A note on continued fractions of quadratic irrationals

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Abstract. *Quadratic irrationals* $\sqrt{D}$ *have a periodic representation in terms of continued fractions. In this paper some relations between n-th approximations of quadratic irrationals are proved. Results are applied to Newton's approximations of quadratic irrationals.*

Key words: *Continued fractions, quadratic irrationals, Newton's aproksimations*

Sažetak. **O verižnim razlomcima korijena prirodnih brojeva**. *Za prirodni broj D iracionalni broj* $\sqrt{D}$ *ima periodički prikaz pomoću verižnih razlomaka. U ovom se članku dokazuju neke relacije izmedju n-tih aproksimacija tih brojeva. Dobiveni su rezultati primijenjeni na Newtonove aproksimacije korijena prirodnih brojeva.*

Ključne riječi: *Verižni razlomci, Newtonove aproksimacije*

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1. Introduction

Let D be a positive integer which is not a square. It is well known that quadratic irrationals $\sqrt{D}$ have a pure periodic representation in terms of continued fractions ([4, Th. 3, p. 294]). Let us denote

$$a_0 + \frac{1}{a_1+} \Big/ \frac{1}{a_2+} \Big/ \cdots \Big/ \frac{1}{a_n+} \Big/ \cdots := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}}}.$$

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Let s be the length of the period. We use the following notation:

$$\sqrt{D} = a_0 + \frac{1}{a_1+} \Big/ \frac{1}{a_2+} \Big/ \cdots \Big/ \frac{1}{a_s+} \Big/ \frac{1}{a_1+} \Big/ \cdots =: (a_0; a_1, a_2, \cdots, a_s).$$

For example

$$\begin{aligned}\sqrt{2} &= (1;2),\\ \sqrt{3} &= (1;1,2),\\ \sqrt{7} &= (1;1,1,1,4)\end{aligned}$$

etc.

We denote the n-th approximation in the standard way

$$R_n = a_0 + \frac{1}{a_1+} \Big/ \frac{1}{a_2+} \Big/ \cdots \Big/ \frac{1}{a_{n-1}} = \frac{P_n}{Q_n}. \tag{1}$$

Then, as it is well known, it holds

$$\begin{array}{ll} P_1 = a_0 & Q_1 = 1; \\ P_2 = a_0 a_1 + 1 & Q_2 = a_1; \\ P_k = a_{k-1} P_{k-1} + P_{k-2} & Q_k = a_{k-1} Q_{k-1} + Q_{k-2}. \end{array} \tag{2}$$

2. Some properties of approximations by continued fractions

Representation

$$\sqrt{D} = (a_0; a_1, \ldots, a_s)$$

of a quadratic irrational has the following property [4, Th. 3, p. 294]

$$a_0 = [\sqrt{D}], \tag{3}$$

$$a_s = 2a_0, \tag{4}$$

$$a_1, \ldots, a_{s-1} \text{ is symmetrical, i.e. } a_i = a_{s-i},\ i = 1, \ldots, s-1. \tag{5}$$

Let us introduce polynomials (p_k) by recursive relations

$$\begin{aligned} p_0 :=&\ \ 1,\\ p_1(x_1) :=&\ \ x_1,\\ p_2(x_1,x_2) :=&\ \ x_1x_2 + 1,\\ p_k(x_1,\ldots,x_k) :=&\ \ x_k p_{k-1}(x_1,\ldots,x_{k-1}) + p_{k-2}(x_1,\ldots,x_{k-2}). \end{aligned} \tag{6}$$

The connection between such polynomials and sequences (P_n), (Q_n) is obvious:

$$P_n = p_n(a_0, a_1, \ldots, a_{n-1}), \tag{7}$$

$$Q_n = p_{n-1}(a_1, \ldots, a_{n-1}), \quad (Q_1 = 1) \tag{8}$$

In the sequel, we shall write polynomials p_n without subscript, since this cannot lead to confusion. Let us prove some lemmas.

Lemma 1. *Polynomials p are symmetric in the sense*

$$p(x_1, \ldots, x_n) = p(x_n, \ldots, x_1). \tag{9}$$

Proof. Easy, by induction. □

Lemma 2. *It holds*

$$p(x_1, x_2, \dots, x_n) = x_1 p(x_2, \dots, x_n) + p(x_3, \dots, x_n). \tag{10}$$

Proof. Using (6) and *Lemma 1.* □

Theorem 1. *It holds, for all positive integers n:*

$$P_{ns} = a_0 Q_{ns} + Q_{ns-1}. \tag{11}$$

and

$$DQ_{ns} = a_0 P_{ns} + P_{ns-1}. \tag{12}$$

Proof. We shall take $n = 1$, the same proof holds for all n. Since $a_s = 2a_0$, we have

$$\begin{aligned}\sqrt{D} &= a_0 + \tfrac{1}{a_1+} \Big/ \tfrac{1}{a_2+} \Big/ \cdots \Big/ \tfrac{1}{a_{s-1}+} \Big/ \tfrac{1}{a_0+\sqrt{D}} \\ &= \frac{p(a_0, a_1, \dots, a_{s-1}, a_0 + \sqrt{D})}{p(a_1, a_2, \dots, a_{s-1}, a_0 + \sqrt{D})} = \frac{(a_0 + \sqrt{D})P_s + P_{s-1}}{(a_0 + \sqrt{D})Q_s + Q_{s-1}}.\end{aligned}$$

From this, it follows

$$a_0 P_s + P_{s-1} - DQ_s = \sqrt{D}(a_0 Q_s + Q_{s-1} - P_s)$$

The assertion follows since $\sqrt{D}$ is irrational. □

Theorem 2. *If s is even and $r = s/2$, then it holds for all positive integers n:*

$$\frac{P_{nr}}{Q_{nr}} = \frac{P_{nr+1} - P_{nr-1}}{Q_{nr+1} - Q_{nr-1}} \tag{13}$$

and

$$D = \frac{P_{nr}}{Q_{nr}} \cdot \frac{P_{nr+1} + P_{nr-1}}{Q_{nr+1} + Q_{nr-1}} \tag{14}$$

Proof. We shall take again $n = 1$. Since $a_{2r} = a_s = 2a_0$, we have

$$\begin{aligned}\sqrt{D} &= \frac{\Big(a_r + \tfrac{1}{a_{r+1}+} \Big/ \cdots \Big/ \tfrac{1}{a_0+\sqrt{D}}\Big)P_r + P_{r-1}}{\Big(a_r + \tfrac{1}{a_{r+1}+} \Big/ \cdots \Big/ \tfrac{1}{a_0+\sqrt{D}}\Big)Q_r + Q_{r-1}} \\ &= \frac{p(a_r, \dots, a_{2r-1}, a_0 + \sqrt{D})P_r + p(a_{r+1}, \dots, a_{2r-1}, a_0 + \sqrt{D})P_{r-1}}{p(a_r, \dots, a_{2r-1}, a_0 + \sqrt{D})Q_r + p(a_{r+1}, \dots, a_{2r-1}, a_0 + \sqrt{D})Q_{r-1}}\end{aligned}$$

The sequence $a_1, \dots, a_{2r-1}$ is symmetric. By *Lemma 1* and 2:

$$\begin{aligned}p(a_r, \dots, a_{2r-1}, a_0 + \sqrt{D}) &= (a_0 + \sqrt{D})p(a_r, \dots, a_{2r-1}) + p(a_r, \dots, a_{2r-2}) \\ &= \sqrt{D}p(a_r, \dots, a_{2r-1}) + p(a_r, \dots, a_{2r-1}, a_0) \\ &= \sqrt{D}p(a_1, \dots, a_r) + p(a_0, \dots, a_r) \\ &= \sqrt{D}Q_{r+1} + P_{r+1}.\end{aligned}$$

Hence,

$$\sqrt{D} = \frac{(\sqrt{D}Q_{r+1} + P_{r+1})P_r + (\sqrt{D}Q_r + P_r)P_{r-1}}{(\sqrt{D}Q_{r+1} + P_{r+1})Q_r + (\sqrt{D}Q_r + P_r)Q_{r-1}}.$$

From this the assertion easily follows. □

3. Newton's approximations

The sequence (R_n) has the best approximation property: between all rationals with the denominator $\leq Q_n$, R_n is the best approximation for $\sqrt{D}$ ([4, Th. 2, p. 290; 3, Th. 181, p. 151]. But, the sequence (R_n) converges to $\sqrt{D}$ very slowly, compared to the Newton's sequence

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{D}{r_n}\right). \tag{15}$$

So, it is interesting to compare the relation between those sequences. As an illustration we give sequences for $\sqrt{2}$, in *Tables* 1 and 2.

n	P_n	Q_n
1	1	1
2	3	2
3	7	5
4	17	12
5	41	29
6	99	70
7	239	169
8	577	408
9	1393	985
10	3363	2378
11	8119	5741
12	19601	13860
13	47321	33461
14	114243	80782
15	275807	195025
16	665857	470832
17	1607521	1136689
⋮		
23	318281039	225058681
24	768398401	543339720
25	1855077841	1311738121
⋮		
31	367296043199	259717522849
32	886731088897	627013566048
33	2140758220993	1513744654945
⋮		
47	489133282872437279	345869461223138161
48	1180872205318713601	835002744095575440
49	2850877693509864481	2015874949414289041
⋮		
63	6513856406668176425230O7	460599203683050495415105
64	1572584048032918633353217	1111984844349868137938112

Table 1: *Sequence of approximations for* $\sqrt{2}$.

Let us see what happens with the Newton's sequences. For the initial values of r_1 we take some approximation given in Table 1. Iterations r_k are represented in the form of rationals, $r_k = \frac{u_k}{v_k}$. In the last column, the value of corresponding iterations in the *Table 1* is given.

k	u_k	v_k	n
1	1	1	1
2	3	2	2
3	17	12	4
4	577	408	8
5	665857	470832	16
6	886731088897	627013566048	32
7	1572584048032918633353217	1111984844349868137938112	64
1	7	5	3
2	99	70	6
3	19601	13860	12
4	768398401	543339720	24
5	1180872205318713601	835002744095575440	48

Table 2: *Newton's sequences for* $\sqrt{2}$*.*

We see that all those approximations appear in the sequence given in *Table 1*. The same is true for initial values arbitrary taken from the *Table 1*, in fact, it holds for all n

$$R_{2n} = \frac{1}{2}\left(R_n + \frac{2}{R_n}\right).$$

This result is proved in [1], p. 440. By inspection through similar table for $\sqrt{3} = (1; 1, 2)$, or $\sqrt{8} = (2; 1, 4)$ we can see that the same is true for the period s of length 2.

Does the same happen with other irrationals $\sqrt{D}$? In general, this depends on the length of the period s. The main result of the paper is the following.

Theorem 3. *Let s be the period of the representation of quadratic irrationals $\sqrt{D}$ in terms of continued fraction and r defined in a way*

$$r = \begin{cases} s, & \text{if } s \text{ is odd,} \\ s/2, & \text{if } s \text{ is even.} \end{cases}$$

Then it holds for all natural n

$$R_{2nr} = \frac{1}{2}\left(R_{nr} + \frac{D}{R_{nr}}\right). \tag{16}$$

Proof. *Case $r = s$.* This case is proved in [1], see also [2]. We give here a different proof using *Theorem 1.*

$$\begin{aligned} R_{2s} &= \frac{\left(a_s + \frac{1}{a_{s+1}+} \Big/ \cdots \Big/ \frac{1}{a_{2s-1}}\right)P_s + P_{s-1}}{\left(a_s + \frac{1}{a_{s+1}+} \Big/ \cdots \Big/ \frac{1}{a_{2s-1}}\right)Q_s + Q_{s-1}} = \frac{\left(a_0 + \frac{P_s}{Q_s}\right)P_s + P_{s-1}}{\left(a_0 + \frac{P_s}{Q_s}\right)Q_s + Q_{s-1}} \\ &= \frac{\dfrac{P_s^2}{Q_s^2} + \dfrac{a_0 P_s + P_{s-1}}{Q_s}}{\dfrac{P_s}{Q_s} + \dfrac{a_0 Q_s + Q_{s-1}}{Q_s}} = \frac{R_s^2 + D}{2R_s}, \end{aligned}$$

The same proof holds for iterations R_{2ns}.

Case $r = s/2$. In a similar way, we can write

$$R_{2r}=\frac{p(a_r,\dots,a_{2r-1})P_r+p(a_{r+1},\dots,a_{2r-1})P_{r-1}}{p(a_r,\dots,a_{2r-1})Q_r+p(a_{r+1},\dots,a_{2r-1})Q_{r-1}}=\frac{Q_{r+1}P_r+Q_rP_{r-1}}{Q_{r+1}Q_r+Q_rQ_{r-1}}=\text{by}(15)$$

$$=\frac{P_r(Q_{r+1}+Q_{r-1})+Q_r(P_{r+1}+P_{r-1})}{2Q_r(Q_{r+1}+Q_{r-1})}=\text{by}(16)=\frac{1}{2}\left(R_r+\frac{D}{R_r}\right).$$

□

4. Some remarks and open questions

For the number $\sqrt{21} = (4; 1, 1, 2, 1, 1, 8)$ (16) holds not only for $r = 3$, but also for $r = 2$ as well. But, for the next number with the same period, $\sqrt{22} = (4; 1, 2, 4, 2, 1, 8)$ (16) holds only for $r = 3$.

The interesting thing happens for $s = 5$. Let us see for example $D = 13$, $\sqrt{D} = (3; 1, 1, 1, 1, 6)$. Then,

$$\frac{1}{2}\left(R_k+\frac{D}{R_k}\right)=\begin{cases}\dfrac{P_{2k}}{Q_{2k}}, & \text{if } k=5n,\\[2mm] \dfrac{P_{2k-2}}{Q_{2k-2}}, & \text{if } k=5n-1,\\[2mm] \dfrac{P_{2k+2}}{Q_{2k+2}}, & \text{if } k=5n+1.\end{cases}$$

If $k = 5n \pm 2$, then R_{2n} is not a standard approximation. The same is true for $\sqrt{29} = (5; 2, 1, 1, 2, 10)$ and for the next number with a period of length 5, $\sqrt{53} = (7; 3, 1, 1, 3, 14)$. But, for $\sqrt{74} = (8; 1, 1, 1, 1, 16)$ (16) holds only for $r = 5$! What can be said for other values of s?

Remark 1. *The following is noted by the referee: If $a > 1$ is odd and $D = a^2+4$, then (17) holds true. Moreover,*

$$\frac{1}{2}\left(R_k+\frac{D}{R_k}\right)=\begin{cases}\dfrac{(a-2)P_{2k+1}+P_{2k}}{(a-2)Q_{2k+1}+Q_{2k}}, & \text{if } k=5n+2,\\[2mm] \dfrac{P_{2k}-(a-2)P_{2k-1}}{Q_{2k}-(a-2)Q_{2k-1}}, & \text{if } k=5n-2.\end{cases}$$

Also, the result for $D = 21$ can be generalized to the numbers of the form $D = a^2-4$, where $a > 3$ is odd.

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