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Legitimate Mathematical Methods

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A thought experiment involving an omniscient being and quantum mechanics is used to justify non-deductive methods in mathematics. The twin prime conjecture is used to illustrate what can be achieved.

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There is a standard view of mathematics that says proofs are the one and only source of evidence and proofs are deductive derivations from first principles. This attitude has a long tradition and there is a comforting surety about it. But occasionally there are voices in opposition, including one that should be particularly influential.

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics. The fact is that in mathematics we still have the same attitude today that in former times one had toward all science, namely we try to derive everything by cogent proofs from the definitions (that is, in ontological terminology, from the essences of things). Perhaps this method, if it claims monopoly, is as wrong in mathematics as it was in physics. (Gödel 1995 [1951], vol. III: 313)

I'm going to argue for the same conclusion, but I will come at it in a very different way. Instead of trying directly to liberalize the notion of evidence in mathematics, I will assume certainty in physics, that is, I will assume that the first principles of quantum mechanics (QM) are just as certain as the Peano axioms (PA), the first principles of arithmetic. The consequence for what counts as legitimate mathematical methods will surprise.

Let's begin with a parable. God parts the clouds and says: "Verily, verily I say unto you, the principles of quantum mechanics are true." Imagine God as you will. I picture Athena, goddess of wisdom and patron of science. But be sure to include her being omniscient and truthful. This means we can now know with certainty that quantum states are represented by vectors in Hilbert space; they evolve according to the

Schrödinger equation; the Born rule will give us the right probabilities for measurement outcomes; and so on. We now have perfect confidence in the truth of the standard principles of QM, which until now were merely empirically well justified. And we also know that anything we can derive from those first principles, such as Heisenberg's uncertainty principle, is unquestionably true, since logic preserves truth.

So far, so good, but we have more questions for God to answer: Is QM complete, in the sense of implying yes or no to every QM question? If P is a consequence of QM, can we derive it in a feasible time? What is the relation of QM to other theories? Do chemistry and biology reduce to QM or not? Other questions will readily come to mind.

We ask, but God won't answer. She smiles benignly then, alas, departs. (Athena frequently helped Odysseus out of a jam, then left him to fend for himself.) Suppose this is how things now stand with us. We now know much with certainty. But we remain either ignorant or only mildly confident of much else in QM. How should we proceed?

Obviously, we should try to construct derivations for as many propositions as possible. But what about the rest? We would probably continue as before. That is, we would continue with a combination of conjectures and experimental testing. Aside from the parts of QM that are clearly certain, it would be business as usual. We would continue to argue over what this involves but details would be more or less the same. There will be experimental probing, hypothesis testing, the use of various statistical techniques, thought experiments, philosophical considerations, and so on.

We would continue to tackle many problems the way we do currently. For example, perhaps there is a derivation of the details of protein folding from the principles of QM, but no such derivation can be found by humans. Calculating the energy levels of complex objects is hopelessly difficult. U_{235} is a many-body problem that can't be exactly solved. Quantum field theory is a relativistic extension of QM, not derivable from it. What about dark energy? Is this even a QM problem? God is no help in answering these questions. We have to carry on as before.

The upshot from all of this is that some physics is certain and some is not, and we will continue to learn about the latter in the same old empirical, fallible, inductive way. Why not demand certainty everywhere in QM? The argument for not doing this, if one is needed, is simple: Pre-God we have lots of justified but fallible beliefs involving QM. Then God tells us that part of this is in fact certain knowledge. Great news. Do we abandon the remaining justified beliefs on the grounds that they are not certain? No, since their status as justified but fallible beliefs remains unchanged from what it was before God certified some of it. In that respect, nothing has changed. The fact that God certifies some of it should not turn us into sceptics about the rest.

Of course, it is still debatable precisely what good scientific method is, but that is a detail that need not trouble us here. Most of QM remains fallible by anybody's lights and should be investigated empirically and inductively. We have certain knowledge of the first principles of QM and their deductive consequences. The rest of QM has the same status as it had before God intervened. Does this have consequences for our knowledge claims elsewhere?

Let's turn to mathematics, where the common attitude is that much of it is certain knowledge (and we don't need God to tell us). I'll stick to an elementary part, basic arithmetic, which, for most of us, is probably as certain as anything could be.

There is a common ideology that goes along with the general attitude about mathematics. Let's assume the Peano axioms (PA), which are a set of rules characterizing the natural numbers. PA says there is a number 0, and for each number there is a successor. Thus, 1 is the successor of 0; 2 is the successor of 1, and so on. There are axioms for addition and multiplication, and for the principle of mathematical induction. These axioms are typically taken to be certainly true, or at least as certain as anything could be. Of course, there are people who claim to doubt them, but there are also people who claim to doubt the law of non-contradiction.[1]

A theorem may be asserted, according to the common ideology, if and only if there is a proof, which is a derivation from the basic axioms. (In practice a sketch of a derivation will suffice, but it is understood that that full details could in principle be provided.) Nothing else should be believed, according to this ideology — a proof is the only evidence allowed.

All of this can be easily illustrated by a famous theorem, first proved in Euclid's *Elements*. The theorem follows from PA. *Prime numbers* are numbers that cannot be factored, that is, they cannot be divided by any numbers except 1 and themselves without remainder. They include: 2, 3, 5, 7, 11, 13, … The rest are *composite numbers*, which are the product of primes. For instance, $4 = 2\times2$, $6 = 2\times3$, $8 = 2\times2\times2$, $9 = 3\times3$, $10 = 2\times5$, $12 = 2\times2\times3$, …, $2093 = 7\times13\times23$, and so on. How many primes are there?

> Theorem: There are infinitely many prime numbers.
>
> Proof: Suppose there are only finitely many primes. Hence, there is a highest p. Let $q = (2\times3\times5\times7\times\ldots\times p) + 1$. If q is a prime, then p is not the highest prime after all. If q is composite, then q is divisible by primes. But none of 2, 3, 5, …, p can divide q, since there is always a remainder of 1. Thus, some prime r must divide q. But $r > p$. Either way, p is not the highest prime. So, the initial

[1] This is perhaps unfair to those who are fictionalists, such as Field (2016) or Leng (2010). I don't wish to debate this issue here. I assume mathematical platonism or some sort of realism from the outset and argue from there. The point of this paper is not about the ontology of mathematics, but rather its legitimate epistemology. What is the best way to acquire objective mathematical knowledge, assuming there is such a thing? (Chess knowledge, by contrast, is not objective.)

assumption that there is a highest prime is false. Thus, there are infinitely many.

Now we have two interesting systems to think about, PA and QM. The first principles of PA and QM (post God) are both certain. Anything we can derive from either we can be sure is true. And yet we treat them differently in a fundamental way. We would be happy to go beyond the certain first principles of QM and continue to use inductive methods to enlarge what we know about the physical realm. But we have been reluctant to do the same with PA. Their epistemic situations are the same, so we should have the same epistemic outlook for each.

The parallel is obvious. In the quantum case (post God), we have two kinds of propositions: (1) QM principles and logical consequences that we can actually derive and (2) all other truths of quantum mechanics that we cannot either practically or in principle derive. In the arithmetic case, we also have two kinds of propositions: (1) PA axioms and logical consequences we can derive from those axioms and (2) all other truths of arithmetic that we cannot either practically or in principle derive.

How should we respond to this schizophrenic methodological attitude? Obviously we should follow the QM example and extend our mathematical knowledge by adding various inductive techniques to PA. This will have profound implications for mathematical practice.

The twin primes conjecture will provide a good example of a more liberal way of proceeding. *Twin primes* are pairs of prime numbers of the form (p, p+2). For instance, (3,5), (5,7), (11,13), (17,19), and so on. How many are there? This is an open problem in number theory in the sense that there is no proof that the number of twin primes is either infinite or finite. Number theorists have been attacking the problem for a long time without finding the answer.[2] It is possible, of course that the problem is unsolvable, in the sense that no proof exists either way. We know from Gödel's incompleteness theorem that such unsolvable problems exist. Euler, who is often quoted on this topic, wondered about the possibility. "Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate." (1710, quoted in Simmons 1992: 276n3).

To proceed, let's take note of the Prime Number Theorem. I will use the standard notation π(n) for the number of primes up to n, e.g., $\pi(10) = 4$. The Prime Number Theorem says: $\pi(n) \approx n/\log n$. That is, the number of primes up to some number n is approximately equal to n divided by the natural log of n. As n gets larger, the approximation becomes more accurate. For example:

[2] The literature on number theory, especially primes, is enormous. Extensive discussions can be found in Hardy and Wright (2008), Ribenboim (1991) and Shanks (1993).

$\pi(100) = 25$	100/log 100 = 21.7
$\pi(1000) = 168$	1000/log 1000 = 144.7
$\pi(\text{billion}) = 5084534$	billion/log billion = 4825494.2

Cramér (1936) developed the idea that primes can be considered as random. If we consider them equiprobably, then the probability that a number less than n is prime is approximately 1/log n. The idea can be tweaked to address obvious problems (eg, half the numbers are even so not prime, aside from 2).

Think of the gap between primes. For instance, the gap between 5 and the next prime 7 is 2; the gap between 11 and 13 is also 2, while the gap between 13 and the next prime 17 is 4, and so on. The apparent randomness of the primes will be reflected in the randomness of the size of the gaps. Since there are infinitely many primes, we can expect the number of gaps of size 2 to occur infinitely often. And that means that primes of the form (p, p+2) will occur infinitely often. In short, *the twin primes conjecture is true*. And it is justified by rather simple but quite compelling inductive means.

The argument is easily generalized to prime pairs of the form (p, p+4), (p, p+6), and so on. There are infinitely many pairs of each of these, as well, since there will be infinitely many gaps of size 4, size 6, and so on. The moral to be drawn from this example is that inductive methods can provide legitimate evidence in mathematics more generally.

I want to stress that the foregoing argument significantly differs from other arguments for inductive methods in mathematics. Besides Gödel who was quoted at the outset, lots of people (including me (Brown 2008, 2017)), have argued for such a conclusion. One of the simplest arguments for a more liberal methodology is the fact that the first principles cannot be proven (without begging the question), so it is in principle hopeless to demand that all our mathematical evidence be based on proofs. Another argument for mathematical fallibility is based on conceptual change. For instance, in the 18^{th} century it was thought that all functions are continuous. The proof for this theorem was flawless. The concept of function, however, changed during the 19^{th} century, so that now we take a function to be an arbitrary association between sets. This allows the radically discontinuous Dirichlet function $f(x)$, which equals 1 or 0, depending on whether x is rational or irrational.

The argument here is quite different in that it assumes that mathematics is in part certain. Specifically, the Peano axioms are taken to be as certain as anything. The argument then follows the lesson of QM resulting from the God thought experiment, namely, that inductive methods should supplement the known-to-be-certain first principles. This is why the God TE at the outset is important; it guarantees the analogy between mathematics and physics, which is the basis of the argument.

Of course, there is no God who guarantees the first principles of QM, and we cannot continue to take those principles to be certain. The thought experiment has done its job and led us to a new way of viewing legitimate mathematical methods. Now we can treat it like Wittgenstein's ladder. Toss it out and agree that even the first principles of QM and PA are fallible, as is all knowledge, but the liberalization in what counts as evidence more than makes up for the loss of certainty.

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References

Brown, J. R. 2008. *Philosophy of Mathematics: Contemporary Introduction to the World of Proofs and Pictures*. 2nd ed. New York: Routledge.

Brown, J. R. 2017. "Proofs and Guarantees." *Mathematical Intelligencer* 39: 47–50.

Cramér, H. 1936. "On the order of magnitude of the difference between consecutive prime numbers." *Acta Arithmetica* 2: 23–46.

Field, H. 2016. *Science Without Numbers*. 2nd ed. Oxford: Oxford University Press.

Gödel, K. 1995 [1951]. *Collected Papers*. Oxford: Oxford University Press.

Hardy, G. H. and Wright, E. M. 2008. *Introduction to the Theory of Numbers*. Oxford: Oxford University Press.

Leng, M. 2010. *Mathematics and Reality*. Oxford: Oxford University Press.

Ribenboim, P. 1991. *The Little Book of Big Primes*. New York: Springer.

Shanks, D. 1993. *Solved and Unsolved Problems in Number Theory*. New York: Chelsea.

Simmons, G. 1992. *Calculus Gems*. New York: McGraw-Hill.