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Abstract

In this note, we give a generalization of the results for the profit maximization problem in the case of the Cobb-Douglas production function presented by Liu in [Appl. Math. Comput. 182 (2006), 1093-1097]. By using geometric programming, we solve a profit maximization problem in the case of the CES production function and show how the results obtained by Liu can be derived from our results.

Key words

profit maximization, Cobb-Douglas technology, CES technology, geometric programming

JEL classification

C60, C65, D21, D24

1. Introduction

One of the most important problems in the theory of the firm is certainly the profit maximization problem. On one hand, traditionally, this problem is solved by differential calculus. On the other hand, the geometric programming (GP) technique was proposed by Liu [7] as a complementary approach in solving the profit maximization problem with Cobb-Douglas technology. Considering a Cobb-Douglas production function, given by

$$f\left(x_1, x_2, ..., x_n\right) = A\prod_{i=1}^{n} x_i^{\varphi_i} , \tag{1}$$

the profit maximization problem becomes

$$\pi^{C-D} = \max_{x_1, x_2, ..., x_n > 0} p\left(A\prod_{i=1}^{n} x_i^{\varphi_i}\right) - \sum_{i=1}^{n} v_i x_i , \tag{2}$$

where $p>0$ is the market price per unit, $A>0$ is the scale of production, $x_i>0$, $v_i>0$, $i = 1, 2, \ldots, n$, are input quantities and input prices respectively, and $\varphi_i>0$, $i = 1, 2, \ldots, n$, are the elasticities of the Cobb-Douglas production function subject to

$$\sum_{i=1}^{n} \varphi_i < 1 . \tag{3}$$

By using signomial geometric programming, Liu [7] obtained the result of problem (2) as follows:

$$\pi^{C-D} = \left(1 - \sum_{i=1}^{n} \varphi_i\right) \left(pA\right)^{1\Big/\left(1-\sum_{i=1}^{n}\varphi_i\right)} \prod_{i=1}^{n} \left(v_i / \varphi_i\right)^{-\varphi_i\Big/\left(1-\sum_{i=1}^{n}\varphi_i\right)} , \tag{4}$$

$$x_i^{C-D} = \pi^{C-D} \varphi_i \Bigg/ \left[v_i \left(1 - \sum_{i=1}^{n} \varphi_i\right)\right], \ i = 1, 2, ..., n . \tag{5}$$

In this note, we solve a profit maximization problem in the case of the CES production function and show how the results obtained by Liu can be derived from our results.

2. Preliminaries

Here, we first introduce the CES production function, defined by

$$\psi\left(x_1, x_2, ..., x_n\right) = A\left(\sum_{i=1}^{n} \alpha_i x_j^{-\rho}\right)^{-\sigma/\rho} , \tag{6}$$

where $\alpha_i>0$, $i = 1, 2, \ldots, n$, are the given allocation coefficients subject to

$$\sum_{i=1}^{n} \alpha_i = 1 , \tag{7}$$

σ is the degree of homogeneity subject to

$$0 < \sigma < 1, \tag{8}$$

and ρ the is substitution coefficient subject to

$$\rho \neq 0, \ \sigma + \rho > 0, \ 1 + \rho > 0 . \tag{9}$$

Constraint (8) is the necessary and sufficient condition for the strict concavity of the CES production function (see [1]). Furthermore, constraint (8) ensures the correct definition of the profit function (see [6]).

Considering the CES production function, the profit maximization problem becomes

$$\pi^{CES} = \max_{x_1, x_2, ..., x_n > 0} p \cdot A\left(\sum_{i=1}^{n} \alpha_i x_j^{-\rho}\right)^{-\sigma/\rho} - \sum_{i=1}^{n} v_i x_i . \tag{10}$$

To solve the problem (10), we will need a power mean inequality.

Lemma 1 [Power mean inequality] Let $n \geq 2,\ n \in \mathbb{N},\ x_i > 0,$ and $w_i > 0,\ i = 1, 2, ..., n,$ such that $\sum_{i=1}^{n} w_i = 1$. Then, for all r>0 the following inequality holds

$$\prod_{i=1}^{n} x^{w_i} \leq \left(\sum_{i=1}^{n} w_i x_i^r \right)^{\frac{1}{r}} . \tag{11}$$

Equality in (11) holds if and only if $x_1 = x_2 = \cdots = x_n$.

The proof of the Lemma 1 can be found in [4].

3. The profit maximization problem with CES technology

According to [2, 3, 5], we can transform problem (10) into a GP problem in two steps as follows.

Step 1. Problem (10) is equivalent to

$$\begin{aligned} &\max_{s,t,x_1,...,x_n>0} \quad t \\ &\text{s.t.} \qquad pAs^{-\sigma/\rho} - \sum_{i=1}^{n} v_i x_i \geq t, \\ &\qquad\qquad \sum_{i=1}^{n} \alpha_i x_j^{-\rho} \leq s . \end{aligned} \tag{12}$$

Step 2. Problem (12) is equivalent to the GP problem

$$\begin{aligned} &\min_{s,t,x_1,...,x_n>0} \quad t^{-1} \\ &\text{s.t.} \qquad p^{-1}A^{-1}ts^{\sigma/\rho} + \sum_{i=1}^{n} p^{-1}A^{-1}v_i s^{\sigma/\rho} x_i \leq 1, \\ &\qquad\qquad \sum_{i=1}^{n} \alpha_i s^{-1} x_j^{-\rho} \leq 1 . \end{aligned} \tag{13}$$

Note that the degree of difficulty of problem (13) is equal to d = *total number of terms – total number of variables* – 1 = (2n+2) - (n+2) - 1 = n–1. According to [5], the corresponding dual of (13) is the following problem:

$$\max_{\substack{\beta,\gamma,\delta_i,\varepsilon_i>0 \\ i=1,2,...,n}} \left(\frac{1}{\beta}\right)^{\beta} \cdot \left(\frac{p^{-1}A^{-1}}{\gamma}\right)^{\gamma} \cdot \prod_{j=1}^{n} \left(\frac{p^{-1}A^{-1}v_j}{\delta_j}\right)^{\delta_j} \cdot \prod_{k=1}^{n} \left(\frac{\alpha_k}{\varepsilon_k}\right)^{\varepsilon_k} \cdot \left(\gamma + \sum_{j=1}^{n} \delta_j\right)^{\gamma + \sum_{j=1}^{n}\delta_j} \cdot \left(\sum_{k=1}^{n} \varepsilon_k\right)^{\sum_{k=1}^{n}\varepsilon_k} \tag{14}$$

s.t. β=1

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma/\rho & \sigma/\rho & \sigma/\rho & \cdots & \sigma/\rho & -1 & -1 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & -\rho & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & -\rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & -\rho \end{bmatrix}_{(n+2)\times(2n+2)} \cdot \begin{bmatrix} \beta \\ \gamma \\ \delta_1 \\ \vdots \\ \delta_n \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{(2n+2)\times 1} = \mathbf{0}, \tag{15}$$

where $\mathbf{0}=\begin{bmatrix}0 & 0 & \cdots & 0\end{bmatrix}^T$ is a null vector with n+2 components, and β, γ, δ_i, ε_i, i = 1, 2, …, n, are dual variables. From (15) we get

$$\beta=\gamma=1, \tag{16}$$

$$\varepsilon_i=\frac{\delta_i}{\rho},\ i=1,2,...,n, \tag{17}$$

$$\sum_{i=1}^{n}\delta_i=\frac{\sigma}{1-\sigma}, \tag{18}$$

$$\sum_{i=1}^{n}\varepsilon_i=\frac{\sigma}{\rho(1-\sigma)}. \tag{19}$$

Using (16)–(19), (14) becomes

$$\max_{\delta_1,\delta_2,...,\delta_n>0}\left(\frac{p^{-1}A^{-1}}{1-\sigma}\right)^{\frac{1}{1-\sigma}}\cdot\left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{\rho(1-\sigma)}}\cdot\left[\prod_{i=1}^{n}\left(v_i\alpha_i^{\frac{1}{\sigma}}\Big/\left(\left(\frac{(1-\sigma)\delta_i}{\sigma}\right)^{\frac{\rho+1}{\rho}}\cdot\left(\frac{\sigma}{1-\sigma}\right)^{\frac{\rho+1}{\rho}}\right)\right)^{\frac{(1-\sigma)\delta_i}{\sigma}}\right]^{\frac{\sigma}{1-\sigma}}. \tag{20}$$

Let $\tilde{\delta}_i=\dfrac{(1-\sigma)\delta_i}{\sigma},\ i=1,2,...,n$. Then, from (18) we have

$$\sum_{i=1}^{n}\tilde{\delta}_i=1. \tag{21}$$

Now, problem (20) becomes

$$M=\max_{\substack{\tilde{\delta}_i>0\\ \sum_{i=1}^{n}\tilde{\delta}_i=1}}\left(\frac{p^{-1}A^{-1}}{1-\sigma}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{-\sigma}{1-\sigma}}\left[\prod_{i=1}^{n}\left(\frac{v_i\alpha_i^{\frac{1}{\rho}}}{\tilde{\delta}_i^{\frac{\rho+1}{\rho}}}\right)^{\tilde{\delta}_i}\right]^{\frac{\sigma}{1-\sigma}}. \tag{22}$$

Let us solve problem (22) by using Lemma 1. According to (11), for all r>0, the following inequality holds:

$$\prod_{i=1}^{n}\left(\frac{v_i\alpha_i^{\frac{1}{\rho}}}{\tilde{\delta}_i^{\frac{\rho+1}{\rho}}}\right)^{\tilde{\delta}_i}\le\left[\sum_{i=1}^{n}\tilde{\delta}_i\cdot\left(\frac{v_i\alpha_i^{\frac{1}{\rho}}}{\tilde{\delta}_i^{\frac{\rho+1}{\rho}}}\right)^{r}\right]^{\frac{1}{r}}. \tag{23}$$

Since $\sigma/(1-\sigma)>0$ and the function $x\mapsto x^{\frac{\sigma}{1-\sigma}}$ is increasing (for x>0), from (22) and (23) we have

$$M\le\left(\frac{p^{-1}A^{-1}}{1-\sigma}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{-\sigma}{1-\sigma}}\left[\sum_{i=1}^{n}v_i^r\alpha_i^{\frac{r}{\rho}}\tilde{\delta}_i^{1-\frac{1+\rho}{\rho}\cdot r}\right]^{\frac{\sigma}{r(1-\sigma)}}, \tag{24}$$

for all r>0. By choosing $r=\dfrac{\rho}{1+\rho}$, from (24) we get

$$M\le\left(\frac{p^{-1}A^{-1}}{1-\sigma}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{-\sigma}{1-\sigma}}\left(\sum_{i=1}^{n}v_i^{\frac{\rho}{1+\rho}}\alpha_i^{\frac{1}{1+\rho}}\right)^{\frac{\sigma}{\frac{\rho}{1+\rho}(1-\sigma)}}. \tag{25}$$

Equality in (23)-(25) holds if and only if

$$\frac{v_i \alpha_i^{\frac{1}{\rho}}}{\tilde{\delta}_i^{\frac{\rho+1}{\rho}}} = \frac{v_1 \alpha_1^{\frac{1}{\rho}}}{\tilde{\delta}_1^{\frac{\rho+1}{\rho}}},\ i = 1, 2, ..., n. \tag{26}$$

From the definition of $\tilde{\delta}_i, i = 1, 2, ..., n,$ and (26) we get

$$\tilde{\delta}_i = \frac{v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}} \alpha_k^{\frac{1}{\rho+1}}},\ i = 1, 2, ..., n. \tag{27}$$

Thus, by definition of the strict global optimum, the strict global maximum *M* of problem (22), and at the same time of problem (14), is given by the expression on the right hand side of the inequality (25), and it is achieved if and only if $\tilde{\delta}_i,\ i = 1, 2, ..., n,$ satisfy (27). In addition, *M* represents the strict global minimum of problem (13). Furthermore, since $\max f = \left(\min f^{-1}\right)^{-1}$ for a positive function *f*, we can find π^{CES} via (12)-(13) as follows

$$\pi^{CES} = M^{-1} = \left(pA(1-\sigma)\right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}} \alpha_i^{\frac{1}{1+\rho}}\right)^{\frac{-\sigma}{\frac{\rho}{1+\rho}(1-\sigma)}}. \tag{28}$$

Now, from (17), (18) and (27) we get the dual variables

$$\delta_i = \frac{\frac{\sigma}{1-\sigma} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}} \alpha_k^{\frac{1}{\rho+1}}},\ \varepsilon_i = \frac{\frac{\sigma}{\rho(1-\sigma)} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}} \alpha_k^{\frac{1}{\rho+1}}},\ i = 1, 2, ..., n. \tag{29}$$

According to [5], from (13), (16)-(19) and (29), we have

$$t^{-1} = \beta M = M \ \Rightarrow\ t = M^{-1}, \tag{30}$$

$$p^{-1} A^{-1} t s^{\frac{\sigma}{\rho}} = \frac{\gamma}{\gamma + \sum_{k=1}^{n} \delta_k} = \frac{1}{1 + \sum_{k=1}^{n} \delta_k}, \tag{31}$$

$$p^{-1} A^{-1} v_i s^{\frac{\sigma}{\rho}} x_i = \frac{\delta_i}{\gamma + \sum_{k=1}^{n} \delta_k},\ i = 1, 2, ..., n, \tag{32}$$

$$\alpha_i s^{-1} x_i^{-\rho} = \varepsilon_i \Big/ \sum_{k=1}^{n} \varepsilon_k,\ i = 1, 2, ..., n, \tag{33}$$

from where we get

$$x_i^{CES} = \frac{\frac{\sigma}{1-\sigma} \left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}} \alpha_k^{\frac{1}{\rho+1}}} \cdot \pi^{CES},\ i = 1, 2, ..., n. \tag{34}$$

Thus, the strict global maximum of the profit maximization problem with CES technology π^{CES} is given by (28), and it is achieved for the input values given by (34).

4. The profit maximization problem with Cobb-Douglas technology

Let us first show how the Cobb-Douglas production function (1) can be obtained from the CES production function (6). Let

$$U = \lim_{\rho \to 0} A\left(\sum_{j=1}^{n} \alpha_j x_j^{-\rho}\right)^{-\frac{\sigma}{\rho}}. \tag{35}$$

Since (7) holds, by taking a natural logarithm and after applying the L'Hospital's rule, from (35) we have

$$\begin{aligned} \ln U &= \ln A + \lim_{\rho \to 0} \frac{-\sigma \ln\left(\sum_{j=1}^{n} \alpha_j x_j^{-\rho}\right)}{\rho} \overset{L'H}{=} \ln A + \lim_{\rho \to 0} \frac{\sigma \sum_{j=1}^{n}\left(x_j^{-\rho} \ln x_j^{\alpha_j}\right)}{\sum_{j=1}^{n} \alpha_j x_j^{-\rho}} = \\ &= \ln A + \frac{\sigma \sum_{j=1}^{n} \ln x_j^{\alpha_j}}{\sum_{j=1}^{n} \alpha_j} = \ln A + \frac{\sum_{j=1}^{n} \ln x_j^{\sigma\alpha_j}}{1} = \ln\left(A \prod_{j=1}^{n} x_j^{\sigma\alpha_j}\right). \end{aligned} \tag{36}$$

Let

$$\varphi_j = \sigma\alpha_j,\ j = 1, 2, ..., n, \tag{37}$$

and note that

$$\sum\nolimits_{j=1}^{n} \varphi_j = \sigma \sum\nolimits_{j=1}^{n} \alpha_j = \sigma < 1. \tag{38}$$

Then, from (36) we have

$$U = \lim_{\rho \to 0} A\left(\sum_{j=1}^{n} \alpha_j x_j^{-\rho}\right)^{-\frac{\sigma}{\rho}} = A \prod_{j=1}^{n} x_j^{\varphi_j}. \tag{39}$$

Thus, the Cobb-Douglas production function given by (1) is the limit when $\rho \to 0$ of the CES production function given by (6).

Further, let us show how $\pi^{C\text{-}D}$ from (4) can be obtained from π^{CES}. From (28) we have

$$\begin{aligned} \lim_{\rho \to 0} \pi^{CES} &= \lim_{\rho \to 0} \left(pA(1-\sigma)\right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}} \alpha^{\frac{1}{1+\rho}}\right)^{\frac{-\sigma}{\frac{\rho}{1+\rho}(1-\sigma)}} . \\ &= \left(pA(1-\sigma)\right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \cdot \Lambda , \end{aligned} \tag{40}$$

where

$$\Lambda = \lim_{\rho \to 0} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}} \alpha^{\frac{1}{1+\rho}}\right)^{\frac{-(1+\rho)\sigma}{\rho(1-\sigma)}}. \tag{41}$$

Since (7) holds, by taking a natural logarithm and after applying L'Hospital's rule, from (41) we have

$$\ln\Lambda = \lim_{\rho\to 0}\frac{-(1+\rho)\sigma\ln\left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}}\alpha_i^{\frac{1}{1+\rho}}\right)}{\rho(1-\sigma)} =$$

$$\overset{L'H}{=}\lim_{\rho\to 0}\frac{-\sigma\ln\left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}}\alpha_i^{\frac{1}{1+\rho}}\right)-(1+\rho)\sigma\left(\frac{1}{(1+\rho)^2}\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}}\alpha_i^{\frac{1}{1+\rho}}\ln\frac{v_i}{\alpha_i}\right)\Big/\left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}}\alpha_i^{\frac{1}{1+\rho}}\right)}{1-\sigma} \tag{42}$$

$$=-\frac{\sigma}{1-\sigma}\sum_{i=1}^{n}\alpha_i\ln\frac{v_i}{\alpha_i}=\ln\prod_{i=1}^{n}\left(\frac{v_i}{\alpha_i}\right)^{-\frac{\sigma\alpha_i}{1-\sigma}}.$$

Thus, from (37) and (40)-(42) we have

$$\begin{aligned}
\lim_{\rho\to 0}\pi^{CES} &= \left(pA(1-\sigma)\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\alpha_i}\right)^{-\frac{\sigma\alpha_i}{1-\sigma}}\\
&= (pA)^{1/\sum_{i=1}^{n}\varphi_i}(1-\sigma)\sigma^{\frac{\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i/\sigma}\right)^{-\frac{\varphi_i}{1-\sum_{i=1}^{n}\varphi_i}}\\
&= (pA)^{1/\sum_{i=1}^{n}\varphi_i}(1-\sigma)\sigma^{\frac{\sigma}{1-\sigma}}\sigma^{\frac{-\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i}\right)^{-\frac{\varphi_i}{1-\sum_{i=1}^{n}\varphi_i}}\\
&= (pA)^{1/\sum_{i=1}^{n}\varphi_i}\left(1-\sum\nolimits_{i=1}^{n}\varphi_i\right)\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i}\right)^{-\frac{\varphi_i}{1-\sum_{i=1}^{n}\varphi_i}}\\
&= \pi^{C-D}.
\end{aligned} \tag{43}$$

Thus, the profit $\pi^{C\text{-}D}$ from (4) is the limit when $\rho\to 0$ of the profit π^{CES} from (28).

Finally, let us show how x_i^{C-D} from (5) can be obtained from x_i^{CES}, $i=1,2,...,n$. Let

$$V_i=\lim_{\rho\to 0}x_i^{CES},\ i=1,2,...,n. \tag{44}$$

Since (7) and (37) hold, from (34), (43) and (44) we have

$$\begin{aligned}
V_i &= \lim_{\rho\to 0}\frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}}\alpha_k^{\frac{1}{\rho+1}}}\cdot\pi^{CES} = \lim_{\rho\to 0}\frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}}\alpha_k^{\frac{1}{\rho+1}}}\cdot\lim_{\rho\to 0}\pi^{CES} =\\
&= \frac{\frac{\sigma}{1-\sigma}\cdot\frac{\alpha_i}{v_i}}{\sum_{k=1}^{n}1\cdot\alpha_k^{1}}\cdot\pi^{C-D} = \frac{\sigma}{1-\sigma}\cdot\frac{\varphi_i/\sigma}{v_i}\cdot\pi^{C-D} = \frac{\pi^{C-D}\varphi_i}{v_i\left(1-\sum_{k=1}^{n}\varphi_k\right)}\\
&= x_i^{C-D}.
\end{aligned} \tag{45}$$

Thus, x_i^{C-D} from (5) is the limit when $\rho\to 0$ of x_i^{CES}, $i=1,2,...,n$, from (34).

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