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Pythagorean 2-tuple linguistic power aggregation operators in multiple attribute decision making

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ABSTRACT

In this paper, we investigate the multiple attribute decision making problems with Pythagorean 2-tuple linguistic information. Then, we utilize power average and power geometric operations to develop some Pythagorean 2-tuple linguistic power aggregation operators: Pythagorean 2-tuple linguistic power weighted average (P2TLPWA) operator, Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator, Pythagorean 2-tuple linguistic power ordered weighted average (P2TLPOWA) operator, Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator, Pythagorean 2-tuple linguistic power hybrid average (P2TLPHA) operator and Pythagorean 2-tuple linguistic power hybrid geometric (P2TLPHG) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the Pythagorean 2-tuple linguistic multiple attribute decision making problems. Finally, a practical example for enterprise resource planning (ERP) system selection is given to verify the developed approach and to demonstrate its practicality and effectiveness.

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Multiple attribute decision making; Pythagorean 2-tuple linguistic set; Pythagorean 2-tuple linguistic power weighted average (P2TLPWA) operator; Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator; enterprise resource planning (ERP) system selection

JELS CLASSIFICATION
C43; C61; D81

1. Introduction

Multiple attribute decision making problems under linguistic information processing environment is an interesting research topic having received more and more attention during the last several years. One of the well-known linguistic information processing models are the 2-tuple linguistic computational model (Beg & Rashid, 2015; Dutta & Guha, 2015; Herrera, Herrera-Viedma 2000a, 2000b; Herrera, Martínez, & Sánchez, 2005; Herrera and Martínez 2001b; Martínez-López, Rodríguez, & Herrera, 2015; Lin, Wei, Wang, & Zhao, 2014; Wu et al., 2015; Zhang & Liu, 2010; Zavadskas & Turskis, 2011; Zavadskas, Turskis, & Kildienė, 2014; Zhang & Chu, 2009). Herrera and Martínez (2001a) show 2-tuple linguistic information processing manner can effectively avoid the loss and distortion of information. Herrera, Herrera-Viedma (2000a)

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developed 2-tuple arithmetic average (TAA) operator, 2-tuple weighted average (TWA) operator, 2-tuple ordered weighted average (TOWA) operator and extended 2-tuple weighted average (ET-WA) operator. Herrera et al. (2005) presented the group decision making model for managing non-homogeneous information processing. Herrera-Viedma, Martinez, Mata, and Chiclana (2005) developed the consensus support system with multi-granular linguistic preference relations. Liao, Li, and Lu (2007) used linguistic information processing model for selecting an ERP system. Herrera, Herrera-Viedma, and Martínez (2008) proposed a fuzzy linguistic methodology to deal with unbalanced linguistic term sets. Wang (2009) presented a 2-tuple fuzzy linguistic evaluation model for selecting appropriate agile manufacturing system. Tai and Chen (2009) developed the intellectual capital evaluation model linguistic variable. Fan, Feng, Sun, and Ou (2009) evaluated knowledge management capability of organizations by using a fuzzy linguistic method. Wei extended TOPSIS method to multiple attribute group decision making with 2-tuple linguistic information. Wei proposed ET-WG and ET-OWG operators for multiple attribute group decision making with 2-tuple linguistic information. Fan and Liu (2010) developed the multi-granularity uncertain linguistic group decision making model. Chang and Wen (2010) developed a novel efficient approach for DFMEA combining 2-tuple and the OWA operator. Jiang and Wei (2014) proposed some Bonferroni mean operators with 2-tuple linguistic information. Xu, Ma, Tao, and Wang (2014) developed some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making. Liu, Lin, and Wu (2014) proposed the dependent interval 2-tuple linguistic aggregation operators for multiple attribute group decision making. Dutta, Guha, and Mesiar (2015) developed a model based on linguistic 2-tuples for dealing with heterogeneous relationship among attributes in multi-expert decision making. Dong & Herrera-Viedma, (2015) proposed the consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation. Wang, Wang, Zhang, and Chen (2015) developed the multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. Qin & Liu, (2016) proposed the 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection. Zhang, Xu, and Wang (2016) developed the consensus reaching model for 2-tuple linguistic multiple attribute group decision making with incomplete weight information.

More recently, Pythagorean fuzzy set (PFS) (Yager, 2013, 2014) has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu (2014) provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number(PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS

(Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Peng and Yang (2015) proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problem with PFNs. Afterwards, Focused on how the notion of "averaging" should be treated in the case of PFNs and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat & Yager, (2014) applied the PFNs in handling the collaborative-based recommender system. Gou, Xu, and Ren (2016) investigate the Properties of Continuous Pythagorean Fuzzy Information. Ren, Xu, and Gou (2016) proposed the Pythagorean fuzzy TODIM approach to multi-criteria decision making. Garg (2016a) proposed the new generalized Pythagorean fuzzy information aggregation by using Einstein Operations. Zeng, Chen, and Li (2016) developed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg (2016b) studied a novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem.

Although, Pythagorean fuzzy set theory has been successfully applied in some areas, the PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. In order to describe the membership degree and the non-membership degree of an element to a linguistic label, which can reflect the decision maker's confidence level when they are making an evaluation, Wei et al.(2017) proposed the concept of Pythagorean 2-tuple linguistic sets(P2TLSs) and some Pythagorean 2-tuple linguistic information aggregating operators to solve this problem based on the Pythagorean fuzzy sets (Yager, 2013-2014) and 2-tuple linguistic information processing model.

From above analysis, we can see that most of the existing Pythagorean 2-tuple linguistic aggregation operators are based on the algebraic product and algebraic sum of P2TLSs to carry the aggregation process. However, all these aggregation operators do not take into account the information about the relationship between the values being fused. To overcome this drawback, motivated by the idea of power average (Yager, 2001) and power geometric operations (Xu & Yager, 2010), in this paper, we develop a series of Pythagorean 2-tuple linguistic power aggregation operators, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other, and study their desirable properties. To do so, the remainder of this paper is set out as follows. In the next section, we shall propose the concept of Pythagorean 2-tuple linguistic set on the basis of the Pythagorean fuzzy set and 2-tuple linguistic information processing model. In Section 3, we shall propose some Pythagorean 2-tuple linguistic power aggregation operators. In Section 4, we shall present we shall propose some Pythagorean 2-tuple linguistic power geometric aggregation operators. In Section 5, based on these operators, we shall present some approaches to multiple attribute decision making with Pythagorean 2-tuple linguistic information. In Section 6, we shall present a numerical example for enterprise resource planning (ERP) system selection with Pythagorean 2-tuple linguistic information in order to illustrate the method proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

In the following, we introduced some basic concepts related to 2-tuple linguistic term sets and Pythagorean fuzzy sets.

2.1. 2-tuple linguistic term sets

Let $S = \{s_i | i = 1, 2, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label,s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics (Herrera & Martínez, 2000a, 2000b; Herrera et al., 2005; Herrera & Martínez, 2001b; Xu, 2004, 2006):

1. The set is ordered:$s_i > s_j$, if $i > j$; (2) Max operator:$\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (3) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$. For example, S can be defined as

$$S = \{s_1 = extremely\ poor, s_2 = very\ poor, s_3 = poor, s_4 = medium,$$
$$s_5 = good, s_6 = very\ good, s_7 = extremely\ good\}$$

Herrera and Martínez (2000a, 2000b) developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple (s_i, α_i), where s_i is a linguistic label from predefined linguistic term set S and α_i is the value of symbolic translation, and $\alpha_i \in [-0.5, 0.5)$.

Definition 1. Let β be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation, $\beta \in [1, t]$, being t the cardinality of S. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [1, t]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation (Herrera & Martínez, 2000a, 2000b).

Definition 2. Let $S = \{s_1, s_2, \ldots, s_t\}$ be a linguistic term set and $\beta \in [1, t]$ is a number value representing the aggregation result of linguistic symbolic. Then the function Δ used to obtain the 2-tuple linguistic information equivalent to β is defined as:

$$\Delta : [1, t] \to S \times [-0.5, 0.5), \tag{1}$$

$$\Delta(\beta) = \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5), \end{cases} \tag{2}$$

where round(.) is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation (Herrera & Martínez, 2000a, 2000b).

Definition 3. Let $S = \{s_1, s_2, \ldots, s_t\}$ be a linguistic term set and (s_i, α_i) be a 2-tuple. There is always a function Δ^{-1} can be defined, such that, from a 2-tuple (s_i, α_i) it return its equivalent numerical value $\beta \in [1, t] \subset R$, which is (Herrera & Martínez, 2000a, 2000b).

$$\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [1, t], \tag{3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta, \tag{4}$$

From Definitions 1 and 2, we can conclude that the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value 0 as symbolic translation:

$$\Delta(s_i) = (s_i, 0), \tag{5}$$

2.2. Pythagorean fuzzy set

Yager, (2014) developed the concept of the Pythagorean fuzzy sets.

Definition 4. (Yager, 2014). Let X be a fix set. A PFS is an object having the form

$$P = \{\langle x, \left(\mu_P(x), \nu_P(x)\right)\rangle | x \in X\} \tag{6}$$

where the function $\mu_P : X \rightarrow [0, 1]$ defines the degree of membership and the function $\nu_P : X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively, and, for every $x \in X$, it holds that

$$\left(\mu_p(x)\right)^2 + \left(\nu_p(x)\right)^2 \leq 1. \tag{7}$$

Definition 5. Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, a score function S of a Pythagorean fuzzy number can be represented as follows:

$$S(\tilde{a}) = \frac{1}{2}(1 + \mu^2 - \nu^2), S(\tilde{a}) \in [0, 1]. \tag{8}$$

Definition 6. Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, an accuracy function H of a Pythagorean fuzzy value can be represented as follows:

$$H(\tilde{a}) = \mu^2 + \nu^2, H(\tilde{a}) \in [0, 1]. \tag{9}$$

to evaluate the degree of accuracy of the Pythagorean fuzzy number $\tilde{a} = (\mu, \nu)$, where $H(\tilde{a}) \in [0, 1]$. The larger the value of $H(\tilde{a})$, the more the degree of accuracy of the Pythagorean fuzzy number $\tilde{a}$.

Based on the score function S and the accuracy function H, in the following, we shall give an order relation between two Pythagorean fuzzy numbers, which is defined as follows:

Definition 7. Let $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ be two Pythagorean fuzzy numbers, $s(\tilde{a}_1) = \frac{1}{2}(1 + (\mu_1)^2 - (\nu_1)^2)$ and $s(\tilde{a}_2) = \frac{1}{2}(1 + (\mu_2)^2 - (\nu_2)^2)$ be the scores of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a}_1) = (\mu_1)^2 + (\nu_1)^2$ and $H(\tilde{a}_2) = (\mu_2)^2 + (\nu_2)^2$ be the accuracy degrees of $\tilde{a}$ and $\tilde{b}$, respectively, then if $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$; if $S(\tilde{a}) = S(\tilde{b})$, then

1. if $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$; (2) if $H(\tilde{a}) < H(\tilde{b})$, $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

Definition 8. (Reformat & Yager, 2014). Let $\tilde{a}_1 = (\mu_1, \nu_1)$, $\tilde{a}_2 = (\mu_2, \nu_2)$, and $\tilde{a} = (\mu, \nu)$ be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \left(\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2(\mu_2)^2}, \nu_1\nu_2\right)$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \left(\mu_1\mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2(\nu_2)^2}\right)$;
3. $\lambda\tilde{a} = (\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda), \lambda > 0$;
4. $(\tilde{a})^\lambda = (\mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda}), \lambda > 0$;
5. $\tilde{a}^c = (\nu, \mu)$.

Based on the Definition 6, we can derive the following properties easily.

Theorem 1. (Reformat & Yager, 2014). Let $\tilde{a}_1 = (\mu_1, \nu_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2)$ be two Pythagorean fuzzy numbers, $\lambda, \lambda_1, \lambda_2 > 0$, then

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$;
3. $\lambda(\tilde{a}_1 \oplus \tilde{a}_2) = \lambda\tilde{a}_1 \oplus \lambda\tilde{a}_2$;
4. $(\tilde{a}_1 \otimes \tilde{a}_2)^\lambda = (\tilde{a}_1)^\lambda \otimes (\tilde{a}_2)^\lambda$;
5. $\lambda_1\tilde{a}_1 \oplus \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1$;
6. $(\tilde{a}_1)^{\lambda_1} \otimes (\tilde{a}_1)^{\lambda_2} = (\tilde{a}_1)^{(\lambda_1 + \lambda_2)}$;
7. $((\tilde{a}_1)^{\lambda_1})^{\lambda_2} = (\tilde{a}_1)^{\lambda_1\lambda_2}$.

2.3. Pythagorean 2-tuple linguistic sets

In the following, Wei et al. proposed the concepts and basic operations of the Pythagorean 2-tuple linguistic sets on the basis of the Pythagorean fuzzy sets (Yager, 2013-2014) and 2-tuple linguistic information processing model (Herrera & Martínez, 2000a; Herrera & Martínez, 2000b).

Definition 9. A Pythagorean 2-tuple linguistic sets A in X is given

$$P = \{\left(s_{\theta(x)}, \rho\right), \left(\mu_P(x), \nu_P(x)\right), x \in X\} \tag{10}$$

where $s_{\theta(a)} \in S$, $\rho \in [-0.5, 0.5), u_P(x) \in [0, 1]$ and $\nu_P(x) \in [0, 1]$, with the condition $0 \leq (u_P(x))^2 + (\nu_P(x))^2 \leq 1, \forall x \in X$. The numbers $\mu_P(x), \nu_P(x)$ represent, respectively, the degree of positive membership, degree of negative membership and degree of negative membership of the element x to linguistic variable $(s_{\theta(x)}, \rho)$. Then for $x \in X$, $\pi_P(x) = \sqrt{1-((u_P(x))^2 + (v_P(x))^2)}$ could be called the degree of refusal membership of the element x to linguistic variable $(s_{\theta(x)}, \rho)$.

For convenience, we call $\tilde{p} = \langle (s_p, \rho), (u_p, v_p) \rangle$ a Pythagorean 2-tuple linguistic number (P2TLN), where $\mu_p \in [0,1], \nu_p \in [0,1], (\mu_p)^2 + (\nu_p)^2 \leq 1,\ s_{\theta(p)} \in S$ and $\rho \in [-0.5, 0.5)$.

Definition 10. Let $\tilde{p} = \langle (s_p, \rho), (u_p, v_p) \rangle$ be a Pythagorean 2-tuple linguistic number (P2TLN), a score function $\tilde{a}$ of a Pythagorean 2-tuple linguistic number can be represented as follows:

$$S(\tilde{p}) = \Delta\left(\Delta^{-1}\left(s_{\theta(p)}, \rho\right) \cdot \frac{1 + (\mu_p)^2 - (\nu_p)^2}{2}\right), \Delta^{-1}\left(S(\tilde{p})\right) \in [1, t]. \tag{11}$$

Definition 11. Let $\tilde{p} = \langle (s_p, \rho), (u_p, v_p) \rangle$ a Pythagorean 2-tuple linguistic number (P2TLN), an accuracy function H of a Pythagorean 2-tuple linguistic number can be represented as follows:

$$H(\tilde{p}) = \Delta\left(\Delta^{-1}\left(s_{\theta(p)}, \rho\right) \cdot \frac{(\mu_p)^2 + (\nu_p)^2}{2}\right), \Delta^{-1}\left(H(\tilde{p})\right) \in [1, t]. \tag{12}$$

to evaluate the degree of accuracy of the Pythagorean 2-tuple linguistic number $\tilde{p} = \langle (s_p, \rho), (u_p, v_p) \rangle$, where $\Delta^{-1}(H(\tilde{p})) \in [1, t]$. The larger the value of $H(\tilde{p})$, the more the degree of accuracy of the Pythagorean 2-tuple linguistic number a.

Based on the score function S and the accuracy function H, in the following, Wei et al. gave an order relation between two Pythagorean 2-tuple linguistic numbers, which is defined as follows:

Definition 12. Let $\tilde{p}_1 = \langle (s_{p_1}, \rho), (u_{p_1}, v_{p_1}) \rangle$ and $\tilde{p}_2 = \langle (s_{p_2}, \rho), (u_{p_2}, v_{p_2}) \rangle$ be two Pythagorean 2-tuple linguistic numbers, $S(\tilde{p}_1) = \Delta(\Delta^{-1}(s_{\theta(p_1)}, \rho_1) \cdot \frac{1+(\mu_{p_1})^2-(\nu_{p_1})^2}{2})$ and $S(\tilde{p}_2) = \Delta(\Delta^{-1}(s_{\theta(p_2)}, \rho_2) \cdot \frac{1+(\mu_{p_2})^2-(\nu_{p_2})^2}{2})$ be the scores of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and let $H(\tilde{p}_1) = \Delta(\Delta^{-1}(s_{\theta(p_1)}, \rho_1) \cdot \frac{(\mu_{p_1})^2+(\nu_{p_1})^2}{2})$ and $H(\tilde{p}_2) = \Delta(\Delta^{-1}(s_{\theta(p_2)}, \rho_2) \cdot \frac{(\mu_{p_2})^2+(\nu_{p_2})^2}{2})$ be the accuracy degrees of $\tilde{p}_1$ and $\tilde{p}_2$, respectively, then if $S(\tilde{p}_1) < S(\tilde{p}_2)$, then $\tilde{p}_1$ is smaller than $\tilde{p}_2$, denoted by $\tilde{p}_1 < \tilde{p}_2$; if $S(\tilde{p}_1) = S(\tilde{p}_2)$, then

1. if $H(\tilde{p}_1) = H(\tilde{p}_2)$, then $\tilde{p}_1$ and $\tilde{p}_2$ represent the same information, denoted by $\tilde{p}_1 = \tilde{p}_2$; (2) if $H(\tilde{p}_1) < H(\tilde{p}_2)$, $\tilde{p}_1$ is smaller than $\tilde{p}_2$, denoted by $\tilde{p}_1 < \tilde{p}_2$.

Motivated by the operations of the 2-tuple linguistic information(Herrera & Martínez, 2000a, 2000b) and Definition 5, in the following, Wei et al. defined some operational laws of Pythagorean 2-tuple linguistic numbers.

Definition 13. Let $\tilde{p}_1 = \langle (s_{p_1}, \rho), (u_{p_1}, v_{p_1}) \rangle$ and $\tilde{p}_2 = \langle (s_{p_2}, \rho), (u_{p_2}, v_{p_2}) \rangle$ be two Pythagorean 2-tuple linguistic numbers, then

$$\tilde{p}_1 \oplus \tilde{p}_2 = \left\langle \Delta\left(\Delta^{-1}\left(s_{\theta(p_1)}, \rho_1\right) + \Delta^{-1}\left(s_{\theta(p_2)}, \rho_2\right)\right), \left(\sqrt{(\mu_{p_1})^2 + (\mu_{p_2})^2 - (\mu_{p_1})^2(\mu_{p_2})^2}, \nu_{p_1}\nu_{p_2}\right) \right\rangle;$$

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left\langle \Delta\left(\Delta^{-1}\left(s_{\theta(p_1)}, \rho_1\right) \cdot \Delta^{-1}\left(s_{\theta(p_2)}, \rho_2\right)\right), \left(\mu_{p_1}\mu_{p_2}, \sqrt{(\nu_{p_1})^2 + (\nu_{p_2})^2 - (\nu_{p_1})^2(\nu_{p_2})^2}\right) \right\rangle;$$

$$\lambda\tilde{p}_1 = \left\langle \Delta\left(\lambda\Delta^{-1}\left(s_{\theta(p_1)}, \rho_1\right)\right), \left(\sqrt{1 - \left(1 - (\mu_{p_1})^2\right)^{\lambda}}, (\nu_{p_1})^{\lambda}\right) \right\rangle;$$

$$(\tilde{p}_1)^{\lambda} = \left\langle \Delta\left(\left(\Delta^{-1}\left(s_{\theta(p_1)}, \rho_1\right)\right)^{\lambda}\right), \left((\mu_{p_1})^{\lambda}, \sqrt{1 - \left(1 - (\nu_{p_1})^2\right)^{\lambda}}\right) \right\rangle;$$

Based on the Definition 13, Wei et al. derived the following properties easily.

Theorem 2. For any two Pythagorean 2-tuple linguistic numbers $\tilde{p}_1 = \langle (s_{p_1}, \rho), (u_{p_1}, v_{p_1}) \rangle$ and $\tilde{p}_2 = \langle (s_{p_2}, \rho), (u_{p_2}, v_{p_2}) \rangle$, it can be proved the calculation rules shown as follows

1. $\tilde{p}_1 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_1$
2. $\tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1$
3. $\lambda(\tilde{p}_1 \oplus \tilde{p}_2) = \lambda\tilde{p}_1 \oplus \lambda\tilde{p}_2, 0 \le \lambda \le 1$
4. $\lambda_1\tilde{p}_1 \oplus \lambda_2\tilde{p}_1 = (\lambda_1 \oplus \lambda_2)\tilde{p}_1, \quad 0 \le \lambda_1, \lambda_2\ , \lambda_1 + \lambda_2 \ \le 1$
5. $\tilde{p}_1^{\lambda_1} \otimes \tilde{p}_1^{\lambda_2} = (\tilde{p}_1)^{\lambda_1 + \lambda_2}, \quad 0 \le \lambda_1, \lambda_2\ , \lambda_1 + \lambda_2 \ \le 1$
6. $\tilde{p}_1^{\lambda_1} \otimes \tilde{p}_2^{\lambda_1} = (\tilde{p}_1 \otimes \tilde{p}_2)^{\lambda_1}, \ \lambda_1 \ge 0.$
7. $((\tilde{p}_1)^{\lambda_1})^{\lambda_2} = (\tilde{p}_1)^{\lambda_1\lambda_2}.$

3. Pythagorean 2-tuple linguistic power aggregation operators

Yager, (2001) developed a nonlinear weighted average aggregation operator called power average (PA) operator, which can be defined as follows:

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}\left(1 + T(a_i)\right)a_i}{\sum_{i=1}^{n}\left(1 + T(a_i)\right)} \tag{13}$$

Where $T(a_i) = \sum_{\substack{j=1 \\ j \ne i}}^{n} Sup(a_i, a_j)$, and $Sup(a, b)$ is the support for a from b, which satisfies the following three properties: (1)$Sup(a, b) \in [0, 1]$;(2)$Sup(a, b) = Sup(b, a)$; (3) $Sup(a, b) \ge Sup(x, y)$, if $|a - b| < |x - y|$. Obvoiusly, the support (Sup) measure is essentially a similarity index. The more similar, the closer two values, and the more they support each other.

In this section, we shall develop some power aggregation operators with Pythagorean 2-tuple linguistic information, such as Pythagorean 2-tuple linguistic power weighted averaging (P2TLPWA) operator, Pythagorean 2-tuple linguistic power ordered weighted averaging (P2TLPOWA) operator and Pythagorean 2-tuple linguistic power hybrid average (P2TLPHA) operator.

Definition 14. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of Pythagorean 2-tuple linguistic numbers. The Pythagorean 2-tuple linguistic power averaging (P2TLPA) operator is a mapping $P^n \to P$ such that

$$\mathrm{P2TLPA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \frac{\overset{n}{\underset{j=1}{\oplus}} \left(\left(1 + T(\tilde{p}_j) \right) \tilde{p}_j \right)}{\sum_{j=1}^{n} \left(1 + T(\tilde{p}_j) \right)} \tag{14}$$

where

$$T(\tilde{p}_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_j, \tilde{p}_i) \tag{15}$$

and $Sup(\tilde{p}_j, \tilde{p}_i)$ is the support for $\tilde{p}_j$ from $\tilde{p}_i$, with the conditions:

1. $Sup(\tilde{p}_j, \tilde{p}_i) \in [0, 1]$;
2. $Sup(\tilde{p}_j, \tilde{p}_i) = Sup(\tilde{p}_i, \tilde{p}_j)$;
3. $Sup(\tilde{p}_j, \tilde{p}_i) \geq Sup(\tilde{p}_s, \tilde{p}_t)$, if $d(\tilde{p}_j, \tilde{p}_i) < d(\tilde{p}_s, \tilde{p}_t)$, where d is a distance measure.

Based on the Definition 14 and Theorem 2, we can get the following result:

Theorem 3. The aggregated value by using P2TLPA operator is also a Pythagorean 2-tuple linguistic numbers, where

$$\begin{aligned} \mathrm{P2TLPA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) &= \frac{\overset{n}{\underset{j=1}{\oplus}} \left(\left(1 + T(\tilde{p}_j) \right) \tilde{p}_j \right)}{\sum_{j=1}^{n} \left(1 + T(\tilde{p}_j) \right)} \\ &= \left\langle \Delta \left(\sum_{j=1}^{n} \frac{\left(1 + T(\tilde{p}_j) \right) \Delta^{-1}(r_j, a_j)}{\sum_{j=1}^{n} \left(1 + T(\tilde{p}_j) \right)} \right), \right. \\ &\left. \left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - (\mu_j)^2 \right)^{\left(1+T(\tilde{p}_j)\right) / \sum_{j=1}^{n} \left(1 + T(\tilde{p}_j) \right)}}, \prod_{j=1}^{n} (\nu_j)^{\left(1+T(\tilde{p}_j)\right) / \sum_{j=1}^{n} \left(1 + T(\tilde{p}_j) \right)} \right) \right\rangle \end{aligned} \tag{16}$$

where

$$T(\tilde{p}_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_j, \tilde{p}_i) \tag{17}$$

Definition 15. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of Pythagorean 2-tuple linguistic numbers, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{p}_j (j = 1, 2, \ldots, n)$, and $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$. The Pythagorean 2-tuple linguistic power weighted averaging (P2TLPWA) operator is a mapping $P^n \rightarrow P$ such that

$$\mathrm{P2TLPWA}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \frac{\bigoplus_{j=1}^{n}\left(\omega_j\left(1+T(\tilde{p}_j)\right)\tilde{p}_j\right)}{\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)} \tag{18}$$

where

$$T(\tilde{p}_j) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \omega_i Sup(\tilde{p}_j, \tilde{p}_i) \tag{19}$$

Based on the Definition 15, Theorem 2 and mathematical induction on n, we can get the following result:

Theorem 4. The aggregated value by using P2TLPWA operator is also a Pythagorean 2-tuple linguistic numbers, where

$$\begin{aligned}
&\mathrm{P2TLPWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \\
&= \frac{\bigoplus_{j=1}^{n}\left(\omega_j\left(1+T(\tilde{p}_j)\right)\tilde{p}_j\right)}{\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)} \\
&= \left\langle \Delta\left(\sum_{j=1}^{n}\frac{\omega_j\left(1+T(\tilde{p}_j)\right)\Delta^{-1}(r_j, a_j)}{\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}\right), \right. \\
&\left(\sqrt{1-\prod_{j=1}^{n}\left(1-(\mu_j)^2\right)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}}, \right. \\
&\left.\left.\prod_{j=1}^{n}(\nu_j)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}\right)\right\rangle
\end{aligned} \tag{20}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{p}_j (j = 1, 2, \ldots, n)$, and $\omega_j > 0$, $\sum_{j=1}^{n}\omega_j = 1$.

It can be easily proved that the P2TLPWA operator has the following properties.

Theorem 5. (Idempotency) If all $\tilde{p}_j (j = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{p}_j = \tilde{p}$ for all j, then

$$\mathrm{P2TLPWA}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p} \tag{21}$$

Theorem 6. (Boundedness) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ be a collection of P2TLNs, and let

$$\tilde{p}^- = \min_j \tilde{p}_j, \tilde{p}^+ = \max_j \tilde{p}_j$$

Then

$$\tilde{p}^- \leq \text{P2TLPWA}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+ \tag{22}$$

Theorem 7. (Monotonicity) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ and $\tilde{p}'_j (j = 1, 2, \ldots, n)$ be two set of P2TLNs, if $\tilde{p}_j \leq \tilde{p}'_j$, for all j, then

$$\text{P2TLPWA}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{P2TLPWA}_\omega(\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_n) \tag{23}$$

Further, we give a Pythagorean 2-tuple linguistic power ordered weighted averaging (P2TLPOWA) operator below:

Definition 16. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of P2TLNs, the Pythagorean 2-tuple linguistic power ordered weighted averaging (P2TLPOWA) operator of dimension n is a mapping P2TLPOWA:$P^n \to P$, that has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$. Furthermore,

$$\begin{aligned}
&\text{P2TLPOWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \\
&= \frac{\bigoplus_{j=1}^{n} \left(w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right) \tilde{p}_{\sigma(j)} \right)}{\sum_{j=1}^n w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right)} \\
&= \left\langle \Delta \left(\sum_{j=1}^{n} \frac{w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right) \Delta^{-1}\left(r_j, a_{\sigma(j)}\right)}{\sum_{j=1}^n w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right)} \right), \right. \\
&\left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{\sigma(j)}\right)^2\right)^{w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right) / \sum_{j=1}^{n} w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right)}}, \right. \\
&\left. \left. \prod_{j=1}^{n} \left(\nu_{\sigma(j)}\right)^{w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right) / \sum_{j=1}^{n} w_j \left(1 + T\left(\tilde{p}_{\sigma(j)}\right)\right)} \right) \right\rangle
\end{aligned} \tag{24}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\tilde{p}_{\sigma(j-1)} \geq \tilde{p}_{\sigma(j)}$ for all $j = 2, \ldots, n$, $w_j (j = 1, 2, \ldots, n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)}, TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\tilde{p}_{\sigma(i)}\right) \tag{25}$$

and $T(\tilde{p}_{\sigma(j)})$ denotes the support of the jth largest Pythagorean 2-tuple linguistic numbers $T(\tilde{p}_{\sigma(j)})$ by all the other Pythagorean 2-tuple linguistic numbers, i.e.,

$$T\left(\tilde{p}_{\sigma(j)}\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup\left(\tilde{p}_{\sigma(j)}, \tilde{p}_{\sigma(i)}\right) \tag{26}$$

Where $\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_{\sigma(j)}, \tilde{p}_{\sigma(i)})$ indicates the support of jth largest Pythagorean 2-tuple linguistic number $\tilde{p}_{\sigma(j)}$ for the ith largest Pythagorean 2-tuple linguistic number $\tilde{p}_{\sigma(i)}$, and g: $[0,1] \to [0,1]$ is a basic unit-interval monotonic(BUM) function, having the properties: g(0)=0, g(1)=1, and $g(x) \geq g(y), if\, x > y$.

It can be easily proved that the P2TLPOWA operator has the following properties.

Theorem 8. (Idempotency) If all $\tilde{p}_j (j = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{p}_j = \tilde{p}$ for all j, then

$$\text{P2TLPOWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p} \tag{27}$$

Theorem 9. (Boundedness) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ be a collection of P2TLNs, and let

$$\tilde{p}^- = \min_j \tilde{p}_j, \tilde{p}^+ = \max_j \tilde{p}_j$$

Then

$$\tilde{p}^- \leq \text{P2TLPOWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+ \tag{28}$$

Theorem 10. (Monotonicity) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ and $\tilde{p}'_j (j = 1, 2, \ldots, n)$ be two set of P2TLNs, if $\tilde{p}_j \leq \tilde{p}'_j$, for all j, then

$$\text{P2TLPOWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{P2TLPOWA}(\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_n) \tag{29}$$

Theorem 11. (Commutativity) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ and $\tilde{p}'_j (j = 1, 2, \ldots, n)$ be two set of P2TLNs, for all j, then

$$\text{P2TLOWA}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \text{P2TLOWA}_w(\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_n) \tag{30}$$

where $\tilde{p}'_j (j = 1, 2, \ldots, n)$ is any permutation of $\tilde{p}_j (j = 1, 2, \ldots, n)$.

From Definitions 15-16, we know that the P2TLPWA operators only weights the Pythagorean 2-tuple linguistic number itself, while the P2TLPOWA operators weights the ordered positions of the Pythagorean 2-tuple linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the P2TLPWA and P2TLPOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the Pythagorean 2-tuple linguistic power hybrid average (P2TLPHA) operator.

Definition 17. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of P2TLNs. A Pythagorean 2-tuple linguistic power hybrid average (P2TLPHA) operator is a mapping P2TLPHA:$P^n \to P$, such that

$$\mathrm{P2TLPHA}_{\omega, w}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \frac{\bigoplus_{j=1}^{n} \left(w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right) \dot{\tilde{p}}_{\sigma(j)} \right)}{\sum_{j=1}^{n} w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)}$$

$$= \left\langle \Delta \left(\frac{\sum_{j=1}^{n} w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right) \Delta^{-1}\left(\dot{r}_j, \dot{a}_{\sigma(j)}\right)}{\sum_{j=1}^{n} w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)} \right), \right.$$

$$\left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\dot{\mu}_{\sigma(j)}\right)^2\right)^{w_j\left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right) / \sum_{j=1}^{n} w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)}}, \right.$$

$$\left. \left. \prod_{j=1}^{n} \left(\dot{\nu}_{\sigma(j)}\right)^{w_j\left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right) / \sum_{j=1}^{n} w_j \left(1 + T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)} \right) \right\rangle \tag{31}$$

where $w = (w_1, w_2, \ldots, w_n)$ is the associated weighting vector, with $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, and $\dot{\tilde{p}}_{\sigma(j)}$ is the j-th largest element of the Pythagorean 2-tuple linguistic arguments $\dot{\tilde{p}}_j (\dot{\tilde{p}}_j = (n\omega_j) \tilde{p}_j, j = 1, 2, \ldots, n), \omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of Pythagorean 2-tuple linguistic arguments $\tilde{p}_j (j = 1, 2, \ldots, n)$, with $\omega_i \in [0, 1], \sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient. And $w_j (j = 1, 2, \ldots, n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)}, TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\dot{\tilde{p}}_{\sigma(i)}\right) \tag{32}$$

and $T(\dot{\tilde{p}}_{\sigma(j)})$ denotes the support of the jth largest Pythagorean 2-tuple linguistic numbers $T(\dot{\tilde{p}}_{\sigma(j)})$ by all the other Pythagorean 2-tuple linguistic numbers, i.e.,

$$T\left(\dot{\tilde{p}}_{\sigma(j)}\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup\left(\dot{\tilde{p}}_{\sigma(j)}, \dot{\tilde{p}}_{\sigma(i)}\right) \tag{33}$$

where $\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\dot{\tilde{p}}_{\sigma(j)}, \dot{\tilde{p}}_{\sigma(i)})$ indicates the support of jth largest Pythagorean 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(j)}$ for the ith largest Pythagorean 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i)}$, and g: $[0, 1] \to [0, 1]$ is a basic unit-interval monotonic(BUM) function, having the properties: g(0)=0, g(1)=1, and $g(x) \geq g(y), if\, x > y$. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then P2TLPHA is reduced to the Pythagorean 2-tuple linguistic power weighted average (P2TLPWA) operator; if $\omega = (1/n, 1/n, \ldots, 1/n)$, then P2TLPHA is reduced to the Pythagorean 2-tuple linguistic power ordered weighted average (P2TLPOWA) operator.

4. Pythagorean 2-tuple linguistic power geometric aggregation operators

Based on the PA operator (Yager, 2001) and geometric mean, in the following, Xu & Yager, (2010) further define a power geometric (PG) operator:

$$PG(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}\left(1+T(a_i)\right)}} \tag{34}$$

Obviously, the PA and PG operators are two nonlinear weighted aggregation tools, whose weighting vectors depend upon the input values and allow values being aggregated to support and reinforce each other, that's to say, the closer a_i and a_j, the more similar they are, and the more they support each other.

In this section, we shall develop some power geometric aggregation operators with Pythagorean 2-tuple linguistic information, such as Pythagorean 2-tuple linguistic power geometric (P2TLPG) operator, Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator, Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator and Pythagorean 2-tuple linguistic power hybrid geometric (P2TLPHG) operator.

Definition 18. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of P2TLNs. The Pythagorean 2-tuple linguistic power geometric (P2TLPG) operator is a mapping $P^n \to P$ such that

$$\text{P2TLPG}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \bigotimes_{j=1}^{n} (\tilde{p}_j)^{\left(1+T(\tilde{p}_j)\right) / \sum_{j=1}^{n}\left(1+T(\tilde{p}_j)\right)} \tag{35}$$

where

$$T(\tilde{p}_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_j, \tilde{p}_i) \tag{36}$$

and $Sup(\tilde{p}_j, \tilde{p}_i)$ is the support for $\tilde{p}_j$ from $\tilde{p}_i$, with the conditions:

1. $Sup(\tilde{p}_j, \tilde{p}_i) \in [0, 1]$;
2. $Sup(\tilde{p}_j, \tilde{p}_i) = Sup(\tilde{p}_i, \tilde{p}_j)$;
3. $Sup(\tilde{p}_j, \tilde{p}_i) \geq Sup(\tilde{p}_s, \tilde{p}_t)$, if $d(\tilde{p}_j, \tilde{p}_i) < d(\tilde{p}_s, \tilde{p}_t)$, where d is a distance measure.

Based on the Definition 18 and Theorem 2, we can get the following result:

Theorem 12. The aggregated value by using P2TLPG operator is also a Pythagorean 2-tuple linguistic numbers, where

$$
\begin{aligned}
&\mathrm{P2TLPG}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \\
&= \overset{n}{\underset{j=1}{\otimes}} (\tilde{p}_j)^{\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\left(1+T(\tilde{p}_j)\right)} \\
&= \left\langle \Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}(r_j, a_j)\right)^{\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\left(1+T(\tilde{p}_j)\right)}\right), \right. \\
&\left. \left(\prod_{j=1}^{n}(\mu_j)^{\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\left(1+T(\tilde{p}_j)\right)}, \sqrt{1-\prod_{j=1}^{n}\left(1-(\nu_j)^2\right)^{\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\left(1+T(\tilde{p}_j)\right)}}\right)\right\rangle
\end{aligned}
\tag{37}
$$

where

$$
T(\tilde{p}_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_j, \tilde{p}_i) \tag{38}
$$

Definition 19. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of P2TLNs, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{p}_j (j = 1, 2, \ldots, n)$, and $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$. The Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator is a mapping $P^n \rightarrow P$ such that

$$
\mathrm{P2TLPWG}_{\omega}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \overset{n}{\underset{j=1}{\otimes}} (\tilde{p}_j)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)} \tag{39}
$$

where

$$
T(\tilde{p}_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \omega_i Sup(\tilde{p}_j, \tilde{p}_i) \tag{40}
$$

Based on the Definition 19, Theorem 2 and mathematical induction on n, we can get the following result:

Theorem 13. The aggregated value by using P2TLPWG operator is also a Pythagorean 2-tuple linguistic numbers, where

$$\begin{aligned}
&\mathrm{P2TLPWG}_{\omega}(\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_n)\\
&=\overset{n}{\underset{j=1}{\otimes}}(\tilde{p}_j)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}\\
&=\left\langle \Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}(r_j,a_j)\right)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}\right),\right.\\
&\left(\prod_{j=1}^{n}(\mu_j)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)},\right.\\
&\left.\left.\sqrt{1-\prod_{j=1}^{n}\left(1-(\nu_j)^2\right)^{\omega_j\left(1+T(\tilde{p}_j)\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_j)\right)}}\right)\right\rangle
\end{aligned} \tag{41}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the weight vector of $\tilde{p}_j(j=1,2,\ldots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$.

It can be easily proved that the P2TLPWG operator has the following properties.

Theorem 14. (Idempotency) If all $\tilde{p}_j(j=1,2,\ldots,n)$ are equal, i.e., $\tilde{p}_j=\tilde{p}$ for all j, then

$$\mathrm{P2TLPWG}_{\omega}(\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_n)=\tilde{p} \tag{42}$$

Theorem 15. (Boundedness) Let $\tilde{p}_j(j=1,2,\ldots,n)$ be a collection of P2TLNs, and let

$$\tilde{p}^-=\min_j \tilde{p}_j, \tilde{p}^+=\max_j \tilde{p}_j$$

Then

$$\tilde{p}^-\le \mathrm{P2TLPWG}_{\omega}(\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_n)\le\tilde{p}^+ \tag{43}$$

Theorem 16. (Monotonicity) Let $\tilde{p}_j(j=1,2,\ldots,n)$ and $\tilde{p}'_j(j=1,2,\ldots,n)$ be two set of P2TLNs, if $\tilde{p}_j\le\tilde{p}'_j$, for all j, then

$$\mathrm{P2TLPWG}_{\omega}(\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_n)\le \mathrm{P2TLPWG}_{\omega}(\tilde{p}'_1,\tilde{p}'_2,\ldots,\tilde{p}'_n) \tag{44}$$

Further, we give a Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator below:

Definition 20. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\nu_j)\rangle(j=1,2,\ldots,n)$ be a collection of P2TLNs, the Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator of dimension n is a mapping P2TLPOWG:$P^n \to P$, that has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^n w_j = 1$. Furthermore,

$$\begin{aligned}
&\mathrm{P2TLPOWG}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \\
&= \bigotimes_{j=1}^{n} \left(\tilde{p}_{\sigma(j)}\right)^{w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)} \\
&= \left\langle \Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{\sigma(j)}, a_{\sigma(j)}\right)\right)^{w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)}\right),\right. \\
&\left(\prod_{j=1}^{n}\left(\mu_{\sigma(j)}\right)^{w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)}\right), \\
&\left.\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\nu_{\sigma(j)}\right)^2\right)^{w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\tilde{p}_{\sigma(j)}\right)\right)}}\right)\right\rangle
\end{aligned} \tag{45}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\tilde{p}_{\sigma(j-1)} \geq \tilde{p}_{\sigma(j)}$ for all $j = 2, \ldots, n$, $w_j (j = 1, 2, \ldots, n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)}, TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\tilde{p}_{\sigma(i)}\right) \tag{46}$$

and $T(\tilde{p}_{\sigma(j)})$ denotes the support of the jth largest Pythagorean 2-tuple linguistic numbers $T(\tilde{p}_{\sigma(j)})$ by all the other Pythagorean 2-tuple linguistic numbers, i.e.,

$$T\left(\tilde{p}_{\sigma(j)}\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup\left(\tilde{p}_{\sigma(j)}, \tilde{p}_{\sigma(i)}\right) \tag{47}$$

Where $\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\tilde{p}_{\sigma(j)}, \tilde{p}_{\sigma(i)})$ indicates the support of jth largest Pythagorean 2-tuple linguistic number $\tilde{p}_{\sigma(j)}$ for the ith largest Pythagorean 2-tuple linguistic number $\tilde{p}_{\sigma(i)}$, and g: $[0, 1] \to [0, 1]$ is a basic unit-interval monotonic(BUM) function, having the properties: g(0)=0, g(1)=1, and $g(x) \geq g(y), if x>y$.

It can be easily proved that the P2TLPOWG operator has the following properties.

Theorem 17. (Idempotency) If all $\tilde{p}_j (j = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{p}_j = \tilde{p}$ for all j, then

$$\mathrm{P2TLPOWG}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p} \tag{48}$$

Theorem 18. (Boundedness) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ be a collection of P2TLNs, and let

$$\tilde{p}^- = \min_j \tilde{p}_j, \tilde{p}^+ = \max_j \tilde{p}_j$$

Then

$$\tilde{p}^{-} \leq \mathrm{P2TLPOWG}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^{+} \tag{49}$$

Theorem 19. (Monotonicity) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ and $\tilde{p}'_j (j = 1, 2, \ldots, n)$ be two set of P2TLNs, if $\tilde{p}_j \leq \tilde{p}'_j$, for all j, then

$$\mathrm{P2TLPOWG}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \mathrm{P2TLPOWG}_w(\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_n) \tag{50}$$

Theorem 20. (Commutativity) Let $\tilde{p}_j (j = 1, 2, \ldots, n)$ and $\tilde{p}'_j (j = 1, 2, \ldots, n)$ be two set of P2TLNs, for all j, then

$$\mathrm{P2TLPOWG}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \mathrm{P2TLPOWG}_w(\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_n) \tag{51}$$

where $\tilde{p}'_j (j = 1, 2, \ldots, n)$ is any permutation of $\tilde{p}_j (j = 1, 2, \ldots, n)$.

From Definitions 19-20, we know that the P2TLPWG operators only weights the Pythagorean 2-tuple linguistic number itself, while the P2TLPOWG operators weights the ordered positions of the Pythagorean 2-tuple linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the P2TLPWG and P2TLPOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the Pythagorean 2-tuple linguistic power hybrid geometric (P2TLPHG) operator.

Definition 21. **A** Pythagorean 2-tuple linguistic power hybrid geometric (P2TLPHG) operator is a mapping P2TLPHG:$P^n \to P$, such that

$$\begin{aligned}
&\mathrm{P2TLPHG}_{w,\omega}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \\
&= \overset{n}{\underset{j=1}{\otimes}} \left(\dot{\tilde{p}}_{\sigma(j)}\right)^{w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)} \\
&= \left\langle \Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(\dot{r}_{\sigma(j)}, \dot{a}_{\sigma(j)}\right)\right)^{w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)}\right), \right. \\
&\left(\prod_{j=1}^{n}\left(\dot{\mu}_{\sigma(j)}\right)^{w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)}\right), \\
&\left.\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\dot{\nu}_{\sigma(j)}\right)^2\right)^{w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)/\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(j)}\right)\right)}}\right\rangle
\end{aligned} \tag{52}$$

where $w = (w_1, w_2, \ldots, w_n)$ is the associated weighting vector, with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $\dot{\tilde{p}}_{\sigma(j)}$ is the j-th largest element of the Pythagorean 2-tuple linguistic arguments $\tilde{p}_j(\dot{\tilde{p}}_j = (\tilde{p}_j)^{n\omega_j}, j = 1, 2, \ldots, n), \omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of Pythagorean 2-tuple linguistic arguments $\tilde{p}_j(j = 1, 2, \ldots, n)$, with $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient. And $w_j(j = 1, 2, \ldots, n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)}, TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\dot{\tilde{p}}_{\sigma(i)}\right) \tag{53}$$

and $T(\dot{\tilde{p}}_{\sigma(j)})$ denotes the support of the jth largest Pythagorean 2-tuple linguistic numbers $T(\dot{\tilde{p}}_{\sigma(j)})$ by all the other Pythagorean 2-tuple linguistic numbers, i.e.,

$$T\left(\dot{\tilde{p}}_{\sigma(j)}\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} Sup\left(\dot{\tilde{p}}_{\sigma(j)}, \dot{\tilde{p}}_{\sigma(i)}\right) \tag{54}$$

where $\sum_{\substack{i=1 \\ i \neq j}}^{n} Sup(\dot{\tilde{p}}_{\sigma(j)}, \dot{\tilde{p}}_{\sigma(i)})$ indicates the support of jth largest Pythagorean 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(j)}$ for the ith largest Pythagorean 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i)}$, and g: $[0,1] \to [0,1]$ is a basic unit-interval monotonic(BUM) function, having the properties: g(0)=0, g(1)=1, and $g(x) \geq g(y), if\, x>y$. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then P2TLPHG is reduced to the Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator; if $\omega = (1/n, 1/n, \ldots, 1/n)$, then P2TLPHG is reduced to the Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator.

5. Models for multiple attribute decision making with pythagorean 2-tuple linguistic information

Based the P2TLPWA (P2TLPWG) operators, in this section, we shall propose the model for multiple attribute decision making with Pythagorean 2-tuple linguistic information. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of the attribute $G_j(j = 1, 2, \ldots, n)$, where $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $\tilde{R} = (\tilde{r}_{ij})_{m \times n} = \langle (s_{ij}, \rho_{ij}), (\mu_{ij}, \nu_{ij}) \rangle_{m \times n}$ is the Pythagorean 2-tuple linguistic decision matrix, where $\tilde{r}_{ij}$ take the form of the Pythagorean 2-tuple linguistic numbers, where μ_{ij} indicates the degree that the alternative A_i satisfies the attribute G_j given by the decision maker, ν_{ij} indicates the degree that the alternative A_i doesn't satisfy the attribute G_j given by the decision maker,$\mu_{ij} \in [0,1]$, $\eta_{ij} \in [0,1]$ $\nu_{ij} \in [0,1]$, $(\mu_{ij})^2 + (\nu_{ij})^2 \leq 1$, $\pi_{ij} = \sqrt{1 - ((\mu_{ij})^2 + (\nu_{ij})^2)}$, $s_{ij} \in S$, $\rho_{ij} \in [-0.5, 0.5)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

In the following, we apply the P2TLPWA (P2TLPWG) operator to the MADM problems with hesitant fuzzy information.

Step 1. Calculate the supports:

$$Sup(\tilde{p}_{ij}, \tilde{p}_{ik}) = 1 - d(\tilde{p}_{ij}, \tilde{p}_{ik}), j, k = 1, 2, \ldots, n. \tag{55}$$

which satisfies the support conditions (1)–(3) in Section 3. Here, without loss of generality, we calculate $d(\tilde{p}_{ij}, \tilde{p}_{ik})$ with the normalized Hamming distance [9]:

$$d(\tilde{p}_{ij}, \tilde{p}_{ik}) = \frac{|\Delta^{-1}(s_{ij}, \rho_{ij}) - \Delta^{-1}(s_{ik}, \rho_{ik})|}{t} \cdot \frac{(|\mu_{ij} - \mu_{ik}| + |\nu_{ij} - \nu_{ik}|)}{2}$$
$$j, k = 1, 2, \ldots, n. \tag{56}$$

Step 2. Utilize the weights $\omega_j (j = 1, 2, \ldots, n)$ of the attribute $G_j (j = 1, 2, \ldots, n)$ to calculate the weighted support $T(\tilde{p}_{ij})$ of the P2TLN $\tilde{p}_{ij}$ by the other P2TLN $\tilde{p}_{ik} (k = 1, 2, \ldots, n, k \neq j)$:

$$T(\tilde{p}_{ij}) = \sum_{\substack{k=1 \\ k \neq j}}^{n} \omega_j Sup(\tilde{p}_{ij}, \tilde{p}_{ik}) \tag{57}$$

and calculate the weight $\xi_{ij} (j = 1, 2, \ldots, n)$ associated with the P2TLN $\tilde{p}_{ij} (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$:

$$\xi_{ij} = \frac{\omega_j \left(1 + T(\tilde{p}_{ij})\right)}{\sum_{j=1}^{n} \omega_j \left(1 + T(\tilde{p}_{ij})\right)}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \tag{58}$$

where $\xi_{ij} \geq 0, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,$ and $\sum_{j=1}^{n} \xi_{ij} = 1, i = 1, 2, \ldots, m.$

Step 3. We utilize the decision information given in matrix $\tilde{P}$, and the P2TLPWA operator

$$\begin{aligned}
\tilde{p}_i = \text{P2TLPWA}_{\omega}(\tilde{p}_{i1}, \tilde{p}_{i2}, \ldots, \tilde{p}_{in}) &= \frac{\bigoplus_{j=1}^{n} \left(\omega_j \left(1 + T(\tilde{p}_{ij})\right) \tilde{p}_{ij}\right)}{\sum_{j=1}^{n} \omega_j \left(1 + T(\tilde{p}_{ij})\right)} \\
&= \left\langle \Delta \left(\sum_{j=1}^{n} \frac{\omega_j \left(1 + T(\tilde{p}_{ij})\right) \Delta^{-1}(s_{ij}, \rho_{ij})}{\sum_{j=1}^{n} \omega_j \left(1 + T(\tilde{p}_{ij})\right)} \right), \right. \\
&\left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - (\mu_{ij})^2\right)^{\omega_j \left(1 + T(\tilde{p}_{ij})\right) / \sum_{j=1}^{n} \omega_j \left(1 + T(\tilde{p}_{ij})\right)}}, \right. \\
&\left. \left. \prod_{j=1}^{n} (\nu_{ij})^{\omega_j \left(1 + T(\tilde{p}_{ij})\right) / \sum_{j=1}^{n} \omega_j \left(1 + T(\tilde{p}_{ij})\right)} \right) \right\rangle
\end{aligned} \tag{59}$$

Or

$$\tilde{p}_i = \text{P2TLPWG}_\omega(\tilde{p}_{i1}, \tilde{p}_{i2}, \ldots, \tilde{p}_{in}) = \overset{n}{\underset{j=1}{\otimes}} (\tilde{p}_{ij})^{\omega_j\left(1+T(\tilde{p}_{ij})\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_{ij})\right)}$$

$$= \left\langle \Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}(s_{ij}, \rho_{ij})\right)^{\omega_j\left(1+T(\tilde{p}_{ij})\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_{ij})\right)}\right), \right.$$

$$\left(\prod_{j=1}^{n}(\mu_{ij})^{\omega_j\left(1+T(\tilde{p}_{ij})\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_{ij})\right)}\right),$$

$$\left.\sqrt{1-\prod_{j=1}^{n}\left(1-(\nu_{ij})^2\right)^{\omega_j\left(1+T(\tilde{p}_{ij})\right)/\sum_{j=1}^{n}\omega_j\left(1+T(\tilde{p}_{ij})\right)}}\right\rangle \tag{60}$$

to derive the overall preference values $\tilde{p}_i (i = 1, 2, \ldots, m)$ of the alternative A_i.

Step 4. Calculate the scores $S(\tilde{p}_i)(i = 1, 2, \ldots, m)$ of the overall Pythagorean 2-tuple linguistic numbers $\tilde{p}_i (i = 1, 2, \ldots, m)$ to rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and then to select the best one(s). If there is no difference between two scores $S(\tilde{p}_i)$ and $S(\tilde{p}_j)$, then we need to calculate the accuracy degrees $H(\tilde{p}_i)$ and $H(\tilde{p}_j)$ of the overall Pythagorean 2-tuple linguistic numbers $\tilde{p}_i$ and $\tilde{p}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(\tilde{p}_i)$ and $H(\tilde{p}_j)$.

Step 5. Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with $S(\tilde{p}_i)(i = 1, 2, \ldots, m)$.

Step 6. End.

6. Numerical example and comparative analysis

6.1. Numerical example

In this section, we utilize a practical multiple attribute decision making problems to illustrate the application of the developed approaches. Suppose an organization plans to implement enterprise resource planning (ERP) system (adapted from Liao et al., 2007). The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project term choose five potential ERP systems $A_i (i = 1, 2, \ldots, 5)$ as candidates. The company employs some external professional organizations (or experts) to aid this decision-making. The project team selects four attributes to evaluate the alternatives: (1) function and technology G_1, (2) strategic fitness G_2, (3) vendor's ability G_3; (4) vendor's reputation G_4. The five possible ERP systems $A_i (i = 1, 2, \ldots, 5)$ are to be evaluated using the Pythagorean 2-tuple linguistic

Table 1. The Pythagorean 2-tuple linguistic decision matrix.

	G_1	G_2
A_1	$<(S_4,0),(0.50,0.80)>$	$<(S_2,0),\ (0.60,0.30)>$
A_2	$<(S_1,0),\ (0.70,0.50)>$	$<(S_4,0),\ (0.70,0.20)>$
A_3	$<(S_5,0),\ (0.60,0.40)>$	$<(S_1,0),\ (0.50,0.70)>$
A_4	$<(S_5,0),\ (0.80,0.10)>$	$<(S_6,0),\ (0.60,0.30)>$
A_5	$<(S_3,0),\ (0.60,0.40)>$	$<(S_1,0),\ (0.40,0.80)>$
	G_3	**G_4**
A_1	$<(S_1,0),\ (0.30,0.60)>$	$<(S_3,0),\ (0.50,0.70)>$
A_2	$<(S_2,0),\ (0.70,0.20)>$	$<(S_4,0),\ (0.40,0.50)>$
A_3	$<(S_4,0),\ (0.50,0.30)>$	$<(S_2,0),\ (0.60,0.30)>$
A_4	$<(S_7,0),\ (0.30,0.40)>$	$<(S_1,0),\ (0.50,0.60)>$
A_5	$<(S_3,0),\ (0.70,0.60)>$	$<(S_1,0),\ (0.50,0.80)>$

Source: Author calculation.

numbers by the decision makers under the above four attributes (whose weighting vector is $\omega = (0.2, 0.1, 0.3, 0.4)$), and construct the following matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times4}$ is shown in Table 1.

In the following, in order to select the most desirable ERP systems, we utilize the P2TLPWA (P2TLPWG) operator to develop an approach to multiple attribute decision making problems with Pythagorean 2-tuple linguistic information, which can be described as following.

Step 1. Utilize (53)–(56) to calculate the weight $\xi_{ij}(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ associated with the P2TLN $\tilde{p}_{ij}(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, which are contained in the matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times4}$ which is shown in Table 1.

$$\xi = \begin{bmatrix} 0.1986 & 0.0996 & 0.3004 & 0.4014 \\ 0.2010 & 0.1005 & 0.3031 & 0.3954 \\ 0.2001 & 0.0980 & 0.3009 & 0.4010 \\ 0.2031 & 0.1030 & 0.3069 & 0.3869 \\ 0.2015 & 0.0988 & 0.3025 & 0.3972 \end{bmatrix}$$

Step 2. According to ξ and Table 1, aggregate all Pythagorean 2-tuple linguistic numbers $\tilde{r}_{ij}(j = 1, 2, \ldots, n)$ by using the P2TLPWA (P2TLPWG) operator to derive the overall Pythagorean 2-tuple linguistic numbers $\tilde{p}_i(i = 1, 2, 3, 4, 5)$ of the alternative A_i. The aggregating results are shown in Table 2.

Step 3. According to the aggregating results shown in Table 2 and the score functions of the ERP systems are shown in Table 3.

Step 4. According to the score functions shown in Table 3 and the comparison formula of score functions, the ordering of the ERP systems are shown in Table 4. Note that " > " means "preferred to". As we can see, depending on the aggregation operators used, the ordering of the ERP systems is slightly different, and the best ERP system is A_4 or A_3.

From Tables 4, we can easily find that these two operators may generate slightly different ranking results. The main reason causing this ranking result difference is that the P2TLPWA operator emphasize the group influences, however the P2TLPWG operator emphasize the individual influences.

Table 2. The aggregating results of the ERP systems by the P2TLPWA (P2TLPWG) operators.

	P2TLPWA	P2TLPWG
A_1	<(s_3,-0.50),(0.46,0.63)>	<(s_2,0.19), (0.44,0.67)>
A_2	<(s_3,-0.21), (0.61,0.35)>	<(s_2,0.45), (0.56,0.40)>
A_3	<(s_3,0.10), (0.56,0.35)>	<(s_3,-0.33), (0.56,0.38)>
A_4	<(s_4,0.17), (0.55,0.34)>	<(s_3,0.03), (0.48,0.43)>
A_5	<(s_2,0.01), (0.58,0. 64)>	<(s_2,-0.26), (0.56,0.69)>

Source: Author calculation.

Table 3. The score functions of the ERP systems.

	P2TLPWA	P2TLPWG
A_1	(s_1,0.02)	(s_1,-0.19)
A_2	(s_2,-0.24)	(s_1,0.42)
A_3	(s_2,-0.14)	(s_2,-0.38)
A_4	(s_3,-0.47)	(s_2,-0.42)
A_5	(s_1,-0.06)	(s_1,-0.27)

Source: Author calculation.

Table 4. Ordering of the ERP systems.

	Ordering
P2TLPWA	$A_4 > A_3 > A_2 > A_1 > A_5$
P2TLPWG	$A_3 > A_4 > A_2 > A_1 > A_5$

Source: Author calculation.

6.2. Comparative analysis

In what follows, we compare our proposed method with other existing methods including the Pythagorean 2-tuple linguistic weighted average (P2TLWA) operator and Pythagorean 2-tuple linguistic weighted geometric (P2TLWG) operator which are proposed as follows:

Definition 22. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \nu_j) \rangle (j = 1, 2, \ldots, n)$ be a collection of Pythagorean 2-tuple linguistic numbers, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\tilde{a}_j (j = 1, 2, \ldots, n)$, and $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$. Then

$$\mathrm{P2TLWA}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \mathop{\oplus}_{j=1}^{n} (\omega_j \tilde{p}_j)$$
$$= \langle \Delta\Big(\sum_{j=1}^{n} \omega_j \Delta^{-1}(r_j, a_j)\Big), \Big(\sqrt{1 - \prod_{j=1}^{n}\Big(1 - (\mu_j)^2\Big)^{w_j}}, \prod_{j=1}^{n} (\nu_j)^{\omega_j}\Big)\rangle \tag{61}$$

$$\mathrm{P2TLWG}_\omega(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \mathop{\otimes}_{j=1}^{n} (\tilde{p}_j)^{\omega_j}$$
$$= \langle \Delta\Big(\prod_{j=1}^{n}\Big(\Delta^{-1}(r_j, a_j)^{\omega_j}\Big)\Big), \Big(\prod_{j=1}^{n} (\mu_j)^{\omega_j}, \sqrt{1 - \prod_{j=1}^{n}\Big(1 - (\nu_j)^2\Big)^{\omega_j}}\Big)\rangle \tag{62}$$

By utilizing the decision information given in matrix $\tilde{R}$, and the P2TLWA and P2TLWG operators, the aggregating results are shown in Table 5.

Table 5. The aggregating results of the ERP systems by the P2TLWA (P2TLWG) operators.

	P2TLWA	P2TLWG
A_1	$<(s_3,-0.50),(0.47,0.46)>$	$<(s_2,0.19), (0.44,0.68)>$
A_2	$<(s_3,-0.20), (0.61,0.35)>$	$<(s_2,0.46), (0.56,0.42)>$
A_3	$<(s_3,0.10), (0.56,0.35)>$	$<(s_3,-0.24), (0.56,0.39)>$
A_4	$<(s_4,0.10), (0.57,0.35)>$	$<(s_3,-0.04), (0.48,0.46)>$
A_5	$<(s_2,0.00), (0.59,0.64)>$	$<(s_2,-0.27), (0.56,0.70)>$

Source: Author calculation.

Table 6. The score functions of the ERP systems.

	P2TLWA	P2TLWG
A_1	$(s_2,-0.48)$	$(s_1,0.31)$
A_2	$(s_2,-0.07)$	$(s_2,-0.38)$
A_3	$(s_2,0.04)$	$(s_2,-0.19)$
A_4	$(s_3,-0.28)$	$(s_2,-0.18)$
A_5	$(s_1,0.35)$	$(s_1,0.14)$

Source: Author calculation.

Table 7. Ordering of the ERP systems.

	Ordering
P2TLWA	$A_4 > A_3 > A_2 > A_1 > A_5$
P2TLWG	$A_4 > A_3 > A_2 > A_1 > A_5$

Source: Author calculation.

According to the aggregating results shown in Table 5 and the score functions of the ERP systems are shown in Table 6.

According to the score functions shown in Table 6 and the comparison formula of score functions, the ordering of the e ERP systems are shown in Table 7. Note that "$>$" means "preferred to". As we can see, depending on the aggregation operators used, the ordering of the ERP systems is same, and the best ERP systems is A_4.

From Tables 4 and 7, we can easily find that these two above mentioned models may generate slightly different ranking results. The main reason causing this ranking result difference is that the P2TLPWA (P2TLPWG) operator can more accurately model the relationships between attributes by introducing the relationship structure of power operations. However, the P2TLWA (P2TLWG) operators which are proposed does not consider such actual situations that some of arguments may be related to all other input arguments.

7. Conclusion

In this paper, we investigate the multiple attribute decision making problems with Pythagorean 2-tuple linguistic information. Then, we utilize power average(Yager, 2001) and power geometric operations(Xu & Yager, 2010) to develop some Pythagorean 2-tuple linguistic power aggregation operators: Pythagorean 2-tuple linguistic power weighted average (P2TLPWA) operator, Pythagorean 2-tuple linguistic power weighted geometric (P2TLPWG) operator, Pythagorean 2-tuple linguistic power ordered weighted average (P2TLPOWA) operator, Pythagorean 2-tuple linguistic power ordered weighted geometric (P2TLPOWG) operator, Pythagorean 2-tuple linguistic power hybrid average (P2TLPHA) operator and Pythagorean 2-tuple linguistic power hybrid geometric (P2TLPHG) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the Pythagorean 2-tuple linguistic multiple attribute decision making problems. Finally, a practical example for enterprise resource planning (ERP) system selection is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, the application of the proposed aggregating operators of P2TLSs needs to be explored in the decision making, risk analysis and many other fields under uncertain environment (Deng & Gao, 2019; Deng, Wang, & Wei, 2019; Gao, Lu, Wei, & Wei, 2018; Han & Liu, 2011; Li & Lu, 2019; Li, Wei, & Lu, 2018; Lin, Zhao, & Wei, 2013; Liu, 2009; Liu, Jin, Zhang, Su, & Wang, 2011; Liu & Zhang, 2010; Liu, Liu, Liu, & Pang, 2017; Lu, Tang, Wei, Wei, & Wei, 2019; Lu & Wei, 2019; Mardani et al. 2015; Merigó, 2008, 2009a, 2009b, 2010; Merigó, Casanovas, & Martínez, 2010; Merigó & Casanovas, 2009; Ngan, 2011; Tang et al., 2019; Wang, Wang, & Wei, 2019; Wang, Gao, & Lu, 2019; Wang, Wang, et al., 2019; Wang, Wang, et al., 2019a,b; Wang, Gao, et al., 2019; Wang, Lu, et al., 2019; Wang, Wei, et al., 2019; Wei, 2018, 2019a, 2019b, 2019c; Wei, Wang, Wei, Wei, & Zhang, 2019a, 2019b; Wei & Wei, 2018; Wu, Gao, & Wei, 2019; Wu, Gao, et al., 2019; Ye, 2009a,b).

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References

Beg, I., & Rashid, T. (2015). Hesitant 2-tuple linguistic information in multiple attributes group decision making. *Journal of Intelligent & Fuzzy Systems*, *30*(1), 109–116. doi:10.3233/IFS-151737

Chang, K. H., & Wen, T. C. (2010). A novel efficient approach for DFMEA combining 2-tuple and the OWA operator. *Expert Systems with Applications*, *37*(3), 2362–2370. doi:10.1016/j.eswa.2009.07.026

Deng, X. M., & Gao, H. (2019). TODIM method for multiple attribute decision making with 2-tuple linguistic Pythagorean fuzzy information. *Journal of Intelligent & Fuzzy Systems*, *37*(2), 1769–1780. doi:10.3233/JIFS-179240

Deng, X. M., Wang, J., & Wei, G. W. (2019). Some 2-tuple linguistic Pythagorean Heronian mean operators and their application to multiple attribute decision-making. *Journal of Experimental & Theoretical Artificial Intelligence*, *31*, 555–574. doi:10.1080/0952813X.2019.1579258

Dong, Y. C., & Herrera-Viedma, E. (2015). Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation. *IEEE Trans. Cybernetics*, *45*(4), 780–792.

Dutta, B., & Guha, D. (2015). Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making. *Applied Soft Computing*, *37*, 166–179. doi:10.1016/j.asoc.2015.08.017

Dutta, B., Guha, D., & Mesiar, R. (2015). A model based on linguistic 2-tuples for dealing with heterogeneous relationship among attributes in multi-expert decision making. *IEEE Transactions on Fuzzy Systems*, *23*(5), 1817–1831. doi:10.1109/TFUZZ.2014.2379291

Fan, Z. P., Feng, B., Sun, Y. H., & Ou, W. (2009). Evaluating knowledge management capability of organizations: A fuzzy linguistic method. *Expert Systems with Applications*, *36*(2), 3346–3354. doi:10.1016/j.eswa.2008.01.052

Fan, Z. P., & Liu, Y. (2010). A method for group decision-making based on multi-granularity uncertain linguistic information. *Expert Systems with Applications*, *37*(5), 4000–4008. doi:10.1016/j.eswa.2009.11.016

Gao, H., Lu, M., Wei, G. W., & Wei, Y. (2018). Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making. *Fundamenta Informaticae*, *159*(4)2018), 385–428. doi:10.3233/FI-2018-1669

Garg, H. (2016). A New generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making. *International Journal of Intelligent Systems*, *31*(9), 886–920. doi:10.1002/int.21809

Garg, H. (2016). A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. *Journal of Intelligent & Fuzzy Systems*, *31*(1), 529–540. doi:10.3233/IFS-162165

Gou, X. J., Xu, Z. S., & Ren, P. J. (2016). The properties of continuous pythagorean fuzzy information. *International Journal of Intelligent Systems*, *31*(5), 401–424. doi:10.1002/int.21788

Han, Z. S., & Liu, P. D. (2011). A fuzzy multi-attribute decision-making method under risk with unknown attribute weights. *Technological and Economic Development of Economy*, *17*(2), 246–258. doi:10.3846/20294913.2011.580575

Herrera, F., & Herrera-Viedma, E. (2000a). Choice functions and mechanisms for linguistic preference relations. *European Journal of Operational Research*, *120*(1), 144–161. doi:10.1016/S0377-2217(98)00383-X

Herrera, F., & Herrera-Viedma, E. (2000b). Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems*, *115*(1), 67–82. doi:10.1016/S0165-0114(99)00024-X

Herrera, F., Herrera-Viedma, E., & Martínez, L. (2008). A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, *16*(2), 354–370. doi:10.1109/TFUZZ.2007.896353

Herrera, F., & Martínez, L. (2001a). The 2-tuple linguistic computational model: Advantages of its linguistic description, accuracy and consistency. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *09*(supp01), 33–49. doi:10.1142/S0218488501000971

Herrera, F., & Martínez, L. (2001b). A model based on linguistic 2-tuple for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Transactions on Systems, Man and Cybernetics, Part B (Cybernetics)*, *31*(2), 227–234. doi:10.1109/3477.915345

Herrera, F., Martınez, L., & Sánchez, P. J. (2005). Managing non-homogeneous information in group decision making. *European Journal of Operational Research*, *166*(1), 115–132. doi:10.1016/j.ejor.2003.11.031

Herrera-Viedma, E., Martinez, L., Mata, F., & Chiclana, F. (2005). A consensus support system model for group decision-making problems with multigranular linguistic preference relations. *IEEE Transactions on Fuzzy Systems*, *13*(5), 644–658. doi:10.1109/TFUZZ.2005.856561

Jiang, X. P., & Wei, G. W. (2014). Some Bonferroni mean operators with 2-tuple linguistic information and their application to multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, *27*, 2153–2162.

Li, Z. X., & Lu, M. (2019). Some novel similarity and distance and measures of Pythagorean fuzzy sets and their applications. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 1781–1799. doi:10.3233/JIFS-179241

Li, Z. X., Wei, G. W., & Lu, M. (2018). Pythagorean fuzzy hamy mean operators in multiple attribute group decision making and their application to supplier selection. *Symmetry*, *10*, 205. doi:10.3390/sym10100505

Liao, X. W., Li, Y., & Lu, B. (2007). A model for selecting an ERP system based on linguistic information processing. *Information Systems*, *32*(7), 1005–1017. doi:10.1016/j.is.2006.10.005

Lin, R., Wei, G. W., Wang, H. J., & Zhao, X. F. (2014). Choquet integrals of weighted triangular fuzzy linguistic information and their applications to multiple attribute decision making. *Journal of Business Economics and Management*, *15*(5), 795–809. doi:10.3846/16111699.2013.773940

Lin, R., Zhao, X. F., & Wei, G. W. (2013). Fuzzy number intuitionistic fuzzy prioritized operators and their application to multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, *24*, 879–888.

Liu, H. C., Lin, Q. L., & Wu, J. (2014). Dependent interval 2-tuple linguistic aggregation operators and their application to multiple attribute group decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *22*(05), 717–736. doi:10.1142/S0218488514500366

Liu, P. D. (2009). Multi-attribute decision-making method research based on interval vague set and TOPSIS method. *Technological and Economic Development of Economy*, *15*(3), 453–463. doi:10.3846/1392-8619.2009.15.453-463

Liu, P. D., Jin, F., Zhang, X., Su, Y., & Wang, M. H. (2011). Research on the multi-attribute decision-making under risk with interval probability based on prospect theory and the uncertain linguistic variables. *Knowledge-Based Systems*, *24*(4), 554–561. doi:10.1016/j.knosys.2011.01.010

Liu, P. D., & Zhang, X. (2010). The study on multi-attribute decision-making with risk based on linguistic variable. *International Journal of Computational Intelligence Systems*, *3*(5), 601–609. doi:10.2991/ijcis.2010.3.5.9

Liu, Z. M., Liu, P. D., Liu, W. D., & Pang, J. Y. (2017). Pythagorean uncertain linguistic partitioned Bonferroni mean operators and their application in multi-attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *32*(3), 2779–2790. doi:10.3233/JIFS-16920

Lu, J. P., & Wei, C. (2019). TODIM method for performance appraisal on social-integration-based rural reconstruction with interval-valued intuitionistic fuzzy information. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 1731–1740. doi:10.3233/JIFS-179236

Lu, J. P., Tang, X. Y., Wei, G. W., Wei, C., & Wei, Y. (2019). Bidirectional project method for dual hesitant Pythagorean fuzzy multiple attribute decision-making and their application to performance assessment of new rural construction. *International Journal of Intelligent Systems*, *34*(8), 1920–1934. doi:10.1002/int.22126

Mardani, A., Jusoh, A., Khalil, M. D. N., Zainab, K., Norhayati, Z., & Alireza, V. (2015). Multiple criteria decision-making techniques and their applications – a review of the literature from 2000 to 2014. *Economic Research-Ekonomska Istraživanja* *28*(1), 516–571. doi:10.1080/1331677X.2015.1075139

Martínez-López, L., Rodríguez, R. M., & Herrera, F. (2015). *The 2-tuple Linguistic model - computing with words in decision making*. Berlin, Germany: **Springer**, pp. 1–168

Merigó, J. M. (2008). *New extensions to the OWA operators and their application in decision making*. (PhD thesis) (in Spanish), Department of Business Administration, University of Barcelona, Spain.

Merigó, J. M. (2009a). Probabilistic decision making with the OWA operator and its application in investment management. in *Proceedings of the IFSA-EUSFLAT International Conference*, Lisbon, Portugal, p. 1364–1369.

Merigó, J. M. (2009b). The probabilistic weighted average operator and its application in decision making. In G. E. Lasker, P. Hruza (Eds.) *Operations Systems Research & Security of Information* (pp. 55–58), Baden-Baden, Germany: The International Institute for Advanced Studies in Systems and Cybernetics.

Merigó, J. M. (2010). Fuzzy decision making with immediate probabilities. *Computers & Industrial Engineering*, *58*(4), 651–657. doi:10.1016/j.cie.2010.01.007

Merigó, J. M., Casanovas, M., & Martínez, L. (2010). Linguistic aggregation operators for linguistic decision making based on the Dempster-Shafer theory of evidence. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *18*(03), 287–304. doi:10.1142/S0218488510006544

Merigó, J. M., & Casanovas, M. (2009). Induced aggregation operators in decision making with the Dempster-Shafer belief structure. *International Journal of Intelligent Systems*, *24*(8), 934–954. doi:10.1002/int.20368

Ngan, S. C. (2011). Decision making with extended fuzzy linguistic computing, with applications to new product development and survey analysis. *Expert Systems with Applications*, *38*, 14052–14059. doi:10.1016/j.eswa.2011.04.213

Peng, X., & Yang, Y. (2015). Some results for Pythagorean Fuzzy Sets. *International Journal of Intelligent Systems*, *30*(11), 1133–1160. doi:10.1002/int.21738

Qin, J. D., & Liu, X. W. (2016). 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection. *Kybernetes*, *45*(1), 2–29. doi:10.1108/K-11-2014-0271

Reformat, M., & Yager, R. R. (2014). Suggesting Recommendations using pythagorean fuzzy sets illustrated using netflix movie data. *IPMU*, 442(1), 546–556.

Ren, P. J., Xu, Z. S., & Gou, X. J. (2016). Pythagorean fuzzy TODIM approach to multi-criteria decision making. *Applied Soft Computing*, *42*, 246–259. doi:10.1016/j.asoc.2015.12.020

Tang, M., Wang, J., Lu, J. P., Wei, G. W., Wei, C., & Wei, Y. (2019). Dual hesitant Pythagorean fuzzy heronian mean operators in multiple attribute decision making. *Mathematics*, *7*, 344.

Tai, W. S., & Chen, C. T. (2009). A new evaluation model for intellectual capital based on computing with linguistic variable. *Expert Systems with Applications*, *36*(2), 3483–3488. doi:10.1016/j.eswa.2008.02.017

Wang, J. Q., Wang, D. D., Zhang, H. Y., & Chen, X. H. (2015). Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. *Soft Computing*, *19*(2), 389–405. doi:10.1007/s00500-014-1259-z

Wang, P., Wang, J., & Wei, G. W. (2019). EDAS method for multiple criteria group decision making under 2-tuple linguistic neutrosophic enviroment. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 1597–1608. doi:10.3233/JIFS-179223

Wang, R. (2019). Research on the application of the financial investment risk appraisal models with some interval number muirhead mean operators. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 1741–1752. doi:10.3233/JIFS-179237

Wang, W. P. (2009). Evaluating new product development performance by fuzzy linguistic computing. *Expert Systems with Applications*, *36*(6), 9759–9766. doi:10.1016/j.eswa.2009.02.034

Wang, J., Gao, H., & Lu, M. (2019). Approaches to strategic supplier selection under interval neutrosophic environment. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 1707–1730. doi:10.3233/JIFS-179235

Wang, J., Gao, H., & Wei, G. W. (2019a). The generalized Dice similarity measures for Pythagorean fuzzy multiple attribute group decision making. *International Journal of Intelligent Systems*, *34*(6), 1158–1183. doi:10.1002/int.22090

Wang, J., Gao, H., & Wei, G. W. (2019b). Some 2-tuple linguistic neutrosophic number Muirhead mean operators and their applications to multiple attribute decision making.

Journal of Experimental & Theoretical Artificial Intelligence, *31*, 409–439. doi:10.1080/0952813X.2018.1552320
Wang, J., Gao, H., Wei, G. W., & Wei, Y. (2019). Methods for multiple-attribute group decision making with q-rung interval-valued orthopair fuzzy information and their applications to the selection of green suppliers. *Symmetry*, *11*(1), 56. doi:10.3390/sym11010056
Wang, J., Lu, J. P., Wei, G. W., Lin, R., & Wei, C. (2019). Models for MADM with single-valued neutrosophic 2-tuple linguistic muirhead mean operators. *Mathematics*, *7*(5), 442. doi:10.3390/math7050442
Wang, J., Wei, G. W., Lu, J. P., Alsaadi, F. E., Hayat, T., Wei, C., & Zhang, Y. (2019). Some q-Rung Orthopair Fuzzy Hamy mean Operators in Multiple Attribute Decision Making and their application to enterprise resource planning systems selection. *International Journal of Intelligent Systems*, *34*(10), 2429–2458. doi:10.1002/int.22155
Wei, G. W. (2018). TODIM method for picture fuzzy multiple attribute decision making. *Informatica*, *29*(3), 555–566. doi:10.15388/Informatica.2018.181
Wei, G. W. (2019a). 2-tuple intuitionistic fuzzy linguistic aggregation operators in multiple attribute decision making. *Iranian Journal of Fuzzy Systems*, *16*, 159–174.
Wei, G. W. (2019b). The generalized dice similarity measures for multiple attribute decision making with hesitant fuzzy linguistic information. *Economic Research-Ekonomska Istraživanja*, *32*(1), 1498–1520. doi:10.1080/1331677X.2019.1637765
Wei, G. W. (2019c). Pythagorean fuzzy hamacher power aggregation operators in multiple attribute decision making. *Fundamenta Informaticae*, *166*(1), 57–85. doi:10.3233/FI-2019-1794
Wei, G. W., Wang, J., Wei, C., Wei, Y., & Zhang, Y. (2019a). Dual hesitant Pythagorean Fuzzy Hamy mean operators in multiple attribute decision making. *IEEE Access*, *7*, 86697–86716. doi:10.1109/ACCESS.2019.2924974
Wei, G. W., Wang, R., Wang, J., Wei, C., & Zhang, Y. (2019b). Methods for Evaluating the Technological Innovation Capability for the High-Tech Enterprises With Generalized Interval Neutrosophic Number Bonferroni Mean Operators. *IEEE Access*, *7*, 86473–86492. doi:10.1109/ACCESS.2019.2925702
Wu, L. P., Gao, H., & Wei, C. (2019). VIKOR method for financing risk assessment of rural tourism projects under interval-valued intuitionistic fuzzy environment. *Journal of Intelligent & Fuzzy Systems*, *37* (2), 2001–2008. doi:10.3233/JIFS-179262
Wu, L. P., Wang, J., & Gao, H. (2019). Models for competiveness evaluation of tourist destination with some interval-valued intuitionistic fuzzy Hamy mean operators. *Journal of Intelligent & Fuzzy Systems*, *36*(6), 5693–5709. doi:10.3233/JIFS-181545
Wu, Q., Wu, P., Zhou, Y. Y., Zhou, L. G., Chen, H. Y., & Ma, X. Y. (2015). Some 2-tuple linguistic generalized power aggregation operators and their applications to multiple attribute group decision making. *Journal of Intelligent & Fuzzy Systems*, *29*(1), 423–436. doi:10.3233/IFS-151609
Xu, Y. J., Ma, F., Tao, F. F., & Wang, H. M. (2014). Some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making. *Knowledge-Based Systems*, *56*, 179–190. doi:10.1016/j.knosys.2013.11.008
Xu, Z. S. (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences*, *166*(1-4), 19–30. doi:10.1016/j.ins.2003.10.006
Xu, Z. S. (2006). A note on linguistic hybrid arithmetic averaging operator in multiple attribute group decision making with linguistic information. *Group Decision and Negotiation*, *15*(6), 593–604. doi:10.1007/s10726-005-9008-4
Xu, Z. S., & Yager, R. R. (2010). Power-geometric operators and their use in group decision making. *IEEE Transactions on Fuzzy Systems*, *18*(1), 94–105.
Yager, R. R. (2001). The power average operator. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, *31*(6), 724–731. doi:10.1109/3468.983429
Yager, R. R. (2013). Pythagorean fuzzy subsets. in: *Proceeding of the Joint IFSA Wprld Congress and NAFIPS Annual Meeting*, Edmonton, Canada, pp. 57−−61.

Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, *22*(4), 958–965. doi:10.1109/TFUZZ.2013.2278989

Ye, J. (2009a). Multicriteria fuzzy decision-making method based on the intuitionistic fuzzy cross-entropy. in: Y.C. Tang, J. Lawry, V.N. Huynh (Eds.), *Proceedings in International Conference on Intelligent Human-Machine Systems and Cybernetics*, 1, IEEE Computer Society, pp. 59–61.

Ye, J. (2009). Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. *Expert Systems with Applications*, *36*(3), 6899–6902. doi:10.1016/j.eswa.2008.08.042

Zavadskas, E. K., & Turskis, Z. (2011). Multiple criteria decision making (MCDM) methods in economics: An overview. *Technological and Economic Development of Economy*, *17*(2), 397–427. doi:10.3846/20294913.2011.593291

Zavadskas, E. K., Turskis, Z., & Kildienė, S. (2014). State of art surveys of overviews on MCDM/MADM methods. *Technological and Economic Development of Economy*, *20*(1), 165–179. doi:10.3846/20294913.2014.892037

Zeng, S. Z., Chen, J. P., & Li, X. S. (2016). A Hybrid Method for Pythagorean Fuzzy Multiple-Criteria Decision Making. *International Journal of Information Technology & Decision Making*, *15*(02), 403–422. doi:10.1142/S0219622016500012

Zhang, W. C., Xu, Y. J., & Wang, H. M. (2016). A consensus reaching model for 2-tuple linguistic multiple attribute group decision making with incomplete weight information. *International Journal of Systems Science*, *47*(2), 389–405. doi:10.1080/00207721.2015.1074761

Zhang, X., & Liu, P. D. (2010). Method for aggregating triangular fuzzy intuitionistic fuzzy information and its application to decision making. *Technological and Economic Development of Economy*, *16*(2), 280–290. doi:10.3846/tede.2010.18

Zhang, X. L., & Xu, Z. S. (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *29*(12), 1061–1078. doi:10.1002/int.21676

Zhang, Z. F., & Chu, X. N. (2009). Fuzzy group decision-making for multi-format and multi-granularity linguistic judgments in quality function deployment. *Expert Systems with Applications*, *36*(5), 9150–9158. doi:10.1016/j.eswa.2008.12.027