A coincidence point theorem for multi-valued contractions*

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Abstract. *A coincidence point theorem for two pairs of mappings is proved.*

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1. Introduction and preliminaries

Let (X, d) be a metric space and let f and g be mappings from X into itself. In [5], S. Sessa defined f and g to be *weakly commuting* if

$$d(gfx, fgx) \leq d(gx, fx)$$

for all $x \in X$. It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the example of [5].

Recently, Jungck [1] extended the concept of weak commutativity in the following way:

Definition 1. *Let f and g be mappings from a metric space (X, d) into itself. The mappings f and g are said to be compatible if*

$$\lim_{n\to\infty}(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

It is obvious that two weakly commuting mappings are compatible, but the converse is not true, see the examples in [1].

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Recently, Kaneko [2] and Singh et al. [6] extended the concepts of weak commutativity and compatibility, see Kaneko et al. [3], for single-valued mappings to the setting of single-valued and multi-valued mappings, respectively.

Now let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on $CB(X)$ induced by the metric d, i.e.,

$$H(A, B) = \max\Big\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\Big\}$$

for $A, B \in CB(X)$, where $d(x, A) = \inf_{y \in A} d(x, y)$.

It is well-known that $(CB(X), H)$ is a metric space, and if (X, d) is complete, then $(CB(X), H)$ is also complete.

The following lemma was proved in Nadler [4].

Lemma 1. *Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there exists a point $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

Definition 2. *Let (X, d) be a metric space and let $f : X \to X$ and $S : X \to CB(X)$ be single-valued and multi-valued mappings, respectively. The mappings f and S are said to be weakly commuting if for all $x \in X$, $fSx \in CB(X)$ and*

$$H(Sfx, fSx) \leq d(fx, Sx),$$

where H is the Hausdorff metric defined on $CB(X)$.

Definition 3. *The mappings f and S are said to be compatible if*

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = z$ for some $z \in X$, where $y_n \in Sx_n$ for $n = 1, 2, \dots$.

Remark 1.

(i) Definition 3 *is slightly different from Kaneko's definition [2].*

(ii) If S is a single-valued mapping on X in Definitions 2 *and* 3*, then* Definitions 2 *and* 3 *become the definitions of weak commutativity and compatibility for single-valued mappings.*

(iii) If the mappings f and S are weakly commuting, then they are compatible, but the converse is not true.

In fact, suppose that f and S are weakly commuting and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in Sx_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = z$ for some $z \in X$. From $d(fx_n, Sx_n) \leq d(fx_n, y_n)$, it follows that $\lim_{n\to\infty} d(fx_n, Sx_n) = 0$. Thus, since f and g are weakly commuting, we have

$$\lim_{n\to\infty} H(Sfx_n, fSx_n) = 0.$$

On the other hand, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0,$$

which means that f and S are compatible.

Example 1. *Let $X = [1, \infty)$ be set with the Euclidean metric d and define $fx = 2x^4 - 1$ and $Sx = [1, x^2]$ for all $x \geq 1$. Note that f and S are continuous and $S(X) = f(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X defined by $x_n = y_n = 1$ for $n = 1, 2, \ldots$. Then we have*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = 1 \in X, \text{ where } y_n \in Sx_n.$$

On the other hand, we can show that $H(fSx_n, Sfx_n) = 2(x_n^4 - 1)^2 \to 0$ if and only $x_n \to 1$ as $n \to \infty$ and so, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0.$$

Therefore, f and T are compatible, but f and T are not weakly commuting at $x = 2$.

2. Main results

Theorem 1. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ be continuous mappings and $S, T : X \to CB(X)$ be H-continuous mappings such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$, the pair S and g are compatible mappings and*

$$\begin{aligned} H^p(Sx, Ty) \quad \leq \quad & \max\{ad(fx, gy)d^{p-1}(fx, Sx), ad(fx, gy)d^{p-1}(gy, Ty), \\ & ad(fx, Sx)d^{p-1}(gy, Ty), cd^{p-1}(fx, Ty)d(gy, Sx)\} \quad (1) \end{aligned}$$

for all $x, y \in X$, where $p \geq 2$ is an integer, $0 < a < 1$ and $c \geq 0$. Then there exists a point $z \in X$ such that $fz \in Sz$ and $gz \in Tz$, i.e., z is a coincidence point of f, S and of g, T. Further, z is unique when $0 < c < 1$.

Proof. Let x_0 be an arbirary point in X. Since $Sx_0 \subseteq g(X)$, there exists a point $x_1 \in X$ such that $gx_1 \in Sx_o$ and so there exists a point $y \in Tx_1$

$$d(gx_1, y) \leq kH(Sx_0, Tx_1),$$

where $k = a^{-1/2} > 1$, which is possible by *Lemma 1*. Since $Tx_1 \subseteq f(X)$, there exists a point $x_2 \in X$ such that $y = fx_2$ and so we have

$$d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Similarly, there exists a point $x_3 \in X$ such that $gx_3 \in Sx_2$ and

$$d(gx_3, fx_2) \leq kH(Sx_2, Tx_1).$$

Inductively, we can obtain a sequence $\{x_n\}$ in X such that

$$\begin{gathered} fx_{2n} \in Tx_{2n}, \quad n \in N, \\ gx_{2n+1} \in Sx_{2n}, \quad n \in N_0 = N \cup \{0\}, \\ d(gx_{2n+1}, fx_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \quad n \in N, \\ d(gx_{2n+1}, fx_{2n}) \leq kH(Sx_{2n}, Tx_{2n+1}), \quad n \in N_0, \end{gathered}$$

where N denotes the set of positive integers. Then, by (1), we have

$$\begin{aligned} d^p(gx_{2n+1}, fx_{2n+2}) &\leq k^p H^p(Sx_{2n}, Tx_{2n+1}) \\ &\leq a^{-p/2} \max\{ad(fx_{2n}, gx_{2n+1})d^{p-1}(fx_{2n}, Sx_{2n}), \\ &\quad ad(fx_{2n}, gx_{2n+1})d^{p-1}(gx_{2n+1}, Tx_{2n+1}), \\ &\quad ad(fx_{2n}, Sx_{2n})d^{p-1}(gx_{2n+1}, Tx_{2n+1}), \\ &\quad cd^{p-1}(fx_{2n}, Tx_{2n+1})d^{p-1}(gx_{2n+1}, fx_{2n+2})\} \\ &\leq a^{-p/2} \max\{ad(fx_{2n}, gx_{2n+1})d^{p-1}(fx_{2n}, gx_{2n+1}), \\ &\quad ad(fx_{2n}, gx_{2n+1})d^{p-1}(gx_{2n+1}, fx_{2n+2}), \\ &\quad ad(fx_{2n}, Sx_{2n})d^{p-1}(gx_{2n+1}, fx_{2n+2}), \\ &\quad cd^{p-1}(fx_{2n}, fx_{2n+2})d^{p-1}(gx_{2n+1}, gx_{2n+1})\}. \end{aligned}$$

Putting $a^{-p/2} = \beta$, we have

$$\begin{aligned} d^p(gx_{2n+1}, fx_{2n+2}) &\leq \beta \max\{ad(fx_{2n}, gx_{2n+1})d^{p-1}(fx_{2n}, gx_{2n+1}), \\ &\quad ad(fx_{2n}, gx_{2n+1})d^{p-1}(gx_{2n+1}, fx_{2n+2}), \\ &\quad ad(fx_{2n}, Sx_{2n})d^{p-1}(gx_{2n+1}, fx_{2n+2}), \\ &\quad cd^{p-1}(fx_{2n}, fx_{2n+2})d^{p-1}(gx_{2n+1}, gx_{2n+1})\}, \\ &\leq \beta ad(fx_{2n}, gx_{2n+1})d^{p-1}(fx_{2n}, gx_{2n+1}), \end{aligned}$$

$$d^p(gx_{2n+1}, fx_{2n+2}) \leq \beta^n a^n d(x_0, gx_1).$$

Since $0 < \beta < 1$, it follows that

$$\{gx_1, fx_2, gx_3, fx_4, \dots, gx_{2n-1}, fx_{2n}, gx_{2n+1}, \dots\}$$

is a Cauchy sequence in X. Since (X, d) is a complete metric space, let

$$\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z.$$

Now, we will prove that z is a coincidence point of f and S. For every $n \in N$, we have

$$d(fgx_{2n+1}, Sz) \leq d(fgx_{2n+1}, Sfx_{2n}) + H(Sfx_{2n}, Sz). \tag{2}$$

It follows from the H-continuity of S that

$$\lim_{n\to\infty} H(Sfx_{2n}, Sz) = 0, \tag{3}$$

since $fx_{2n} \to z$ as $n \to \infty$. Since f and S are compatible mappings and

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} y_n = z,$$

where $y_n = gx_{2n+1} \in Sx_{2n}$ and $z_n = x_{2n}$, we have

$$\lim_{n\to\infty} d(fy_n, Sfz_n) = \lim_{n\to\infty} d(fgx_{2n+1}, Sfx_{2n}) = 0. \tag{4}$$

Thus, from equations (2), (3) and (4), we have

$$\lim_{n\to\infty} d(fgx_{2n+1}, Sz) = 0$$

and so

$$d(fz, Sz) \leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz).$$

Letting n tend to infinity, it follows that $d(fz, Sz) = 0$. This implies that $fz \in Sz$, since Sz is a closed subset of X. Thus z is a coincidence point of f and S. Similarly, we can prove that z is a coincidence point of g and T. This completes the proof of the theorem. □

Letting $f = g$ be the identity mapping on X, in *Theorem 1*, we have the following corollary:

Corollary 1. *Let (X, d) be a complete metric space and let $S, T : X \mapsto CB(X)$ be H-continuous multi-valued mappings such that*

$$\begin{aligned} H^p(Sx, Ty) \quad \leq \quad & \max\{ad(x,y)d^{p-1}(y,Ty), ad(x,y)d^{p-1}(y,Ty), \\ & ad(x,Sx)d^{p-1}(y,Ty), cd^{p-1}(x,Ty)d(y,Sx)\} \end{aligned}$$

for all $x, y \in X$, where $p \geq 2$, $0 < a < 1$, $c > 0$. Then S and T have a common fixed point z in X.

Putting $f = g$ and $S = T$ in *Theorem 1*, we have the following corollary:

Corollary 2. *Let (X, d) be a complete metric space, let $f : X \to X$ be a continuous mapping and let $S : X \to CB(X)$ be an H-continuous mapping such that $S(X) \subset g(X)$, and*

$$\begin{aligned} H^p(Sx, Sy) \quad \leq \quad & \max\{ad(fx,fy)d^{p-1}(fx,Sx), ad(fx,fy)d^{p-1}(gy,Sy), \\ & cd^{p-1}(fx,Sy)d(fy,Sx)\} \end{aligned}$$

for all $x, y \in X$, where $p \geq 2$ is an integer, $0 < a < 1$ and $c \geq 0$. Then there exists a coincidence point z of f and S.

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