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Explicit forms for three integrals in Wand et al.*

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Abstract. We derive explicit forms for the three integrals used in Kim and Wand [3] and Wand, Ormerody, Padoan and Frühwirth [7]. The explicit forms involve known special functions for which in-built routines are available.
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1. Introduction

Article [3] gave an explicit form of expectation propagation for a simple statistical model, while [7] studied mean field variational Bayes for elaborate distributions. Their studies involved the following three integrals. The first integral is

$$\mathcal{A}(p,q,r,s,t,u) = \int_{\mathbb{R}} \frac{x^p\ \exp\left\{qx - rx^2\right\}}{\left(t+sx+x^2\right)^u}\ \mathrm{d}x \tag{1}$$

for $p \geq 0, q, s \in \mathbb{R}$, $r, u > 0$ and $s^2 < 4t$; see equation (2.1) on page 552 of [3]. The second integral is

$$\mathcal{B}(p,q,r,s,t,u) = \int_{\mathbb{R}} \frac{x^p\ \exp\left\{qx - r\mathrm{e}^x - s\ \mathrm{e}^x/\left(t+\mathrm{e}^x\right)\right\}}{\left(t+\mathrm{e}^x\right)^u}\ \mathrm{d}x \tag{2}$$

for $p, s \geq 0, q \in \mathbb{R}$ and $r, t, u > 0$; see equation (2.1) on page 552 of [3]. The third integral is

$$\mathcal{I}(p,q,r,s) = \int_{\mathbb{R}} x^p\ \exp\left\{qx - rx^2 - s\ \mathrm{e}^{-x}\right\}\ \mathrm{d}x \tag{3}$$

for $p \geq 0, q \in \mathbb{R}$ and $s, r > 0$; see page 851 in [7].

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Both publications [3] and [7] provided explicit forms for none of the three integrals. In this note, we provide explicit forms for all three integrals. The forms involve known special functions and in-built routines for computing them are available in the literature.

The organization of this note is the following. Section 2 gives two explicit forms for (1). Section 3 gives one explicit form for (2). Section 4 gives two explicit forms for (3).

2. The integral $\mathcal{A}$

We present an approach in which we use the Grünwald-Letnikov fractional derivative. The Grünwald-Letnikov fractional derivative of order ν with respect to the argument x of a suitable function f is defined by [6]

$$\mathbb{D}_x^\nu[f] := \lim_{h\downarrow 0} \frac{1}{h^\nu} \sum_{m=0}^\infty (-1)^m \binom{\nu}{m} f\left(x+(\nu-m)h\right),$$

where $h \downarrow 0$ means that in approaching zero h remains positive. As is well-known (see, for example, [4]), the Grünwald-Letnikov fractional derivative $\mathbb{D}_x^\nu$ of order ν of the exponential function is

$$\mathbb{D}_x^\nu\left[\mathrm{e}^{\alpha x}\right] = \alpha^\nu \mathrm{e}^{\alpha x}. \tag{4}$$

Firstly, consider the well-known integral

$$\mathscr{I}(\alpha,\beta) = \int_{\mathbb{R}} \mathrm{e}^{\alpha x-\beta x^2}\ \mathrm{d}x = \sqrt{\frac{\pi}{\beta}}\ \exp\left\{\frac{\alpha^2}{4\beta}\right\}, \qquad \Re(\beta)>0.$$

Obviously, we have

$$\mathbb{D}_\alpha^p\left[\mathscr{I}(\alpha,\beta)\right] = \int_{\mathbb{R}} x^p \mathrm{e}^{\alpha x-\beta x^2}\ \mathrm{d}x.$$

On the other hand,

$$\mathbb{D}_\alpha^p\left[\mathscr{I}(\alpha,\beta)\right] = \int_0^\infty x^p \mathrm{e}^{\alpha x-\beta x^2}\ \mathrm{d}x + \int_{-\infty}^0 x^p \mathrm{e}^{\alpha x-\beta x^2}\ \mathrm{d}x =: I^+ + I^-.$$

Using equation (13), page 313 of [1],

$$I^+ = (2\beta)^{-\frac{p+1}{2}}\ \Gamma(p+1)\ \exp\left\{\frac{\alpha^2}{8\beta}\right\}\ D_{-p-1}\left(-\frac{\alpha}{\sqrt{2\beta}}\right),$$

where $D_\mu(\cdot)$ denotes the parabolic cylinder function of order μ (see, for example, [2]), and the constraint $p+1>0$ should be satisfied (which is definitely a weaker assumption than the assumed $p>0$ by Kim and Wand [3]). Accordingly, we have

$$I^- = (-1)^p(2\beta)^{-\frac{p+1}{2}}\ \Gamma(p+1)\ \exp\left\{\frac{\alpha^2}{8\beta}\right\}\ D_{-p-1}\left(\frac{\alpha}{\sqrt{2\beta}}\right).$$

Therefore,

$$\mathbb{D}^p_\alpha\left[\mathscr{I}(\alpha,\beta)\right] = (2\beta)^{-\frac{p+1}{2}}\,\Gamma(p+1)\,\exp\left\{\frac{\alpha^2}{8\beta}\right\}$$
$$\times\left[D_{-p-1}\left(-\frac{\alpha}{\sqrt{2\beta}}\right) + (-1)^p D_{-p-1}\left(\frac{\alpha}{\sqrt{2\beta}}\right)\right].$$

Now, introduce a parameter $a>0$ and specify $\alpha = q - as \in \mathbb{R}$ and $\beta = r + a$, the latter evidently positive; therefore nothing harms the assumption on the parameter space of (p,q,r,s,t). Considering now the integral

$$\mathrm{e}^{-at}\mathbb{D}^p_{q-as}\left[\mathscr{I}(q-as,r+a)\right] = \int_{\mathbb{R}} x^p \mathrm{e}^{qx-rx^2-a\left(t+sx+x^2\right)}\,\mathrm{d}x,$$

we conclude that

$$\mathcal{A}(p,q,r,s,t,u) = (-1)^u\,\mathbb{D}^{-u}_a\left[\mathrm{e}^{-at}\mathbb{D}^p_{q-as}\left[\mathscr{I}(q-as,r+a)\right]\right]\Big|_{a=0}.$$

This formula proves the following result.

Proposition 1. *For all $p\geq 0, q,s\in\mathbb{R}$, $r,u>0$ and $s^2<4t$, we have*

$$\mathcal{A}(p,q,r,s,t,u) = \mathrm{e}^{\mathrm{i}\pi u}\,2^{-\frac{p+1}{2}}\,\Gamma(p+1)\,\lim_{a\downarrow 0}\mathbb{D}^{-u}_a\Bigg[(r+a)^{-\frac{p+1}{2}}\,\exp\left\{\frac{(q-as)^2}{8(r+a)} - at\right\}$$
$$\times\left\{D_{-p-1}\left(\frac{as-q}{\sqrt{2(r+a)}}\right) + \mathrm{e}^{\mathrm{i}\pi p}\,D_{-p-1}\left(\frac{q-as}{\sqrt{2(r+a)}}\right)\right\}\Bigg].$$

We now present another approach to calculating integral $\mathcal{A}$. Kummer's (or confluent hypergeometric) function series definition is

$${}_1F_1(a,c,z) = \sum_{n\geq 0}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}.$$

The parabolic cylinder function D_ν is expressible in terms of the Tricomi confluent hypergeometric function, *viz.*

$$U(a;c;z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\,{}_1F_1(a;c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\,{}_1F_1(1+a-c;2-c;z),$$

as (see equations (2) and (4) on page 117 of [2])

$$D_\nu(z) = 2^{\frac{\nu}{2}}\,\mathrm{e}^{-\frac{z^2}{4}}\,U\left(-\frac{\nu}{2},\frac{1}{2},\frac{z^2}{2}\right) = 2^{\frac{\nu-1}{2}}\,z\,\mathrm{e}^{-\frac{z^2}{4}}\,U\left(\frac{1-\nu}{2},\frac{3}{2},\frac{z^2}{2}\right), \qquad (5)$$

where in both cases $-\pi < 2\arg(z) \leq \pi$. Applying the first formula in (5), we obtain

Proposition 2. *For the same parameter space as in the previous proposition, we have*

$$\mathcal{A}(p,q,r,s,t,u) = 2^{-(p+1)}\,\mathrm{e}^{\mathrm{i}\pi u}\,\left(1+\mathrm{e}^{\mathrm{i}\pi p}\right)\,\Gamma(p+1)$$
$$\times\lim_{a\downarrow 0}\mathbb{D}^{-u}_a\left[\frac{\mathrm{e}^{-at}}{(r+a)^{\frac{p+1}{2}}}\,U\left(\frac{p+1}{2},\frac{1}{2},\frac{(q-as)^2}{4(r+a)}\right)\right].$$

3. The integral $\mathcal{B}$

We apply again the Grünwald-Letnikov fractional derivative (4) of the exponential function. Firstly, we eliminate the denominator and the power x^p in the integrand, namely,

$$\mathcal{B}(p,q,r,s,t,u) = (-1)^u\ \mathbb{D}_s^u \mathbb{D}_{q-u}^p \left[\int_{\mathbb{R}} \exp\left\{ (q-u)x - r\mathrm{e}^x - \frac{s\ \mathrm{e}^x}{t+\mathrm{e}^x} \right\}\ \mathrm{d}x \right].$$

Now, we transform the inner integral $I(s)$ and use the Maclaurin expansion with respect to s of the appropriate exponential term to obtain

$$\begin{aligned}
&\mathcal{B}(p,q,r,s,t,u)\\
&\quad= (-1)^u\ \mathbb{D}_s^u \left[\mathrm{e}^{-s}\ \mathbb{D}_{q-u}^p \left[\int_{\mathbb{R}} \exp\left\{ (q-u)x - r\mathrm{e}^x + \frac{s\ t}{t+\mathrm{e}^x} \right\}\ \mathrm{d}x \right]\right]\\
&\quad= (-1)^u\ \mathbb{D}_s^u \left[\mathrm{e}^{-s}\ \mathbb{D}_{q-u}^p \left[\sum_{n\ge 0} \frac{(st)^n}{n!} \int_{\mathbb{R}} \frac{\mathrm{e}^{(q-u)x - r\mathrm{e}^x}}{(t+\mathrm{e}^x)^n}\ \mathrm{d}x \right]\right]\\
&\quad= (-1)^u \sum_{n\ge 0} \frac{t^n}{n!}\ \mathbb{D}_s^u \left[\mathrm{e}^{-s}\ s^n\ \mathbb{D}_{q-u}^p \left[\int_{\mathbb{R}_+} \frac{y^{q-u-1}\mathrm{e}^{-ry}}{(t+y)^n}\ \mathrm{d}y \right]\right]\\
&\quad= (-1)^u \sum_{n\ge 0} \frac{1}{n!}\ \mathbb{D}_s^u \Big[\mathrm{e}^{-s}\ s^n\ \mathbb{D}_w^p \big[\Gamma(w)\ t^w\ U(w, w+1-n, r\ t) \big]_{w=q-u} \Big], \qquad (6)
\end{aligned}$$

where in (6) the Laplace transform formula (see equation (2.1.3.1) on page 18 of [5])

$$\int_{\mathbb{R}_+} \frac{\mathrm{e}^{-px}\ x^{\alpha-1}}{(x+z)^\rho}\ \mathrm{d}x = \Gamma(\alpha)\ z^{\alpha-\rho}\ U(\alpha, \alpha+1-\rho, p\ z)$$

was used, which holds for all $\Re(\alpha) > 0$, $\Re(p) > 0$ and $|\arg(z)| < \pi$. This proves the following result.

Proposition 3. *For all $p, s \ge 0, q \in \mathbb{R}$, $r, t, u > 0$ and $q > -u$, we have*

$$\begin{aligned}
\mathcal{B}(p,q,r,s,t,u) = \mathrm{e}^{\mathrm{i}\pi u} \sum_{n\ge 0} \frac{1}{n!} \mathbb{D}_s^u \Big[\mathrm{e}^{-s}\ s^n \\
\times \mathbb{D}_{q-u}^p \big[\Gamma(q-u)\ t^{q-u}\ U(q-u, q-u+1-n, r\ t) \big] \Big]. \qquad (7)
\end{aligned}$$

Unfortunately, our method holds true for $q - u > 0$ only since $\Gamma(q-u)$ in (7).

4. The integral $\mathcal{I}$

This time it is enough to split the integration domain into positive and negative reals and take the Maclaurin expansion in both sub-integrals of the exponential expression $\exp\{-s\mathrm{e}^{-x}\}$. We obtain the following

Proposition 4. *For all $p \geq 0, q \in \mathbb{R}$, $s, r > 0$, we have*

$$\mathcal{I}(p,q,r,s) = \frac{\Gamma(p+1)}{(2r)^{\frac{p+1}{2}}} \sum_{n\geq 0} \frac{(-s)^n}{n!} \left\{ D_{-p-1}\left(\frac{n-q}{\sqrt{2r}}\right) + \mathrm{e}^{\mathrm{i}\pi p}\, D_{-p-1}\left(\frac{q-n}{\sqrt{2r}}\right)\right\}. \quad (8)$$

Moreover, the following computable series representation holds:

$$\mathcal{I}(p,q,r,s) = \frac{\Gamma(p+1)}{(4r)^{\frac{p+1}{2}}} \left(1+\mathrm{e}^{\mathrm{i}\pi p}\right) \sum_{n\geq 0} \frac{(-s)^n}{n!}\, U\left(\frac{p+1}{2}, \frac{1}{2}, \frac{(q-n)^2}{4r}\right).$$

Expression (8) could be deduced by some aspects of the discussion on pages 851-852 of [7]. However, there the authors' approach to quadratures for $I(p,q,r,s)$ was completely different.

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References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of Integral Transforms*, volume I, McGraw-Hill, New York, 1954. (Based, in part, on notes left by Harry Bateman).

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, volume II, McGraw-Hill, New York, 1954. (Based, in part, on notes left by Harry Bateman).

[3] A. S. I. Kim, M. P. Wand, *The explicit form of expectation propagation for a simple statistical model*, Electron. J. Stat. **10**(2016), 550–581.

[4] M. D. Ortigueira, *A coherent approach to non-integer order derivatives*, Signal Process. **86**(2006), 2505–2515.

[5] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and series. Direct Laplace transforms*, volume 4, Gordon and Breach, New York, 1992.

[6] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, New York, 1993.

[7] M. P. Wand, J. T. Ormerody, S. A. Padoan, R. Frühwirth, *Mean field variational Bayes for elaborate distributions*, Bayesian Anal. **6**(2011), 847–900.