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AUTOMORPHISM GROUPS OF FINITE RINGS OF CHARACTERISTIC p^2 AND p^3

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Abstract. In this paper we describe the group of automorphisms of a completely primary finite ring R of characteristic p^2 or p^3 with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$; the annihilator of $\mathcal{J}$ coincides with $\mathcal{J}^2$; and the maximal Galois (coefficient) subring R_o of R lies in the center of R.

1. Introduction

Throughout this paper we will assume that all rings are finite, associative (but generally not commutative) with identities, denoted by $1 \neq 0$, that ring homomorphisms preserve 1, a ring and its subrings have the same 1 and that modules are unital. Recall that an Artinian ring R with radical $\mathcal{J}$ is called primary if $R/\mathcal{J}$ is simple and is called completely primary if $R/\mathcal{J}$ is a division ring. The object of this paper is to describe explicitly, the group of automorphisms of a completely primary finite ring R of characteristic p^2 or p^3 such that if $\mathcal{J}$ is the Jacobson radical of R, then $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, the annihilator of $\mathcal{J}$ coincides with $\mathcal{J}^2$ and the coefficient subring R_o lies in the center of R. The automorphisms of R are determined by their images on the generators of the additive group of R and on the invertible element b of order $p^r - 1$ of the Galois subring R_o of R. This supplements the author's earlier work [1] on rings of characteristic p. We freely use the definitions and notations introduced in [1, 2, 3, 5].

Let R be a completely primary finite ring, $\mathcal{J}$ the set of all zero divisors in R, p a prime, k, n and r be positive integers. Then the following results will be assumed (see [5]): $|R| = p^{nr}$, $\mathcal{J}$ is the Jacobson radical of R, $\mathcal{J}^n = (0)$,

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$|\mathcal{J}| = p^{(n-1)r}$, $R/\mathcal{J} \cong GF(p^r)$, the finite field of p^r elements and $charR = p^k$, where $1 \leq k \leq n$; the group of units G_R is a semi-direct product $G_R = (1+\mathcal{J})\times_\theta \langle b\rangle$, of its normal subgroup $1+\mathcal{J}$ of order $p^{(n-1)r}$ by a cyclic subgroup $\langle b\rangle$ of order $p^r - 1$. If $n = k$, it is known that, up to isomorphism, there is precisely one completely primary ring of order p^{rk} having characteristic p^k and residue field $GF(p^r)$. It is called the *Galois ring* $GR(p^{rk},\ p^k)$ and a concrete model is the quotient $\mathbb{Z}_{p^k}[X]/(f)$, where f is a monic polynomial of degree r, irreducible modulo p. Any such polynomial will do: the rings are all isomorphic. Trivial cases are $GR(p^n,\ p^n) = \mathbb{Z}_{p^n}$ and $GR(p^n,\ p) = \mathbb{F}_{p^n}$. In fact, $R = \mathbb{Z}_{p^n}[b]$, where b is an element of R of multiplicative order $p^r - 1$; $\mathcal{J} = pR$ and $Aut(R) \cong Aut(R/pR)$ (see [5, Proposition 2]).

Let R be a completely primary finite ring, $|R/\mathcal{J}| = p^r$ and $charR = p^k$. Then it can be deduced from [4] that R has a coefficient subring R_o of the form $GR(p^{kr},\ p^k)$ which is clearly a maximal Galois subring of R. Moreover, if $R_o^{'}$ is another coefficient subring of R then there exists an invertible element x in R such that $R_o^{'} = xR_ox^{-1}$ (see [5, Theorem 8]). Furthermore, there exist $m_1, \ldots, m_h \in \mathcal{J}$ and $\sigma_1, \ldots, \sigma_h \in Aut(R_o)$ such that $R = R_o \oplus \sum_{i=1}^{h} R_om_i$ (as R_o-modules), $m_ir_o = r_o^{\sigma_i}m_i$, for all $r_o \in R_o$ and any $i = 1, \ldots, h$ (use the decomposition of $R_o \otimes_{\mathbb{Z}} R_o$ in terms of $Aut(R_o)$ and apply the fact that R is a module over $R_o \otimes_{\mathbb{Z}} R_o$). Moreover, $\sigma_1, \ldots, \sigma_h$ are uniquely determined by R and R_o. We call σ_i the automorphism associated with m_i and $\sigma_1, \ldots, \sigma_h$ the associated automorphisms of R with respect to R_o.

Now, let $R_o = \mathbb{Z}_{p^k}[b]$ be a coefficient subring of R of order p^{kr} and characteristic p^k and let $K_o = \langle b\rangle \cup \{0\}$, denote the set of coset representatives of $\mathcal{J}$ in R. Then it is easy to show that every element of R_o can be written uniquely as $\sum_{i=0}^{k} a_ip^i$, where $a_i \in K_o$.

2. Cube zero radical completely primary finite rings

We now assume that R is a completely primary finite ring with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \neq (0)$. These rings were studied by the author who gave their constructions for all characteristics, and for details of the general background, the reader is referred to [2] and [3]. Since R is such that $\mathcal{J}^3 = (0)$, then by one of the above results, $charR$ is either p, p^2 or p^3. The ring R contains a coefficient subring R_o with $charR_o = charR$, and with R_o/pR_o equal to $R/\mathcal{J}$. Moreover, R_o is a Galois ring of the form $GR(p^{kr},\ p^k)$, $k = 1,\ 2$ or 3. Let $ann(\mathcal{J})$ denote the two-sided annihilator of $\mathcal{J}$ in R. Of course $ann(\mathcal{J})$ is an ideal of R. Because $\mathcal{J}^3 = (0)$, it follows easily that $\mathcal{J}^2 \subseteq ann(\mathcal{J})$.

We know from the above results that $R = R_o \oplus \sum_{i=1}^{h} R_om_i$, where $m_i \in \mathcal{J}$, and that there exist automorphisms $\sigma_i \in Aut(R_o)$ $(i = 1, \ldots, h)$ such that

$m_i r_o = r_o^{\sigma_i} m_i$, for all $r_o \in R_o$ and for all $i = 1, \ldots, h$; and the number h and the automorphisms $\sigma_1, \ldots, \sigma_h$ are uniquely determined by R and R_o. Again, because $\mathcal{J}^3 = (0)$, we have that $p^2 m_i = 0$, for all $m_i \in \mathcal{J}$. Further, $pm_i = 0$ for all $m_i \in ann(\mathcal{J})$. In particular, $pm_i = 0$ for all $m_i \in \mathcal{J}^2$.

2.1. *A construction of rings of characteristic p^2 and p^3.* Let R_o be the Galois ring $GR(p^{2r},\ p^2)$ or $GR(p^{3r},\ p^3)$. Let s, d, t be integers with either $1 \leq 1+t \leq s^2$ or $1 \leq d+t \leq s^2$ if $charR_o = p^2$ and $1 \leq 1+d+t \leq s^2$ if $charR_o = p^3$. Let V be an R_o/pR_o-space which when considered as an R_o-module has a generating set $\{v_1, \ldots, v_t\}$ and let U be an R_o-module with an R_o-module generating set $\{u_1,\ \ldots,\ u_s\}$; and suppose that $d \geq 0$ of the u_i are such that $pu_i \neq 0$. Since R_o is commutative, we can think of them as both left and right R_o-modules.

Let (a_{ij}^l), for $l = 0,\ 1, \ldots, t$ or $l = 1,\ 2, \ldots, d+t$ be $s \times s$ linearly independent matrices with entries in R_o/pR_o if $charR_o = p^2$ and $l = 0,\ 1, \ldots, d+t$ be $s \times s$ linearly independent matrices with entries in R_o/pR_o if $charR_o = p^3$.

On the additive group $R = R_o \oplus U \oplus V$ we define multiplication by the following relations:

$$u_i u_j = a_{ij}^o p^f + \sum_{l=1}^{d} a_{ij}^l p u_l + \sum_{k=1}^{t} a_{ij}^{d+t} v_k;$$

$$u_i v_k = v_k u_i = u_i u_j u_\lambda = p v_k = v_l v_k = v_k v_l = 0; \qquad (2.1)$$

$$u_i \alpha = \alpha u_i,\ v_k \alpha = \alpha v_k;\ (1 \leq i,\ j,\ \lambda \leq s;\ 1 \leq l \leq d;\ 1 \leq k \leq t);$$

where $\alpha,\ a_{ij}^o,\ a_{ij}^l,\ a_{ij}^{d+k} \in R_o/\mathcal{J}_o$, and $f = 1$ or 2, depending on whether $charR_o = p^2$ or p^3.

By the above relations, R is a completely primary finite ring of characteristic p^2 or p^3 in which the maximal Galois subring lies in $Z(R)$, the center of R, and with Jacobson radical $\mathcal{J} = pR \oplus U \oplus V$, $ann(\mathcal{J}) = \mathcal{J}^2$, $\mathcal{J}^2 = pR \oplus V$ or $\mathcal{J}^2 = pU \oplus V$(if $charR = p^2$); $\mathcal{J}^2 = p^2 R \oplus pU \oplus V$ (if $charR = p^3$), and $\mathcal{J}^3 = (0)$. We call (a_{ij}^l) the *structural matrices* of the ring R and the numbers $p,\ n,\ r,\ s,\ d$ and t *invariants* of the ring R.

Throughout, we need the following result proved in [2, Theorem 6.1]

Theorem 2.1. *Let R be a ring. Then R is a completely primary finite ring of characteristic p^2 or p^3 in which the maximal Galois subring lies in $Z(R)$, with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, annihilator of $\mathcal{J}$ coincides with $\mathcal{J}^2$ if and only if R is isomorphic to one of the rings given by the relations in (2.1).*

Remark 2.2. We know that $R = R_o \oplus R_o m_1 \oplus \ldots \oplus R_o m_h$, where $m_i \in \mathcal{J}$; and that $\mathcal{J} = pR_o \oplus R_o m_1 \oplus \ldots \oplus R_o m_h$. Since $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 = ann(\mathcal{J})$, with $\mathcal{J}^2 \neq (0)$, we can write

$$\{m_1, \ldots, m_h\} = \{u_1, \ldots, u_s,\ v_1, \ldots, v_t\}$$

where, $u_1, \ldots, u_s \in \mathcal{J} - \mathcal{J}^2$ and $v_1, \ldots, v_t \in \mathcal{J}^2$, so that $s + t = h$.

In view of the above considerations and by 1.8 of [2], the non-zero elements of

$$\{1,\ p,\ u_1, \ldots, u_s,\ pu_1, \ldots, pu_s,\ v_1, \ldots,\ v_t\} \tag{2.2}$$

form a "basis" for R over K_o.

Since $pm = 0$, for all $m \in \mathcal{J}^2$, it is easy to check that if $charR = p^2$, then either

(i) $p \in \mathcal{J}^2$; or
(ii) $p \in \mathcal{J} - \mathcal{J}^2$.

For clarity of our work, we consider the two cases separately in the rest of the paper.

Remark 2.3. Suppose that $charR = p^2$ and p lies in $\mathcal{J}^2$. In this case, (2.2) becomes

$$\{1,\ p,\ u_1, \ldots, u_s,\ v_1, \ldots,\ v_t\};$$

and by [2, Proposition 3.2], $1 \leq 1 + t \leq s^2$. Hence, every element of R may be written uniquely as

$$a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k;\ (a_o,\ a_1,\ b_i,\ c_k \in K_o);$$

and therefore,

$$u_i u_j = a_{ij}^o p + \sum_{k=1}^{t} a_{ij}^k v_k,$$

where $a_{ij}^o,\ a_{ij}^k \in R_o/pR_o$.

Remark 2.4. If $charR = p^2$ and p lies in $\mathcal{J} - \mathcal{J}^2$, suppose that $d \geq 0$ is the number of the elements pu_i in (2.2) which are not zero and suppose, without loss of generality, that $pu_1, \ldots, pu_d$ are the d non-zero elements. Then, (2.2) becomes

$$\{1,\ p,\ u_1, \ldots, u_s,\ pu_1, \ldots, pu_d,\ v_1, \ldots,\ v_t\}$$

and by [2, Proposition 3.2], we have $1 \leq d + t \leq s^2$. Hence, every element of R may be written uniquely as

$$a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k;\ (a_o,\ a_1,\ b_i,\ c_l,\ d_k \in K_o)$$

and

$$u_i u_j = \sum_{l=1}^{d} a_{ij}^l p u_l + \sum_{k=1}^{t} a_{ij}^{k+d} v_k,$$

where $a_{ij}^l,\ a_{ij}^{k+d} \in R_o/pR_o$.

REMARK 2.5. If the characteristic of R is p^3, the argument is similar to that given in the case where $charR = p^2$. However, in this case, $p \in \mathcal{J} - \mathcal{J}^2$ and $p^2 \in \mathcal{J}^2$ and an arbitrary element in R is of the form

$$a_o + a_1 p + a_2 p^2 + \sum_{i}^{s} b_i u_i + \sum_{l=1}^{d} c_l p u_l + \sum_{k}^{t} d_k v_k,$$

and we define multiplication by

$$u_i u_j = a_{ij}^{o} p^2 + \sum_{l=1}^{d} a_{ij}^{l} p u_l + \sum_{k=1}^{t} a_{ij}^{k+d} v_k;$$

and the only parameters left in defining R are the $s \times s$ linearly independent structural matrices $A_l = (a_{ij}^{l})$ over R_o/pR_o, for all $l = 0,\ 1, \ldots, t{+}1, \ldots, t{+}d$.

3. The group of automorphisms

First note that since R is generated by b, u_i and v_k, it is sufficient to give the images of these elements to completely determine the automorphisms. Next, any automorphism from R to R when reduced to R_o must fix R_o. So to determine this group, we first show that the Galois subring R_o of R and the ideal $\mathcal{J}^2$ given by $\mathcal{J}^2 = pU \oplus V$ (if $charR = p^2$, and $p \in \mathcal{J} - \mathcal{J}^2$), are invariant under any automorphism $\phi \in Aut(R)$. We then compute the image of the rest of the generators, by a fixed element of $Aut(R)$.

LEMMA 3.1. *Let $\phi \in Aut(R)$. Then $\phi(R_o)$ is a maximal Galois subring of R which is equal to R_o.*

PROOF. Suppose there is an automorphism $\phi : R \to R$. It is obvious that $\phi(R_o)$ is a maximal Galois subring of R so that there exists an invertible element $x \in R$ such that $x\phi(R_o)x^{-1} = R_o$.

Now, consider the map $\psi : R \to R$ given by $r \mapsto x\phi(r)x^{-1}$. Then, clearly, ψ is an automorphism of R which sends R_o to itself. □

LEMMA 3.2. *Let $\phi \in Aut(R)$ and suppose that $charR = p^2$ and $p \in \mathcal{J} - \mathcal{J}^2$. Then $\phi(\mathcal{J}^2) = \mathcal{J}^2$.*

PROOF. This follows easily since for any $v \in \mathcal{J}^2$, we have $\phi(v) \in \mathcal{J}^2$ because $[\phi(v)]^2 = \phi(v^2) = 0$. □

REMARK 3.3. Following the above two results, we remark that if $charR = p^2$ and $p \in \mathcal{J}^2$, then $\phi(pR_o) \subset \mathcal{J}^2$; and if $charR = p^3$, then $\phi(p^2R_o) \subseteq \mathcal{J}^2$.

LEMMA 3.4. *Let R be a ring of Theorem 2.1 and let $\phi \in Aut(R)$. Then for each $j = 1, \ldots, s$; each $k = 1, \ldots, t$; and each $l = 1, \ldots, d$;*

$$\phi(u_j) = \sum_{i=1}^{2} a_{ij} p^i + \sum_{\mu=1}^{s} b_{\mu j} u_\mu + \sum_{l=1}^{d} c_{lj} p u_l + \sum_{k=1}^{t} d_{kj} v_k;$$

$$\phi(v_k) = \sum_{i=1}^{2} e_{ik}p^i + \sum_{\eta=1}^{d} g_{\eta k}pu_\eta + \sum_{\rho=1}^{t} f_{\rho k}v_\rho,$$

and

$$\phi(pu_l) = a_{1l}p^2 + \sum_{\mu=1}^{d} b_{\mu l}pu_\mu,$$

where $a_{ij},\ b_{\mu l},\ c_{lj},\ d_{kj},\ e_{ik},\ g_{\eta k},\ f_{\rho k} \in R_o/pR_o$*; and for* $r_o \in R_o$, $\phi(r_o) = r_o^\sigma$, *for some* $\sigma \in Aut(R_o)$.

Proof. Since

$$u_j \in \mathcal{J} = pR_o \oplus U \oplus V = pR_o \oplus \sum_{j=1}^{s} R_o u_\mu \oplus \sum_{k=1}^{t} R_o v_k,$$

for all $i = 1, \ldots, s$; and

$$v_k \in \mathcal{J}^2 = pR_o \oplus V = pR_o \oplus \sum_{\rho=1}^{t} R_o v_\rho,$$

or

$$v_k \in \mathcal{J}^2 = pU \oplus V = \sum_{\eta=1}^{d} R_o pu_\eta \oplus \sum_{\rho=1}^{t} R_o v_\rho,$$

(if $charR = p^2$), or

$$v_k \in \mathcal{J}^2 = p^2R_o \oplus pU \oplus V = p^2R_o \oplus \sum_{\eta=1}^{d} R_o pu_\eta \oplus \sum_{\rho=1}^{t} R_o v_\rho,$$

(if $charR = p^3$) for all $\rho = 1, \ldots, t$ and all $\eta = 1, \ldots, d$; the result follows.

The last part may be deduced from Lemma 3.1 since $\phi|_{R_o} = \sigma \in Aut(R_o)$. □

Remark 3.5. In Lemma 3.4, if $charR = p^2$ and $p \in \mathcal{J}^2$, then the coefficients of p^2, pu_l, pu_η, and pu_μ are all equal to zero; and if $charR = p^2$ and $p \in \mathcal{J} - \mathcal{J}^2$, the scalars a_{2j}, e_{ij} and the coefficient of p^2, are all zero.

3.1. *Notation.* We first establish some notation that will be useful in the rest of the paper. So, let R be a ring of Theorem 2.1. If $\sigma \in Aut(R_o)$ and $x \in G_R$, the group of unit elements in R, define the mappings α_σ, ψ_x from R to R as follows:

$$\alpha_\sigma\Big(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l pu_l + \sum_{k=1}^{t} d_k v_k\Big)$$
$$= \sum_{i=0}^{2} a_i^\sigma p^i + \sum_{j=1}^{s} b_j^\sigma u_j + \sum_{l=1}^{d} c_l^\sigma pu_l + \sum_{k=1}^{t} d_k^\sigma v_k,$$

and

$$\psi_x(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = x(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k)x^{-1}.$$

Also, if

$$\psi(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = \sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j \varphi(u_j) + \sum_{l=1}^{d} c_l p \varphi(u_l) + \sum_{k=1}^{t} d_k \phi(v_k),$$

where $\varphi \in Aut_{R_o/pR_o}(U)$ and $\phi \in Aut_{R_o/pR_o}(V)$, let $\psi\sigma = \psi\alpha_\sigma$; if

$$\beta(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = \sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{i=1}^{2}\sum_{j=1}^{s} a_{ij} b_j p^i + \sum_{l=1}^{d}\sum_{j=1}^{s} c_{lj} b_j p u_l + \sum_{k=1}^{t}\sum_{j=1}^{s} b_j d_{kj} v_k + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k,$$

where $a_{ij},\ c_{lj},\ d_{kj} \in R_o/pR_o$, let $\beta\sigma = \beta\alpha_\sigma$; if

$$\gamma(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = \sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{l=1}^{d} a_{1l} c_l p^2 + \sum_{k=1}^{t} d_k v_k,$$

where $a_{1l} \in R_o/pR_o$, let $\gamma\sigma = \gamma\alpha_\sigma$; and if

$$\delta(\sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = \sum_{i=0}^{2} a_i p^i + \sum_{j=1}^{s} b_j u_j + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k + \sum_{\eta=1}^{d}\sum_{k=1}^{t} d_k g_{\eta k} p u_\eta + \sum_{i=1}^{2}\sum_{k=1}^{t} d_k e_{ik} p^i,$$

where $e_{ik},\ g_{\eta k} \in R_o/pR_o$, let $\delta\sigma = \delta\alpha_\sigma$. Finally, if $A = (a_{ij})$, define $A^\sigma = (a_{ij}^\sigma)$.

Due to some similarities of these rings, we present in this paper, detailed proofs of results on rings of characteristic p^2 in which $p \in \mathcal{J}^2$. The other two cases may be proved in a similar manner with minor modifications.

We start with the following.

3.2. *Rings in which* $p \in \mathcal{J}^2$.

THEOREM 3.6. *Let R be a ring of Theorem 2.1 and of characteristic p^2 in which $p \in \mathcal{J}^2$, with the invariants p, n, r, s, and t. Then, $\psi \in Aut(R)$ if and only if*

$$\begin{aligned}
&\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) \\
&\quad = x a_o^\sigma x^{-1} + x a_1^\sigma x^{-1} p + \sum_{i=1}^{s} a_{1i} x b_i^\sigma x^{-1} p + \sum_{k=1}^{t} e_{1k} x c_k^\sigma x^{-1} p \\
&\qquad + \sum_{i=1}^{s} x b_i^\sigma x^{-1} \varphi(u_i) + \sum_{k=1}^{t} \sum_{i=1}^{s} d_{ki} x b_i^\sigma x^{-1} v_k + \sum_{k=1}^{t} x c_k^\sigma x^{-1} \phi(v_k),
\end{aligned}$$

where $\sigma \in Aut(R_o)$, $x \in G_R$, $\varphi \in Aut_{R_o/pR_o}(U)$, $\phi \in Aut_{R_o/pR_o}(V)$; a_{1i}, d_{ki}, $e_{1k} \in R_o/pR_o$.

PROOF. Let $\psi \in Aut(R)$. Then there exists $x \in G_R$ such that $\psi(R_o) = x R_o x^{-1}$, and hence, $\psi(r) = x r^\sigma x^{-1}$, for any $r \in R_o$, for some automorphism σ of R_o. Since

$$R = \psi(R_o) \oplus \sum \psi(R_o)\psi(u_i) \oplus \sum \psi(R_o)\psi(v_k)$$

and conjugation is an automorphism of R,

$$R = R_o \oplus \sum R_o x^{-1}\psi(u_i) x \oplus \sum R_o x^{-1}\psi(v_k) x.$$

But $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, hence, $x^{-1}\psi(u_i)x = \alpha_i \psi(u_i)$ and $x^{-1}\psi(v_k)x = \beta_k \psi(v_k)$, where α_i, $\beta_k \in R_o/pR_o$, for all $i = 1, \ldots, s$; $k = 1, \ldots, t$.

Thus,

$$R = R_o \oplus \sum R_o \alpha_i \psi(u_i) \oplus \sum R_o \beta_k \psi(v_k)$$

and hence,

$$R = R_o \oplus \sum R_o \psi(u_i) \oplus \sum R_o \psi(v_k).$$

Therefore, for any $i \in \{1, \ldots, s\}$ and any $k \in \{1, \ldots, t\}$, $\psi(u_i) = \varphi(u_i) + a_{1i} p + \sum d_{ki} v_k$ and $\psi(v_k) = e_{1k} p + \phi(v_k)$, where $\varphi \in Aut_{R_o/pR_o}(U)$; $\phi \in Aut_{R_o/pR_o}(V)$; and a_{1i}, d_{ki}, $e_{1k} \in R_o/pR_o$.

Conversely, let ψ be as defined above. We need to check that for every $r = a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k \in R$,

$$\begin{aligned}\theta : r \quad \mapsto \quad & a_o^{\sigma} + a_1^{\sigma} p + \sum_{i=1}^{s} a_{1i} b_i^{\sigma} p + \sum_{k=1}^{t} e_{1k} c_k^{\sigma} p + \\ & \sum_{i=1}^{s} b_i^{\sigma} \varphi(u_i) + \sum_{k=1}^{t} \sum_{i=1}^{s} d_{ki} b_i^{\sigma} v_k + \sum_{k=1}^{t} c_k^{\sigma} \phi(v_k),\end{aligned}$$

where $\varphi(u_i) = x^{-1}\theta(u_i)x$, and $\phi(v_k) = x^{-1}\theta(v_k)x$, is an automorphism of R.

So, let $s = d_o + d_1 p + \sum_{i=1}^{s} e_i u_i + \sum_{k=1}^{t} f_k v_k$ be another element in R. Then,

$$\begin{aligned}\theta : s \quad \mapsto \quad & d_o^{\sigma} + d_1^{\sigma} p + \sum_{i=1}^{s} a_{1i} e_i^{\sigma} p + \sum_{k=1}^{t} e_{1k} f_k^{\sigma} p + \\ & \sum_{i=1}^{s} e_i^{\sigma} \varphi(u_i) + \sum_{k=1}^{t} \sum_{i=1}^{s} d_{ki} e_i^{\sigma} v_k + \sum_{k=1}^{t} f_k^{\sigma} \phi(v_k).\end{aligned}$$

Now,

$$\begin{aligned}\theta(r)\theta(s) \quad = \quad & a_o^{\sigma} d_o^{\sigma} + [a_o^{\sigma} d_1^{\sigma} + a_1^{\sigma} d_o^{\sigma}]p + \sum_{i=1}^{s} [a_o^{\sigma} a_{1i} e_i^{\sigma} + a_{1i} b_i^{\sigma} d_o^{\sigma}]p \\ & + \sum_{k=1}^{t} [a_o^{\sigma} e_{1k} f_k^{\sigma} + e_{1k} c_k^{\sigma} d_o^{\sigma}]p + \sum_{i=1}^{s} [a_o^{\sigma} e_i^{\sigma} + b_i^{\sigma} d_o^{\sigma}]\varphi(u_i) \\ & + \sum_{k=1}^{t} \sum_{i=1}^{s} [a_o^{\sigma} d_{ki} e_i^{\sigma} + d_{ki} b_i^{\sigma} d_o^{\sigma}] v_k + \sum_{k=i}^{t} [a_o^{\sigma} f_k^{\sigma} + c_k^{\sigma} d_o^{\sigma}]\phi(v_k) \\ & + \sum_{i,j=1}^{s} b_i^{\sigma} e_j^{\sigma} \varphi(u_i)\varphi(u_j).\end{aligned}$$

On the other hand,

$$\begin{aligned}\theta(rs) \quad = \quad & (a_o d_o)^{\sigma} + (a_o d_1 + a_1 d_o)^{\sigma} p + \sum_{i=1}^{s} a_{1i}(a_o e_i + b_i d_o)^{\sigma} p \\ & + \sum_{k=1}^{t} e_{1k}(a_o f_k + c_k d_o)^{\sigma} p + \sum_{i=1}^{s} (a_o e_i + b_i d_o)^{\sigma} \varphi(u_i) \\ & + \sum_{k=1}^{t} \sum_{i=1}^{s} d_{ki}(a_o e_i + b_i d_o)^{\sigma} v_k + \sum_{k=1}^{t} (a_o f_k + c_k d_o)^{\sigma} \phi(v_k) \\ & + \sum_{i,j=1}^{s} (b_i e_j a_{ij}^{o})^{\sigma} p + \sum_{k=1}^{t} \sum_{i,j=1}^{s} (b_i e_j a_{ij}^{k})^{\sigma} \phi(v_k).\end{aligned}$$

From the above equalities, we deduce that

$$(a_{ij}^o)^\sigma p + \sum_{k=1}^{t} (a_{ij}^k)^\sigma \phi(v_k) = \sum_{i,j=1}^{s} \varphi(u_i)\varphi(u_j). \tag{3.1}$$

Now, it is obvious that $\psi = \psi_x \theta$ and hence, ψ is an automorphism of R. □

From the assumptions that $\sigma \in Aut(R_o)$, $x \in G_R$, $\varphi \in Aut_{R_o/pR_o}(U)$ and $\phi \in Aut_{R_o/pR_o}(V)$ one obtains the following: $\varphi(u_i) = \sum_{\nu=1}^{s} b_{\nu i} u_\nu$ and $\phi(v_k) = \sum_{\rho=1}^{t} c_{\rho k} v_\rho$, with $b_{\nu i},\ c_{\rho k} \in R_o/pR_o$.

Hence, (3.1) implies that

$$(a_{ij}^o)^\sigma p + \sum_{\rho,k=1}^{t} c_{\rho k} (a_{ij}^k)^\sigma v_\rho = \sum_{\rho=0}^{t} \sum_{\nu,\mu=1}^{s} b_{\nu i} b_{\mu j} a_{\nu\mu}^\rho v_\rho$$

or

$$\sum_{\rho,k=0}^{t} c_{\rho k} (a_{ij}^k)^\sigma v_\rho = \sum_{\rho=0}^{t} \sum_{\nu,\mu=1}^{s} b_{\nu i} b_{\mu j} a_{\nu\mu}^\rho v_\rho,$$

where $c_{oo} = 1$, $c_{1k} = e_{1k}$ and $v_o = p$. It follows that

$$\sum_{\nu,\mu=1}^{s} b_{\nu i} b_{\mu j} a_{\nu\mu}^\rho = \sum_{k=0}^{t} c_{\rho k} \psi(a_{ij}^k)\ (\rho = 0,\ 1, \ldots, t). \tag{3.2}$$

Hence, in matrix form, (3.2) implies that

$$B^T A_\rho B = \sum_{k=0}^{t} c_{\rho k} A_k^\sigma\ (\rho = 0,\ 1, \ldots, t),$$

where $\sigma \in Aut(R_o)$, $B \in GL(s, R_o/pR_o)$ and $C = (c_{k\rho}) \in GL(1+t, R_o/pR_o)$.

Conversely, suppose there exist $\sigma \in Aut(R_o)$, $B \in GL(s, R_o/pR_o)$ and $C = (c_{\rho k}) \in GL(1+t, R_o/pR_o)$, with

$$B^T A_\rho B = \sum_{k=0}^{t} c_{\rho k} A_k^\sigma\ (\rho = 0,\ 1, \ldots, t),$$

where $c_{oo} = 1$, $c_{1k} = e_{1k}$.

Consider the map $\psi : R \to R$ defined by

$$\begin{aligned} \psi(a_o + a_1 p + \sum_i b_i u_i \sum_k c_k v_k) \quad = \quad & a_o^\sigma + [a_1^\sigma + \sum_i a_{1i} b_i^\sigma + \sum_k e_{1k} c_k^\sigma] p + \\ & \sum_\nu \sum_i b_i^\sigma b_{\nu i} u_\nu + \\ & \sum_\rho [\sum_i b_i^\sigma d_{\rho i} + \sum_k c_k^\sigma c_{\rho k}] v_\rho. \end{aligned}$$

Then it is routine to verify that ψ is a homomorphism from R to R and that it preserves the identity element.

But $Ker\psi$ consists of all elements

$$a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k \in R$$

such that

$$a_o^\sigma + [a_1^\sigma + \sum_i a_{1i} b_i^\sigma + \sum_k e_{1k} c_k^\sigma] p + \sum_\nu \sum_i b_i^\sigma b_{\nu i} u_\nu + \sum_\rho [\sum_i b_i^\sigma d_{\rho i} + \sum_k c_k^\sigma c_{\rho k}] v_\rho = 0;$$

which implies that

$$a_o^\sigma + [a_1^\sigma + \sum_i a_{1i} b_i^\sigma + \sum_k e_{1k} c_k^\sigma] p = 0,$$

$$\sum_\nu \sum_i b_i^\sigma b_{\nu i} u_\nu = 0$$

and

$$\sum_\rho [\sum_i b_i^\sigma d_{\rho i} + \sum_k c_k^\sigma c_{\rho k}] v_\rho = 0.$$

Now,

$$\sum_\nu \sum_i b_i^\sigma b_{\nu i} u_\nu = 0 \text{ implies that } \sum_i b_i^\sigma b_{\nu i} = 0, \text{ for every } \nu = 1, \ldots, s;$$

since $\{u_i, \ldots, u_s\}$ is linearly independent over R_o/pR_o. Further, $(b_{\nu i})$ is invertible, so that the homogeneous system $\sum_i b_i^\sigma b_{\nu i} = 0;\ \nu = 1, \ldots, s$, has the trivial solution as its unique solution and hence, $b_i = 0$ (for every $i = 1, \ldots, s$) since $\sigma \in Aut(R_o)$.

Similarly, $c_k = 0$ for every $k = 1, \ldots, t$ since $(c_{\rho k})_{t \times t}$ is invertible. Hence,

$$a_o^\sigma + [a_1^\sigma + \sum_i a_{1i} b_i^\sigma + \sum_k e_{1k} c_k^\sigma] p = 0,$$

with $c_k = 0$ for every $k = 1, \ldots, t$ and $b_i = 0$ for every $i = 1, \ldots, s$ implies that $a_o^\sigma + a_1^\sigma p = 0$, so that $a_1^\sigma p = -a_o^\sigma$. But $a_1^\sigma p \in pR_o$, implying that $a_o^\sigma \in pR_o$, a contradiction, since $a_o \in K_o$. Hence, $a_o = a_1 = 0$.

Hence, $Ker\psi = (0)$ and therefore, ψ is injective, and since R is finite, ψ is also surjective. Thus, ψ is an automorphism of R.

We have thus proved the following:

PROPOSITION 3.7. *Let R be a ring of Theorem 2.1 and of characteristic p^2 with the invariants $p,\ n,\ r,\ s,\ t$. Then ψ is an automorphism of R if and only if there exist $\sigma \in Aut(R_o)$, $B \in GL(s, R_o/pR_o)$ and $C = (c_{\rho k}) \in GL(1+t, R_o/pR_o)$ such that $B^T A_\rho B = \sum_{k=0}^{t} c_{\rho k} A_k^\sigma$, where A_ρ and A_k are structural matrices for R and $c_{oo} = 1$, $c_{1k} = e_{1k}$.*

Thus, the set of elements $\sigma \in Aut(R_o/pR_o)$, $C = (c_{\rho k}) \in GL(1 + t, R_o/pR_o)$, $B \in GL(s, R_o/pR_o)$ and $1 + \mathcal{J} \cong pR_o \oplus U \oplus V$, determines all the automorphisms of the ring R.

Consider the set of equations $B^T A_\rho B = \sum_{k=1}^{t} c_{\rho k} A_k^\sigma$ given in Proposition 3.7, with $B = (b_{ij}) \in GL(s, R_o/pR_o)$. Then, it is easy to see that $B = (b_{ij})$ is the transition matrix between the bases $(\overline{u_i})$ of $\mathcal{J}/\mathcal{J}^2$. Equally, $C = (c_{\rho k})$ is the transition matrix between the bases (v_k) $(k = 0,\ 1, \ldots, t)$ of $\mathcal{J}^2$. By calculating $u_\nu u_\mu$ (the images of the u_i under ψ) and comparing coefficients of (v_ρ) $(\rho = 0,\ 1, \ldots, t)$ (the images of the v_k under ψ) we obtain equations, which in matrix form, are $B^T A_\rho B = \sum_{k=1}^{t} c_{\rho k} A_k^\sigma$.

The problem of determining the groups of automorphisms of our rings amounts to classifying $(1 + t)-$tuples of linearly independent matrices A_o, $A_1, \ldots,\ A_t$ under the above relation, $B,\ C$ being arbitrary invertible matrices, σ being an arbitrary automorphism and $1 + \mathcal{J}$ being the normal subgroup of G_R of order $p^{(n-1)r}$.

Let $\mathcal{A}$ be the set of all $(1 + t)-$tuples $(A_o,\ A_1, \ldots, A_t)$ of $s \times s$ matrices over R_o/pR_o. The group $GL(s, R_o/pR_o)$ acts on $\mathcal{A}$ by "congruence":

$$(A_o,\ A_1, \ldots, A_t) \cdot B = (B^T A_o B,\ B^T A_1 B, \ldots, B^T A_t B)$$

and on the left via

$$\begin{aligned} &C \cdot (A_o,\ A_1, \ldots, A_t) \\ &\quad = (c_{1o} A_o^\sigma + c_{11} A_1^\sigma + \cdots + c_{1t} A_t^\sigma, \ldots, c_{to} A_o^\sigma + c_{t1} A_1^\sigma + \cdots + c_{tt} A_t^\sigma), \end{aligned}$$

where $C = (c_{\rho k})$. Thus, these two actions are permutable and define a (left) action of $G = GL(s, R_o/pR_o) \times GL(1 + t, R_o/pR_o)$ on $\mathcal{A}$:

$$(B,\ C) \cdot (A_o,\ A_1, \ldots, A_t) = C \cdot (A_o^\sigma, \ldots, A_t^\sigma) \cdot B^{-1},$$

for some fixed automorphism σ. By restriction, G acts on the subset Y consisting of $(1 + t)-$tuples $A_0,\ A_1, \ldots, A_t$, linearly independent. This amounts to studying the "congruence" action (via B) on $GL(s, R_o/pR_o)$ on the set $\mathcal{Y}$ of $1 + t-$dimensional subspaces of $\mathbb{M}_{s\times s}(R_o/pR_o)$, C just representing a change of basis in a given space. In the same way, the whole action of G on $\mathcal{A}$ may be represented as an action of $GL(1 + t, R_o/pR_o)$ on the set $\mathbf{A}$ of subspaces of dimension $\leq 1 + t$. We may call two $(1 + t)$-tuples in the same G-orbit as *equivalent*.

3.3. *Rings in which $p \in \mathcal{J} - \mathcal{J}^2$.*

Theorem 3.8. *Let R be a ring of Theorem 2.1 and of characteristic p^2 in which $p \in \mathcal{J} - \mathcal{J}^2$, with the invariants $p,\ n,\ r,\ s,\ t,$ and d. Then, $\psi \in Aut(R)$*

if and only if

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k) = x a_o^\sigma x^{-1} + x a_1^\sigma x^{-1} p$$
$$+ \sum_{i=1}^{s} x b_i^\sigma x^{-1} \varphi(u_i) + \sum_{l=1}^{d} \sum_{i=1}^{s} b_{li} x b_i^\sigma x^{-1} p u_l + \sum_{\nu=1}^{t} \sum_{i=1}^{s} c_{\nu i} x b_i^\sigma x^{-1} v_\nu$$
$$+ \sum_{l=1}^{d} x c_l^\sigma x^{-1} p \varphi(u_l) + \sum_{k=1}^{t} x d_k^\sigma x^{-1} \phi(v_k) + \sum_{\eta=1}^{d} \sum_{k=1}^{t} d_{\eta k} x d_k^\sigma x^{-1} p u_l,$$

where $\sigma \in Aut(R_o)$, $x \in G_R$, $\varphi \in Aut_{R_o/pR_o}(U)$, $\phi \in Aut_{R_o/pR_o}(V)$; b_{li}, $c_{\nu i}$, $d_{\eta k} \in R_o/pR_o$.

Like we did for Theorem 3.6, we deduce from this the following:

Proposition 3.9. *Let R be a ring of characteristic p^2 in which R_o lies in the center, with the invariants p, n, r, s, t, d and in which p does not lie in $\mathcal{J}^2$. Then ψ is an automorphism of R if and only if there exist $\sigma \in Aut(R_o)$, $B \in GL(s, R_o/pR_o)$ and $C = (c_{\rho k}) \in GL(t + d, R_o/pR_o)$ such that $B^T A_\rho B = \sum_{k=1}^{t+d} c_{\rho k} A_k^\sigma$, where A_k and A_ρ are structural matrices for R.*

3.4. *Rings of characteristic p^3.* We now consider the case of rings of Theorem 2.1 and of characteristic p^3.

Theorem 3.10. *Let R be a ring of Theorem 2.1 and of characteristic p^3 with the invariants p, n, r, s, t, d.Then $\psi \in Aut(R)$ if and only if*

$$\psi(a_o + a_1 p + a_2 p^2 + \sum_{i=1}^{s} b_i u_i + \sum_{l=1}^{d} c_l p u_l + \sum_{k=1}^{t} d_k v_k)$$
$$= x a_o^\sigma x^{-1} + x a_1^\sigma x^{-1} p + x a_2^\sigma x^{-1} p^2 + \sum_{i=1}^{s} b_{oi} x b_i^\sigma x^{-1} p^2 + \sum_{l=1}^{d} g_{ol} x c_l^\sigma x^{-1} p^2$$
$$+ \sum_{k=1}^{t} e_{ok} x d_k^\sigma x^{-1} p^2 + \sum_{i=1}^{s} x b_i^\sigma x^{-1} \varphi(u_i) + \sum_{l=1}^{d} \sum_{i=1}^{s} b_{li} x b_i^\sigma x^{-1} p u_l$$
$$+ \sum_{\nu=1}^{t} \sum_{i=1}^{s} c_{\nu i} x b_i^\sigma x^{-1} v_\nu + \sum_{l=1}^{d} x c_l^\sigma x^{-1} p \varphi(u_l) + \sum_{k=1}^{t} x d_k^\sigma x^{-1} \phi(v_k)$$
$$+ \sum_{\eta=1}^{d} \sum_{k=1}^{t} d_{\eta k} x d_k^\sigma x^{-1} p u_\eta,$$

where $\sigma \in Aut(R_o)$, $x \in G_R$, $\varphi \in Aut_{R_o/pR_o}(U)$, $\phi \in Aut_{R_o/pR_o}(V)$; b_{oi}, e_{ok}, g_{ol}, b_{li}, $c_{\nu i}$, $d_{\eta k} \in R_o/pR_o$.

From this we deduce the following matrix version of the result.

Proposition 3.11. *Let R be a ring of Theorem 2.1 and of characteristic p^3 with the invariants p, n, r, s, t, d. Then ψ is an automorphism of R if and only if there exist $\sigma \in Aut(R_o)$, $B \in GL(s, R_o/pR_o)$ and $C = (c_{\rho k}) \in GL(1+t+d, R_o/pR_o)$ such that $B^T D_\rho B = \sum_{k=0}^{t+d} c_{\rho k} A_k^\sigma$, where A_k and A_ρ are structural matrices for R.*

4. The main results

We now describe explicitly, the group of automorphisms of the ring R. In what follows, we provide the proof for the case when the characteristic of R is p^2 and $p \in \mathcal{J}^2$; while the proofs for the other cases may be obtained through minor modifications of the proof of Theorem 4.1.

Theorem 4.1. *Let R be a ring of Theorem 2.1, of characteristic p^2 in which $p \in \mathcal{J}^2$ and with the invariants p, n, r, s, t. Then*

$$Aut(R) \cong [\mathbb{M}_{(1+t)\times s}(R_o/pR_o) \times \mathbb{M}_{1\times t}(R_o/pR_o) \times (pR_o \oplus U \oplus V)] \times_{\theta_2} [Aut(R_o) \times_{\theta_1} (GL(s, R_o/pR_o) \times GL(t, R_o/pR_o))].$$

Proof. Let G be the subgroup of $Aut(R)$ which contains all the automorphisms ψ defined by

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = a_o^\sigma + a_1^\sigma p + \sum_{i=1}^{s} b_i^\sigma \varphi(u_i) + \sum_{k=1}^{t} c_k^\sigma \phi(v_k),$$

where $\sigma \in Aut(R_o)$, $\varphi \in Aut_{R_o/pRo}(U)$ and $\phi \in Aut_{R_o/pR_o}(V)$.

Let G_0 be the subgroup of G which contains all the automorphisms α_σ such that

$$\alpha_\sigma(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = a_o^\sigma + a_1^\sigma p + \sum_{i=1}^{s} b_i^\sigma u_i + \sum_{k=1}^{t} c_k^\sigma v_k,$$

where $\sigma \in Aut(R_o)$. Then $G_0 \cong Aut(R_o)$. Let G_1 be the subgroup of G which contains all the automorphisms ψ such that

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = a_o + a_1 p + \sum_{i=1}^{s} b_i \varphi(u_i) + \sum_{k=1}^{t} c_k v_k,$$

where $\varphi \in Aut_{R_o/pR_o}(U)$; and let G_2 be the subgroup of G which contains all the automorphisms ψ such that

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k \phi(v_k),$$

where $\phi \in Aut_{R_o/pR_o}(V)$. Then G_1 and G_2 are subgroups of G and $G_1 \times G_2$ is a direct product. Moreover, $G_1 \cong Aut_{R_o/pR_o}(U) \cong GL(s, R_o/pR_o)$ and $G_2 \cong Aut_{R_o/pR_o}(V) \cong GL(t, R_o/pR_o)$.

Finally, let H be the subgroup of $Aut(R)$ containing all the automorphisms ψ defined by

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = x(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{i=1}^{s} b_i a_{1i} p + \sum_{\rho=1}^{t}\sum_{i=1}^{s} b_i d_{\rho i} v_\rho + \sum_{k=1}^{t} c_k v_k + \sum_{k=1}^{t} c_k e_{1k} p)x^{-1},$$

where $x \in 1 + \mathcal{J}$, a_{1i}, $d_{\rho i}$, $e_{1k} \in R_o/pR_o$; H_1 be the subgroup of H which contains all the automorphisms ψ defined by

$$\psi(a_o+a_1p+\sum_{i=1}^{s} b_i u_i+\sum_{k=1}^{t} c_k v_k) = a_o+a_1p+\sum_{i=1}^{s} b_i u_i+\sum_{\rho=0}^{t}\sum_{i=1}^{s} b_i d_{\rho i} p+\sum_{k=1}^{t} c_k v_k,$$

where $d_{\rho i} \in R_o/pR_o$ and $v_o = p$; H_2 be the subgroup of H which contains all the automorphisms ψ such that

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k + \sum_{k=1}^{t} c_k e_{1k} p,$$

where $e_{1k} \in R_o/pR_o$; and let H_3 be the subgroup of H which contains all the automorphisms ψ such that

$$\psi(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k) = x(a_o + a_1 p + \sum_{i=1}^{s} b_i u_i + \sum_{k=1}^{t} c_k v_k)x^{-1},$$

where $x \in 1 + \mathcal{J} \subset G_R$. Then it is easy to check that the direct product $H = H_1 \times H_2 \times H_3$ and the semidirect product $G = (G_1 \times G_2) \times_{\theta_2} G_0$ are subgroups of $Aut(R)$, where if $\varphi \in G_1 \times G_2$ and $\alpha_\sigma \in G_0$, then $\theta_2(\alpha_\sigma)(\varphi) = \varphi\sigma$.

Let $\varphi \in H \cap G$. Since every element of H is either fixing R_o elementwise or sending R_o to another maximal Galois subring of R and $\varphi \in G$, φ fixes R_o elementwise.

Let $\varphi = \beta\psi_x$, where $\beta \in H_1 \times H_2$ and $\psi_x \in H_3$. Since $x \in 1 + \mathcal{J}$, clearly, $\varphi = \beta\psi_x = \beta$. Since $\beta \in G$, $\beta(U) = U$ and $\beta(V) = V$. But the only element of $H_1 \times H_2$ which fixes U and V is the identity. Thus, $\varphi = id_R$ and hence, $H \cap G = id_R$. Now, it is easy to see that $Aut(R) = H \times_{\theta_1} G$, where if $\beta\psi_x \in H_1 \times H_2$ and $\varphi\alpha_\sigma \in G$, then $\theta_1(\varphi\alpha_\sigma)(\beta\psi_x) = \beta_\sigma\varphi_{\psi\alpha_\sigma}(x)$. It is trivial to check that the mappings $g : H_1 \mapsto \mathbb{M}_{(1+t)\times s}(R_o/pR_o)$ given by $g(\beta_M) = \sum_{\rho=0}^{t} d_{\rho i} v_\rho$ and $h : H_2 \mapsto \mathbb{M}_{1\times t}(R_o/pR_o)$ given by $h(\beta_M) = e_{1k}p$, are isomorphisms, and hence, combining with $f : H_3 \to pR_o \oplus U \oplus V$, we obtain an isomorphism $H \cong \mathbb{M}_{(1+t)\times s}(R_o/pR_o) \times \mathbb{M}_{1\times t}(R_o/pR_o) \times (pR_o \oplus U \oplus V)$.

Hence,

$$Aut(R) \cong [\mathbb{M}_{(1+t)\times s}(R_o/pR_o) \times \mathbb{M}_{1\times t}(R_o/pR_o) \times (pR_o \oplus U \oplus V)] \times_{\theta_2} [Aut(R_o) \times_{\theta_1} (GL(s, R_o/pR_o) \times GL(t, R_o/pR_o))],$$

where

$$\theta_1(\sigma)(B,\ C)\cdot(A_0,\ldots,A_t) = C\cdot(A_o^\sigma,\ldots,A_t^\sigma)\cdot C^{-1};$$

and

$$\theta_2(\sigma,\ B,\ C)(A_0,\ldots,A_t) = (B^TA_0B,\ B^TA_1B,\ldots,B^TA_tB).$$

□

Theorem 4.2. *Let R be a ring of Theorem 2.1, of characteristic p^2 in which $p \in \mathcal{J} - \mathcal{J}^2$ and with the invariants $p,\ n,\ r,\ s,\ t,\ d$. Then*

$$Aut(R) \cong [\mathbb{M}_{(d+t)\times s}(R_o/pR_o) \times \mathbb{M}_{d\times t}(R_o/pR_o) \times (pR_o \oplus U \oplus V)] \times_{\theta_2} [Aut(R_o) \times_{\theta_1} (GL(s, R_o/pR_o) \times GL(t, R_o/pR_o) \times GL(d, R_o/pR_o))].$$

Proof. Modify the proof of Theorem 4.1. □

Theorem 4.3. *Let R be a ring of Theorem 2.1 and of characteristic p^3 with the invariants $p,\ n,\ r,\ s,\ t,\ d$. Then $Aut(R)$ is isomorphic to*

$$[\mathbb{M}_{(1+d+t)\times s}(K) \times \mathbb{M}_{(1+t)\times d}(K) \times \mathbb{M}_{1\times d}(K) \times (pR_o \oplus U \oplus V)] \times_{\theta_2} [Aut(R_o) \times_{\theta_1} (GL(s, K) \times GL(t, K) \times GL(d, K))];$$

where $K = R_o/pR_o$.

Proof. Similar to Theorem 4.1 with some modifications. □

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