MATHEMATICAL COMMUNICATIONS
Math. Commun. **26**(2021), 107–119

Approximate optimal control of Volterra-Fredholm integral equations based on parametrization and variational iteration method

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Received June 28, 2019; accepted December 16, 2020

Abstract. This article presents appropriate hybrid methods for solving optimal control problems ruled by Volterra-Fredholm integral equations. The techniques are grounded on variational iteration together with a shooting method like procedure and parametrization methods for resolving optimal control problems ruled by Volterra-Fredholm integral equations. The resulting value shows that the proposed method is trustworthy, able to provide analytic treatment that clarifies such equations and usable for a large class of nonlinear optimal control problems governed by integral equations.

AMS subject classifications: 49J21, 65R20

Key words: optimal control problems, Volterra-Fredholm integral equations, modified variational iteration method, parametrization methods

1. Introduction

Optimal control problems ruled by integral equations appear in different scientific fields, for instance, in 1976, Kamien and Muller designed the capital replacement problem by an optimal control problem with an integral state equation [14]. Since finding solutions for such issues is highly important, scholars have paid attention to these matters. Nevertheless, there are several practical dilemmas that are more complicated to be solved analytically. Therefore, it is necessary to present a new computational strategy to defeat these dilemmas.

In articles [4, 5, 18, 23], some numerical methods are presented to solve optimal control problems described by integral equations. Recently numerical solutions based on perturbation and parametrization [6], homotopy analysis and parametrization[3], Legendre polynomials [26], Bernstein Polynomials [24], sinc wavelet and parametrization [15], triangular functions [16] and the hybrid of block pulse functions and Legendre polynomials [17], efficient Chebyshev collocation methods [28], a relaxation approach [13], hybrid functions of Bernstein polynomials and block-pulse functions

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[21], have been used to solve optimal control problems governed by integral equations.

The variational iteration method (VIM) was suggested by He ([9]-[11]) as a revision of an overall Lagrange multiplier method [12]. The VIM method is an iteration method that includes a generic Lagrange multiplier. The value of the generic Lagrange multiplier is chosen using variational theory. The VIM method has been extensively used to solve various issues, and there are different modifications proposed to overcome disabilities arising in the solution method ([1, 7]).

Vinokurov [27] and Medhin [19] depicted Pontryagins extremum principle to the optimization of the controlled integral equation. In those articles, the results are displayed in an intricate system involving integral and difficult terms. The intricate system, except in very particular cases, is so hard to be solved. On the other hand, in Vinokurov (1969), there are some fundamental mistakes illustrated by Neustadt in [22]. So, to compose a methodology to avoid the solution of the stated intricate system is the main purpose of all computational aspects of optimal control problems ruled by integral equation systems. One significant goal of this article is to provide an appropriate method to solve these types of problems. So, we offered a new hybrid direct method based on variational iteration [9, 10] and the parametrization method [25] for solving the optimal control problem of Volterra-Fredholm integral equations, in which the control variable is a continuous polynomial. In fact, we approximate the control variables and then using a modified version of VIM, we compute the state variables. Of course, we also calculate the state variables by VIM method and then compare the results. The control variables can be approximated by choosing an appropriate function with finitely many unknown parameters. Three examples are given to show the suitability and efficiency of the suggested method. Section 2 in this paper is devoted to the basic idea of the VIM. In Section 3, methods for solving optimal control problems with Volterra-Fredholm integral equations have been discussed. The convergence analysis has been explored in Section 4. Some test examples are given in Section 5. Finally, the conclusion is given in Section 6.

2. Variational iteration method

Consider a differential equation as follows:

$$Lv + Nv = g(t),$$

where L is a linear operator, N a non-linear operator and $g(x)$ the source inhomogeneous term. Due to the structure of the VIM method, correction functional can be constructed as follows:

$$v_{n+1} = v_n + \int_0^t \lambda(s)(Lv_n(s) + N\tilde{v}_n(s) - g(s))ds,$$

where λ is a general Lagrangian multiplier that can be characterized optimally via variational theory. The symbol n means the nth order approximation and $\tilde{v}_n$ is regarded as a restricted variation, i.e., $\delta\tilde{v}_n = 0$ [9]. The consecutive approximations $v_{n+1}(t), n \geq 0$ of the solution $v(t)$ will be easily achieved upon using the obtained

Lagrange multiplier and by applying any optional function v_0. So, the solution

$$v(t) = \lim_{n\to\infty} v_n(t).$$

3. Solving optimal control problems with Volterra-Fredholm integral equations based on the VIM

Here we present a direct hybrid method to solve optimal control problems governed by Volterra-Fredholm integral equations. To do so, we consider $z_k(t)$ as a polynomial basis, which is dense in the space of $C([0,1])$. Since the control function $u(t)$ is a continuous function, it can be approximated with the finite combination from elements of this basis. Now, consider the optimal control problem ruled by a Volterra-Fredholm integral equation as follows:

$$Minimize\, J(x,u) = \int_0^T f_0(t, x(t), u(t))dt; \tag{1}$$

then

$$x'(t) = y(t) + \int_0^t k_1(s,t,x(s),u(s))ds + \int_0^T k_2(s,t,x(s),u(s))ds, \tag{2}$$

$$x(0) = x_0, \qquad x(T) = x_T, \tag{3}$$

where $k_1, k_2 \in L^2([0,T]\times[0,T]\times R\times R)$ and $f_0 \in C([0,T]\times R\times R)$. So, without loss of generality, we assume that $T = 1$.

To respond to such problems by the proposed method, equation (2) is solved by the VIM and the control function is considered as follows:

$$u(t) = \sum_{j=0}^{k} a_j z_j(t), \tag{4}$$

where a_j defines unknown parameters, and $z_j(t)$ are some polynomial functions. Based on He's variational iteration method, the correction functional for (2) can be constructed as follows:

$$\begin{aligned} x_{m+1}(t) = x_m(t) + \int_0^t \lambda(\tau)[x_m'(\tau) - y(\tau) - \int_0^\tau k_1(s,t,x_m(s),\sum_{j=0}^{k} a_j z_j(s))ds \\ - \int_0^T k_2(s,t,x_m(s),\sum_{j=0}^{k} a_j z_j(s))ds]d\tau. \end{aligned} \tag{5}$$

By taking variation with respect to x_m and considering the restricted variations

$$\delta y = \delta k_1(s,t,\tilde{x}_m(s),\sum_{j=0}^{k} a_j z_j(s)) = \delta k_2(s,t,\tilde{x}_m(s),\sum_{j=0}^{k} a_j z_j(s)) = 0,$$

we get

$$\delta x_{m+1}(t) = \delta x_m(t) + \lambda(\tau)\delta x_m(t)|_{\tau=t} - \int_0^t \lambda'(\tau)\delta x_m(t)d\tau.$$

The following stationary condition is obtained:

$$1 + \lambda(\tau) = 0, \qquad \lambda'(\tau) = 0. \tag{6}$$

The resultant equation (6) is $\lambda(\tau) = -1$. Substituting from Lagrange multipliers into the correction functional equations (5) results in the following iterative formula:

$$\begin{aligned} x_{m+1} = x_m - \int_0^t [x'_m(\tau) - y(\tau) - \int_0^\tau k_1(s, t, x_m(s), \sum_{j=0}^k a_j z_j(s))ds \\ - \int_0^T k_2(s, t, x_m(s), \sum_{j=0}^k a_j z_j(s))ds]d\tau. \end{aligned} \tag{7}$$

And the exact solution is eventually obtained as follows:

$$x(t, a_0, \ldots, a_k) = \lim_{m\to\infty} x_m(t, a_0, \ldots, a_k). \tag{8}$$

Now by substituting (8) and (4) in (1), an approximate solution of the optimal control problem by the Volterra-Fredholm integral equation is obtained by

$$min_{(a_0,\ldots,a_k)} J_k(a_0, \ldots, a_k) = \int_0^T (f_0(t, x(t, a_0, \ldots, a_k), \sum_{j=0}^k a_j z_j(t))dt, \tag{9}$$

subject to

$$x(T, a_0, \ldots, a_k) = x_T,$$

optimization problem (9) can be substituted by the following minimization problem:

$$Minimize\,(J_k(a_0, \ldots, a_k) + (x(T, a_0, \ldots, a_k) - x_T)^2).$$

The stop criterion is considered as follows:

$$||J^*_{k+1} - J^*_k|| < \epsilon, \tag{10}$$

where J^*_k is the optimal value of (9) in the kth iteration.

Mathematica optimization Toolbox can solve optimization problem (9) for a commanded small positive number ϵ that should be chosen according to the accuracy desired.

4. Convergence of the proposed method

The convergence of the stated method has been investigated in this part. We conceptualize U as the collection of admissible control functions

$$U = \{u : [0, 1] \to W | u(.) \in C[0, 1]\},$$

where $W \subseteq R^m$ is a compact set.

Definition 1.

The pair (x,u) is called an admissible pair if it satisfies (2) and (3). We describe ξ as the set of admissible pairs. Explain ξ^m and ξ_k^m as follows:

$$\xi^m := \{(x_m(.;u), u(.))|u \in U\},$$
$$\xi_k^m := \{(x_m(.;u_k), u_k(.))|u_k \in P_k \cap U\},$$

where P_k is the set of all polynomials of degree at most k. We define

$$\alpha_k^m := inf_{(x_m,u_k)\in\xi_k^m} J(x_m,u_k), \qquad \alpha^m := inf_{(x_m,u)\in\xi^m} J(x_m,u).$$

Assumption 1. *We assume α_k^m, α^m exist for all $m,k \in N$.*

Theorem 1. *Suppose that $x_0(t) = x_0$ and the iterative sequence $\{x_m(t)\}$ obtained from (7) converges to $x(t)$; then $x(t)$ is the solution of Eq. (2).*

Proof. Taking limits in the iterative formula in (7), it follows that

$$\lim_{m\to\infty} x_{m+1} = \lim_{m\to\infty} x_m - \int_0^t \lim_{m\to\infty}[x'_m(\tau) - y(\tau)\int_0^\tau k_1(s,t,x_m(s),\sum_{j=0}^k a_j z_j(s))ds$$
$$- \int_0^T k_2(s,t,x_m(s),\sum_{j=0}^k a_j z_j(s))ds]d\tau.$$

And thus

$$\int_0^t [x'(\tau) - y(\tau) - \int_0^\tau k_1(s,t,x(s),\sum_{j=0}^k a_j z_j(s))ds - \int_0^T k_2(s,t,x(s),\sum_{j=0}^k a_j z_j(s))ds]d\tau = 0.$$

Then differentiation of both sides with respect to t yields

$$x'(t) = y(t) + \int_0^t k_1(s,t,x(s),\sum_{j=0}^k a_j z_j(s))ds + \int_0^T k_2(s,t,x(s)\sum_{j=0}^k a_j z_j(s))ds.$$

Clearly, $x(t)$ satisfies (2). Moreover, if $t = 0$, then form (7), $x_{m+1}(0) = x_m(0)$, for every $m \geq 0$. Thus $x_0(0) = x_m(0) = x_0$. Hence, $x(t)$ is the solution of Eq. (2) and the proof is complete. □

Lemma 1. *The following relations hold:*

- $\alpha_1^m \geq \alpha_2^m \geq \cdots \geq \alpha_k^m \geq \cdots \geq \alpha^m$;
- $\lim_{k\to\infty} \alpha_k^m = \alpha^m$;
- $\lim_{m\to\infty}\lim_{k\to\infty} \alpha_k^m = \alpha$, in which $\alpha = inf_{(x,u)\in\xi} J(x,u) \equiv J(x^*,u^*)$.

Proof. The proof can be found in [15]. □

Equation (7) has immediately relied on the integration. So if the right-hand side integration (7) was nonlinear, it would be time-consuming and very difficult to be answered even in the first few iterations of the VIM method. To dominate these blemishes, if x, u, k_1 and k_2 are analytic functions, the following improvement can be used [8]:

$$x_{n+1}(t) = x_n(t) - \int_0^t (T_n(s) - T_{n-1}(s))ds, \tag{11}$$

where $n \geq 0$ and $T_n(s)$ is the n-th order Taylor expansion of

$$x'(t) - y(t) - \int_0^t k_1(s,t,x(s),u(s))ds - \int_0^T k_2(s,t,x(s),u(s))ds$$

around t_0. Also $x(t_0) = x_0$ and $T_{-1}(s) = 0$. This improvement can remove the time-consuming and repeated calculations from the VIM method. The integrands in (11) are subtractions of two successive Taylor expansions, that is the n-th term of Taylor expansion. Thus, the complicated integrations in (7) turn into the integration of the n-th term of Taylor expansion.

Algorithm of the method

Stage 1: Pick $\epsilon > 0$ for the optimal precision and take $m = 1$ and fix q.
Stage 2: Consider $k = 1$.
Stage 3. Substitute $u(t)$ by (4).
Stage 4. Calculate $x_m(t, a_0, a_1, \ldots, a_k)$ by (7).
Stage 5. Compute $J_k^* = inf J_k$ in (9), if $k = 1$, go to stage 7; otherwise go to stage 6.
Stage 6. If the stopping criteria (10) hold, stop; otherwise, go to stage 7
Stage 7. If $k > q$, go to stage 8; otherwise $k = k + 1$ go to stage 3.
Stage 8. $m = m + 1$ go to stage 2.

5. Numerical examples

In this part, we try to solve three numerical examples of optimal control problems ruled by integral equations with the hybrid modified variational iteration and the parametrization method. The following notations have been applied to analyze the error of the method:

$$||E_x||_2^2 = ||x - x^*||_2^2 = \int_a^b (x(t) - x^*(t))^2 dt,$$
$$||E_u||_2^2 = ||u - u^*||_2^2 = \int_a^b (u(t) - u^*(t))^2 dt,$$

where x^*, u^* are the exact solutions, and x, u are the approximation solutions obtained using the current proposed methods. We use the following notations in the tables:
VIMP: Variational iteration and parametrization method.
MVIMP: Modified variational iteration and parametrization method.
HPMP: Homotopy perturbation and parametrization method.

Example 1. *Consider the following optimal control problem*

$$J(x,u) = \int_0^1 (x(t) - t^2 - 1)^2 + (u(t) - t^2)^2 dt, \tag{12}$$

governed by

$$x'(t) = 2t + \int_0^t s^2 x'(s)u(s)ds - \int_0^1 s^2 t^6 x'(s)u(s)ds, \tag{13}$$

the boundary conditions are

$$x(0) = 1, \qquad x(1) = 2.$$

To solve this we consider the control function as follows:

$$u = \sum_{j=0}^{k} a_j t^j.$$

And then we solve system (13) by means of the VIM method. According to the method of the solution presented in Section 3, we make a correction functional in the following form:

$$\begin{aligned} x_{m+1}(t) = x_m(t) - \int_0^t [x'_m(\tau) - 2\tau - \int_0^\tau s^2 x'_m(s)(\sum_{j=0}^{k} a_j s^j)ds \\ - \int_0^1 s^2 t^6 x'_m(s)(\sum_{j=0}^{k} a_j s^j)ds]d\tau. \end{aligned} \tag{14}$$

Successive approximations $x_m(t)$ will be achieved by starting with the initial approximation $x_0(t) = 1$ in Eq. (14). We suppose

$$x(t, a_0, \ldots, a_k) \approx x_3(t, a_0, \ldots, a_k). \tag{15}$$

Now by substituting (15) and $u = \sum_{j=0}^{k} a_j t^j$ into (12) we have

$$\begin{aligned} min_{(a_0,\ldots,a_k)} J_k(a_0, \ldots, a_k) = \int_0^1 \left((x_3(t, a_0, \ldots, a_k) - t^2 - 1)^2 + ((\sum_{j=0}^{k} a_j t^j) - t^2)^2 \right) dt, \\ s.t \;\; x(1, a_0, \ldots, a_k) = 2. \end{aligned} \tag{16}$$

We can solve the following minimization problem instead of optimization problem (16)

$$Minimize\,(J_k(a_0, \ldots, a_k) + (x(1, a_0, \ldots, a_k) - 2)^2). \tag{17}$$

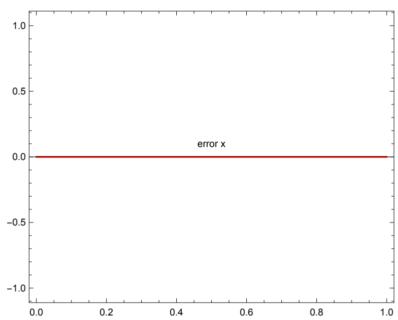
It is possible to respond well to Eq. (17) by using the conventional optimization Toolboxes (FindMinimum) by Mathematica. The precise optimal trajectory and control functions are $x(t) = t^2 + 1$ and $u(t) = t^2$, respectively. The results achieved by VIMP and MVIMP have been compared with the HPMP approach [6] for j^*_k in Table 2, where $\epsilon = 10^{-17}$. The errors of x and u have been reported in Table 1. Also, the figure of the errors of $x(t)$ and $u(t)$ for the MVIMP strategy are outlined in Figure 1, where $k = 2, m = 3$.

Method	$\|E_x\|_2^2$	$\|E_u\|_2^2$
VIMP	9.225661×10^{-24}	2.22452×10^{-20}
MVIMP	0	2.3242×10^{-20}

Table 1: *Errors of $x(t)$ and $u(t)$ for $m = 3$ and $k = 2$*

Itr	VIMP	MVIMP	HPMP
$m = 3, k = 1$	5.55915×10^{-3}	5.55556×10^{-3}	2.62×10^{-2}
$m = 3, k = 2$	0	2.77556×10^{-17}	1.1130×10^{-7}
$m = 3, k = 3$	-2.77556×10^{-17}	-3.7692×10^{-17}	2.2843×10^{-8}

Table 2: *Comparison of numerical results for J_k^* in Example 1*



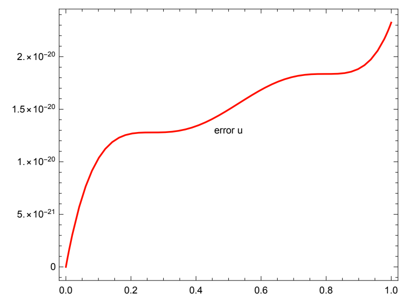


Figure 1: *Figure of the errors of $x(t)$ and $u(t)$ for $(m = 3, k = 2)$*

Example 2. *Consider the following optimal control problem, in which $k_1 = 0$:*

$$MinJ(x, u) = \int_0^1 (x(t) - u(t))^2 dt,$$

subject to

$$x'(t) = e^t - \frac{1}{3}t + \int_0^1 (s^2 t e^{-2s} x'(s) u(s)) ds$$

with boundary conditions

$$x(0) = 1, \qquad x(1) = e. \tag{18}$$

The precise optimal trajectory and control functions are $x(t) = e^t$ and $u(t) = e^t$, respectively. The results gained by VIMP and MVIMP methods have been compared with HPMP approaches [6] for j_k^* in Table 4, where $\epsilon = 10^{-5}$. The errors of x and u for Example 2 have been reported in Table 3. Also, the figure of the errors of $x(t)$ and $u(t)$ for the MVIMP method are illustrated in Figure 2, where $k = 3, m = 2$.

Method	$\|E_x\|_2^2$	$\|E_u\|_2^2$
VIMP	7.67558×10^{-5}	4.91304×10^{-5}
MVIMP	2.49883×10^{-14}	1.09746×10^{-7}

Table 3: *Errors of $x(t)$ and $u(t)$ for $k = 3, m = 2$*

Itr	VIMP	MVIMP	HPMP
$m = 2, k = 1$	3.75946×10^{-3}	3.93439×10^{-3}	4.5×10^{-3}
$m = 2, k = 2$	3.10278×10^{-5}	2.78368×10^{-5}	4.2147×10^{-4}
$m = 2, k = 3$	3.30991×10^{-6}	1.09746×10^{-7}	4.0135×10^{-4}

Table 4: *Comparison of numerical values for J_k^* in Example 2*

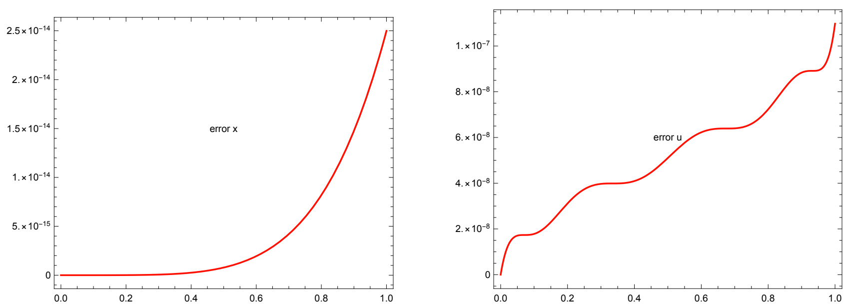


Figure 2: *Figure of the errors of $x(t)$ and $u(t)$ for $(m = 2, k = 3)$*

Example 3. *Consider the optimal control problem as follows, in which $k_2 = 0$,*

$$MinJ(x,u) = \int_0^1 (x(t) - \sin(t))^2 + (u(t) - t^2)^2 dt, \tag{19}$$

subject to

$$x(t) = g(t) + \int_0^t (tsx^3(s) + s^2u^2(s))ds, \tag{20}$$

where $g(t) = \sin(t) - \frac{1}{7}t^7 + \frac{1}{3}t^2 \sin^2(t)\cos(t) + \frac{2}{3}t^2\cos(t) - \frac{1}{9}t\sin^3(t) - \frac{2}{3}t\sin(t)$, with boundary conditions

$$x(0) = 0, \qquad x(1) = 0.841471.$$

Equation (20) is equivalent to the following equation:

$$x'(t) - g'(t) - \left(t^2x^3(t) + t^2u^2(t) + \int_0^t sx^3(s)ds\right) = 0.$$

The consecutive formula (7) to solve this problem is as follows:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - \int_0^t \left(x'(s) - g'(s) - \left(s^2x^3(s) + s^2u^2(s) + \int_0^s \tau x^3(\tau)d\tau\right)\right) ds, \\ x(0) &= 0. \end{aligned} \tag{21}$$

By the VIM method, the integration cannot be calculated for more than two repetitions. Therefore, the VIM formula (21) is not a suitable solution to this problem.

So we have used a modified version of VIM as described above. Denote the n-th order Taylor expansions of iteration as:

$$T_n(t) = x'_n(t) - g'_n(t) - \left(t^2 x_n^3(t) + t^2 u^2(t) + \int_0^t x_n^3(s)ds.\right)$$

Using the modified VIM formula (11), and $u = \sum_{j=0}^{3} a_j t^j$, the following results are obtained:

$$\begin{aligned}
x_0(t) &= 0,\\
x_1(t) &= t,\\
x_2(t) &= t,\\
x_3(t) &= t - \frac{t^3}{6} + \frac{1}{3}t^3 a_0^2,\\
x_4(t) &= t - \frac{t^3}{6} + \frac{1}{3}t^3 a_0^2 + \frac{1}{2}t^4 a_0 a_1,\\
x_5(t) &= t - \frac{t^3}{6} + \frac{1}{3}t^3 a_0^2 + \frac{1}{2}t^4 a_0 a_1 + \frac{1}{120}t^5 + \frac{1}{5}t^5 a_1^2 + \frac{2}{5}t^5 a_0 a_2.\\
&\vdots
\end{aligned}$$

Now with substituting $x(t)$ and $u(t) = \sum_{j=0}^{k} a_j t^j$ into (19) we have

$$\begin{aligned}
min_{(a_0,\ldots,a_k)} J_k(a_0,\ldots,a_k) &= \int_0^1 (x_{10}(t,a_0,\ldots,a_k) - \sin(t))^2 + ((\sum_{j=0}^{k} a_j t^j) - t^2)^2 dt,\\
s.t \quad & x(1,a_0,\ldots,a_2) = 0.841471. \qquad (22)
\end{aligned}$$

We can solve the following minimization problem instead of optimization problem (22):

$$Minimize\,(J_k(a_0,\ldots,a_k) + (x(1,a_0,\ldots,a_k) - 0.841471)^2). \qquad (23)$$

Equation (23) can be solved using conventional optimization Toolboxes (FindMinimum) by Mathematica. The precise optimal trajectory and control functions are $x(t) = \sin(t)$ and $u(t) = t^2$, respectively. The computed results of applying the algorithm for J_k^* with respect to $\epsilon = 10^{-13}$ have been explored in Table 5. The errors of x and u for Example 3 have been reported in Table 6. Table 7 shows the numerical results obtained with Legendre polynomials in [26] presented for comparison. Also, the figure of the errors of $x(t)$ and $u(t)$ for the MVIMP method are depicted in Figure 3, where $k = 3, m = 10$.

Itr	$k=1$	$k=2$	$k=3$
$m=8$	0.00556831	3.7649×10^{-13}	3.64668×10^{-13}
$m=9$	0.00562433	1.59471×10^{-13}	1.59432×10^{-13}
$m=10$	0.0056075	1.59435×10^{-13}	-3.52691×10^{-17}

Table 5: *Numerical results for* J_k^* *in Example 3*

Itr	$\|E_x\|_2^2$	$\|E_u\|_2^2$
$m=10,\ k=3$	-2.17404×10^{-15}	1.02966×10^{-18}

Table 6: *Errors of* $x(t)$ *and* $u(t)$ *for Example 3*

Itr	$\|E_x\|_2^2$	$\|E_u\|_2^2$	J^*
$m=4$	9.5×10^{-7}	1.2×10^{-7}	7.6×10^{-11}
$m=6$	1.1×10^{-9}	8.5×10^{-11}	1.7×10^{-16}

Table 7: *Numerical results from [26] with Legendre polynomials for Example 3*

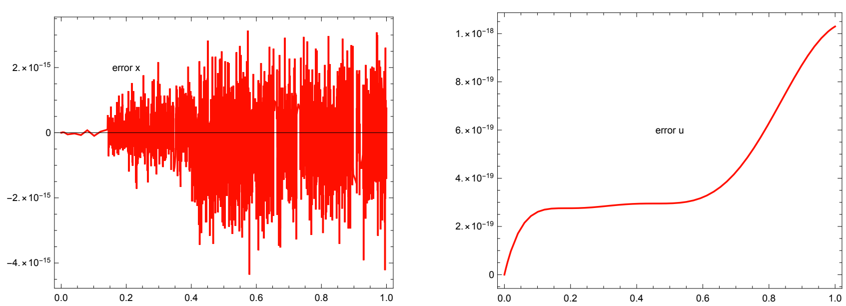


Figure 3: *Figure of the errors of* $x(t)$ *and* $u(t)$ *for* $(m = 10, k = 3)$

6. Conclusions

This article illustrates efficient hybrid methods due to finding the minimum of optimal control problem of Volterra-Fredholm integral equations. Our approach was based on variational iteration and parametrization methods. However, the main disadvantage of VIM is generally a huge term and the consuming time to compute, so it needs large computer memory and time. Responding to this problem, we propose a modified version of VIM. As shown in the results, the MVIM strategy is more efficient and cost-effective than the VIM method, and where the VIM method does not answer MVIM is very effective. Finally, the article provides three numerical examples to illustrate the feasibility and effectiveness of our suggested approach.

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