvariant, continuous negative definite function on G, and hence one can assume that $\psi(e) = 0$.

Remark 2. *Given a unitary representation $(\pi, \mathcal{H})$ of G and a vector $\xi \in \mathcal{H}$, the function $g \longmapsto ||\pi(g)\xi - \xi||^2$ is continuous negative definite on G. This follows from the fact that $||\pi(g)\xi - \xi||^2 = 2||\xi||^2 - 2\mathrm{Re}\,\langle \pi(g)\xi, \xi \rangle$.*

An important characterization of bounded, continuous negative definite functions on G is given in [2, Corollary 2.10.3] as follows:

Theorem 2. *Let ψ be a normalized, real-valued, continuous negative definite function on G. Then ψ is bounded on G if and only if there exists a continuous positive definite function φ on G, such that $\psi(g) = \varphi(e) - \varphi(g)$, for every $g \in G$.*

Remark 3. *One can obviously see that* $\mathrm{Re}\,(\psi(g)-\psi(e)) \geq 0$, *for every $g \in G$. Moreover, the real-valued function $\psi_1(g) = \mathrm{Re}\,(\psi(g) - \psi(e))$ is negative definite, and ψ is bounded on G if and only if the same holds for ψ_1 (cf. [3, Section 7.14]). Therefore, the previous characterization can be extended to all complex-valued negative definite functions on G.*

Definition 5. *Let $\mathcal{H}$ be an affine real Hilbert space,* $\mathrm{Is}(\mathcal{H})$ *the group of affine isometries of $\mathcal{H}$, and G a Hausdorff topological group. An affine isometric action of G on $\mathcal{H}$ is a strongly continuous group homomorphism α from G to* $\mathrm{Is}(\mathcal{H})$.

Remark 4 (cf. [2, Proposition 2.10.2])**.** *A topological group G has property (FH) if every affine isometric action of G on a real Hilbert space has an invariant vector. Given a real-valued, normalized negative definite function ψ on G, there exist a real Hilbert space $\mathcal{H}_\psi$, and an affine isometric action α_ψ of G on $\mathcal{H}_\psi$ such that $\psi(g) = ||\alpha_\psi(g)0||_\psi^2$, for every $g \in G$.*

The characterization of bounded negative definite functions by means of Kazhdan's property (T) was established in 1977 for locally compact topological groups by Guichardet [7] and Delorme [5]. It was later proved in 1981 using another method by Akemann and Walter [1]. In [2, Theorem 2.10.4 and Theorem 2.12.4], it is proved that Kazhdan's property (T) implies property (FH) on Hausdorff topological groups or equivalently that every negative definite function is bounded on such groups. This result is stated in Theorem 3 below.

Theorem 3. *Let G be a Hausdorff topological group, and ψ a real-valued, continuous negative definite function on G. If G has Kazhdan's property (T), then ψ is bounded on G.*

Proposition 1. *Let G be a Hausdorff topological group, and $(\pi, \mathcal{H})$ a unitary representation of G having a unit vector $\xi \in \mathcal{H}$ such that there exists $\alpha \in \,]0, 1[$ satisfying $\sup_{g\in G} ||\pi(g)\xi - \xi|| < \alpha$. Then, the unitary representation π has a non-zero G-invariant vector.*

Proof. The Bruhat-Tits fixed point theorem implies that for every affine isometric action of G on a Hilbert space, every closed bounded invariant convex subset contains a fixed point. In our case, the orbit $\pi(G)\xi$ is contained in the closed α-ball around ξ which does not contain 0. Hence, the existence of a fixed point follows immediately from the Bruhat-Tits theorem. □

Remark 5. *If $(G_n)_n$ is an increasing sequence of σ-compact groups, then the direct limit group G_∞ is also σ-compact. In fact, for every n, there exists an increasing sequence $(Q_{n,p})_p$ of compact subsets of G_n such that $G_n = \bigcup_p Q_{n,p}$. It follows that $G_\infty = \bigcup_{p=1}^{+\infty} K_p$, where $K_p = \bigcup_{n=1}^{p} Q_{n,p}$.*

The converse statement of Theorem 3 was stated in terms of property (FH) for σ-compact locally compact topological groups (cf. [2, Theorem 2.12.4 and Proposition

2.4.5], but the given proof can be adapted for direct limits of such groups. For the sake of completeness, we present the proof in terms of negative definite functions.

Theorem 4. *Let G_∞ be the countable direct limit of an increasing sequence of σ-compact, locally compact groups $(G_n)_n$. Then, G_∞ has property (T) if and only if every continuous negative definite function is bounded on G_∞.*

Proof. The necessary condition is an immediate consequence of Theorem 3 and Remark 3. As for the sufficient condition, by Remark 5, there exists an increasing sequence $(K_p)_p$ of compact subsets of G_∞ such that $G_\infty = \bigcup_p K_p$. If G_∞ does not have Kazhdan's property (T), then there exists a unitary representation $(\pi, \mathcal{H})$ of G_∞ which has almost G_∞-invariant vectors but does not have non-zero G_∞-invariant ones. By Remark 1, it follows that for every $p \in \mathbb{N}$, there exists a unit vector $\xi_p \in \mathcal{H}$ such that $||\pi(g)\xi_p - \xi_p|| < \frac{1}{2^p}$, for every $g \in K_p$. Thus, for every $g \in G_\infty$, there exists $p_0 \in \mathbb{N}$, such that $g \in K_{p_0}$, and so $||\pi(g)\xi_p - \xi_p||^2 < \frac{1}{4^p}$, for every $p \geq p_0$. As a result,

$$\sum_{p=p_0}^{+\infty} 2^p \, ||\pi(g)\xi_p - \xi_p||^2 \leq \sum_{p=p_0}^{+\infty} \frac{1}{2^p} < +\infty.$$

In consequence, the series $\sum_{p\geq 1} 2^p \, ||\pi(g)\xi_p - \xi_p||^2$ is uniformly convergent on G_∞. Hence, the function

$$\psi(g) = \sum_{p=1}^{\infty} 2^p \, ||\pi(g)\xi_p - \xi_p||^2,$$

is well-defined, continuous and negative definite on G_∞. Moreover, as π does not have non-zero G_∞-invariant vectors, by Proposition 1, there exists a sequence $(g_p)_p$ in G_∞ such that $||\pi(g_p)\xi_p - \xi_p|| \geq \frac{1}{2}$, for every $p \in \mathbb{N}$. Therefore, for every $p \in \mathbb{N}$,

$$\psi(g_p) \geq 2^p \, ||\pi(g_p)\xi_p - \xi_p|| \geq 2^{p-1}.$$

It follows that

$$\lim_{p\to+\infty} \psi(g_p) = +\infty,$$

and hence the negative definite function ψ is unbounded on G_∞. □

3. Negative definite functions on $SL_\infty(\mathbb{F})$

In this section, we prove that the infinite dimensional special linear group $SL_\infty(\mathbb{F})$ has Kazhdan's property (T). Then, we deduce the boundedness of every continuous negative definite function on this group. Moreover, we derive a geometric property concerning continuous affine isometric actions of $SL_\infty(\mathbb{F})$. This generalizes the previous result proved by the author in [14] for a related Olshanski spherical pair.

Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ be the quaternion field, and let $SL_n(\mathbb{F})$ be the special linear group of rank n with $n = 3, 4, \ldots$. For every n, we canonically embed $SL_{n-1}(\mathbb{F})$ into $SL_n(\mathbb{F})$ using the mapping:

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding countable direct limit group is

$$SL_\infty(\mathbb{F}) = \bigcup_{n\geq 3} SL_n(\mathbb{F}).$$

Given an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{F}^n$, the previous embedding is also realized by identifying $SL_{n-1}(\mathbb{F})$ with the subgroup in $SL_n(\mathbb{F})$ fixing the n^{th}-basis vector u_n. We regard the infinite dimensional special linear group $SL_\infty(\mathbb{F})$ as the group of infinite unimodular invertible matrices with finitely many entries distinct from the Kronecker delta. For every n and every $1 \leq i \neq j \leq n$, an elementary matrix $e_{ij}(x) \in SL_n(\mathbb{F})$ is defined by

$$e_{ij}(x) = I_n + x\,\delta_{ij},$$

where I_n is the identity matrix of rank n, and δ_{ij} is the square matrix of rank n with one at the (i, j)-entry and 0 elsewhere.

Lemma 1 (see [8, § 6.7, Lemma 1]). *The group $SL_n(\mathbb{F})$ is generated by the set of all elementary matrices $e_{ij}(x)$ for $x \in \mathbb{F}$ and $1 \leq i \neq j \leq n$.*

The following proposition is crucial since it ensures the existence of invariant vectors for any unitary representation of the infinite dimensional special linear group $SL_\infty(\mathbb{F})$.

Proposition 2. *Let $(\pi, \mathcal{H})$ be a unitary representation of $SL_n(\mathbb{F})$, and ξ a $SL_{n-1}(\mathbb{F})$-invariant vector. Then ξ is $SL_n(\mathbb{F})$-invariant.*

Proof. Let us assume that ξ is $SL_{n-1}(\mathbb{F})$-invariant with respect to $(\pi, \mathcal{H})$. Using the fact that for every $y \in \mathbb{F}$ and every $i, j \in \{1, \ldots, n-1\}$ with $i \neq j$,

$$e_{ij}(y) = e_{in}(y)\, e_{nj}(1)\, e_{in}(-y)\, e_{nj}(-1),$$

it follows by Lemma 1 that it is sufficient to prove the invariance of ξ by all elementary matrices $e_{in}(x)$ and $e_{ni}(x)$ for $x \in \mathbb{F}$ and $i \in \{1, \ldots, n-1\}$. Let $x \in \mathbb{F}$ be fixed and let $(\lambda_k)_k$ be a sequence in $\mathbb{F} \setminus \{0\}$ which converges to 0. Let us fix $i \in \{1, \ldots, n-1\}$. Choose $\ell \in \{1, \ldots, n-1\} \setminus \{i\}$, and define the diagonal matrix g_k in $SL_{n-1}(\mathbb{F})$ as follows: $g_k u_i = \lambda_k u_i$, $g_k u_\ell = \lambda_k^{-1} u_\ell$ and $g_k u_j = u_j$, for every $j \in \{1, \ldots, n\} \setminus \{i, \ell\}$. Then,

$$g_k\, e_{in}(x)\, g_k^{-1} = e_{in}(\lambda_k x), \quad \text{and} \quad g_k^{-1} e_{ni}(x)\, g_k = e_{ni}(\lambda_k x).$$

Since ξ is $SL_{n-1}(\mathbb{F})$-invariant, it follows that

$$\begin{aligned} ||\pi(e_{in}(x))\xi - \xi|| &= ||\pi(e_{in}(x))\pi(g_k^{-1})\xi - \pi(g_k^{-1})\xi|| \\ &= ||\pi(g_k)\pi(e_{in}(x))\pi(g_k^{-1})\xi - \xi|| \\ &= ||\pi(e_{in}(\lambda_k x))\xi - \xi||. \end{aligned}$$

Given that π is strongly continuous, and $(e_{in}(\lambda_k x))_k$ converges to the identity operator I_n, one concludes that $\pi(e_{in}(x))\xi = \xi$. In the same way as above, one can prove that $\pi(e_{ni}(x))\xi = \xi$. □

It is well known that for every $n \geq 3$, the finite dimensional, non-compact Lie group $SL_n(\mathbb{F})$ has Kazhdan's property (T) (cf. [9]). In what follows we prove that the same holds for the countable direct limit group $SL_\infty(\mathbb{F})$.

Theorem 5. *The direct limit group $SL_\infty(\mathbb{F})$ has Kazhdan's property (T).*

Proof. Let π be a unitary representation of $SL_\infty(\mathbb{F})$ in a Hilbert space $\mathcal{H}$, and assume that π has almost $SL_\infty(\mathbb{F})$-invariant vectors. Since for every $n \geq 3$, $SL_n(\mathbb{F})$ is a closed subgroup of $SL_\infty(\mathbb{F})$, the representation π remains a continuous unitary representation of $SL_n(\mathbb{F})$ in $\mathcal{H}$. On the other hand, as $SL_3(\mathbb{F})$ has Kazhdan's property (T), the unitary representation π has a non-zero $SL_3(\mathbb{F})$-invariant vector $\xi_0 \in \mathcal{H}$. It follows by Proposition 2 that ξ_0 is $SL_n(\mathbb{F})$-invariant for every $n \geq 3$. Hence, ξ_0 is $SL_\infty(\mathbb{F})$-invariant. This proves that the direct limit group $SL_\infty(\mathbb{F})$ has Kazhdan's property (T). $\square$

The following result follows from Theorem 5 and Theorem 4.

Corollary 1. *Every continuous negative definite function on $SL_\infty(\mathbb{F})$ is bounded.*

For every n, let K_n be a compact subgroup of $SL_n(\mathbb{F})$ such that $(SL_n(\mathbb{F}), K_n)$ is a Gelfand pair. Furthermore, let us assume that for every n, K_n is a closed subgroup of K_{n+1} such that $K_n = K_{n+1} \cap SL_n(\mathbb{F})$. Hence, the countable direct limit subgroup K_∞ is well-defined, and is closed in $SL_\infty(\mathbb{F})$. Moreover, the resulting pair $(SL_\infty(\mathbb{F}), K_\infty)$ is an Olshanski spherical pair. By [11, Theorem 3.15], every K_∞-biinvariant, continuous positive definite function on $SL_\infty(\mathbb{F})$ has a unique integral representation via a bounded positive measure μ on the spherical dual Ω of $SL_\infty(\mathbb{F})$:

$$\varphi(g) = \int_\Omega \omega(g)\, \mu(d\omega).$$

The following result is an immediate consequence of the preceding integral representation, Corollary 1, and Theorem 2.

Theorem 6. *Let ψ be a K_∞-biinvariant, continuous negative definite function on $SL_\infty(\mathbb{F})$. Then ψ is bounded, and has the following integral representation:*

$$\psi(g) = \psi(e_\infty) + \int_\Omega (1 - \omega(g))\, \nu(d\omega),$$

where e_∞ is a neutral element of $SL_\infty(\mathbb{F})$, and ν is a unique bounded positive measure on Ω.

Remark 6. *Theorem 6 generalizes the integral representation of negative definite functions given in [14] for the Olshanski spherical pair $(SL_\infty(\mathbb{C}), SU_\infty(\mathbb{C}))$.*

In the context of geometric group theory, a real-valued, normalized negative definite function ψ can be written as $\psi(g) = ||\alpha(g)0||^2$, where α is an affine isometric action on a real Hilbert space $\mathcal{H}$ (see Remark 4). In this context, the boundedness of ψ is equivalent to the existence of an invariant vector for the affine isometric action α (cf. [2, Corollary 2.10.3]).

The following geometric conclusion can be derived from Theorem 6.

Corollary 2. *Every affine isometric action of $SL_\infty(\mathbb{F})$ on a real Hilbert space with a K_∞-invariant vector also has an $SL_\infty(\mathbb{F})$-invariant vector.*

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