GLASNIK MATEMATIČKI
Vol. 43(63)(2008), 41 – 57

VERTEX OPERATOR ALGEBRAS ASSOCIATED TO CERTAIN ADMISSIBLE MODULES FOR AFFINE LIE ALGEBRAS OF TYPE A

Ozren Perše
University of Zagreb, Croatia

Abstract. Let $L(-\frac{1}{2}(l+1),0)$ be the simple vertex operator algebra associated to an affine Lie algebra of type $A_l^{(1)}$ with the lowest admissible half-integer level $-\frac{1}{2}(l+1)$, for even l. We study the category of weak modules for that vertex operator algebra which are in category $\mathcal{O}$ as modules for the associated affine Lie algebra. We classify irreducible objects in that category and prove semisimplicity of that category.

1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $\hat{\mathfrak{g}}$ the associated affine Lie algebra. For any complex number $k \neq -h^\vee$, denote by $L(k,0)$ the simple vertex operator algebra associated to $\hat{\mathfrak{g}}$ with level k. The representation theory of $L(k,0)$ heavily depends on the choice of level k. If k is a positive integer, $L(k,0)$ is a rational vertex operator algebra (cf. [12, 28]), i.e. the category of $\mathbb{Z}_+$-graded weak $L(k,0)$-modules is semisimple. Irreducible objects in that category are integrable highest weight $\hat{\mathfrak{g}}$-modules of level k ([12, 22]). The corresponding associative algebra $A(L(k,0))$, defined in [28], is finite-dimensional (cf. [18]). In some cases such as $k \notin \mathbb{Q}$ or $k < -h^\vee$ (studied in [19, 20]), categories of $L(k,0)$-modules have significantly different structure then categories of $L(k,0)$-modules for a positive integer k. However, there are examples of rational levels k such that the category of weak $L(k,0)$-modules

2000 *Mathematics Subject Classification.* 17B69, 17B67.
Key words and phrases. Vertex operator algebra, affine Kac-Moody algebra, modular invariant representation.

which are in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-modules, has similar structure as the category of $\mathbb{Z}_+$-graded weak $L(k,0)$-modules for positive integer levels k. These are so called admissible levels, defined in [16, 17].

The case of vertex operator algebras associated to affine Lie algebras of type $C_l^{(1)}$ with admissible half-integer levels has been studied in [1, 2]. Vertex operator algebras associated to affine Lie algebras of type $A_1^{(1)}$ with arbitrary admissible level have been studied in [4, 7]. In these cases vertex operator algebra $L(k,0)$ has finitely many irreducible weak modules from the category $\mathcal{O}$ and every weak $L(k,0)$-module from the category $\mathcal{O}$ is completely reducible. One can say that these vertex operator algebras are rational in the category $\mathcal{O}$. In [4], authors gave a conjecture that vertex operator algebras $L(k,0)$, for all admissible levels k, are rational in the category $\mathcal{O}$. In the case of vertex operator algebras associated to affine Lie algebras of type $B_l^{(1)}$ with admissible half-integer levels, certain parts of this conjecture were verified in [25]. Admissible modules for affine Lie algebras were also recently studied in [3, 8, 13, 26, 27]. Vertex operator algebras associated to certain affine Lie algebras with non-admissible negative integer levels have been studied in [5].

When k is an admissible level, vertex operator algebra $L(k,0)$ is a quotient of the generalized Verma module by the maximal ideal generated by one singular vector. The formula for this singular vector is very complicated for general admissible level k (cf. [23]). But for some special cases of affine Lie algebras and half-integer admissible levels k, this singular vector has conformal weight 2, and the formula for this vector is relatively simple (cf. [1, 2, 25]). In this paper we study one similar special case, for which we verify the conjecture from [4].

We consider the case of an affine Lie algebra of type $A_l^{(1)}$ and the corresponding vertex operator algebra $L(-\frac{1}{2}(l+1),0)$, for even l. We show that $-\frac{1}{2}(l+1)$ is an admissible level for this affine Lie algebra. The results on admissible modules from [16] imply that $L(-\frac{1}{2}(l+1),0)$ is a quotient of the generalized Verma module by the maximal ideal generated by a singular vector of conformal weight 2. Using results from [12, 28], we can identify the corresponding associative algebra $A(L(-\frac{1}{2}(l+1),0))$ with a certain quotient of $U(\mathfrak{g})$. Algebra $A(L(-\frac{1}{2}(l+1),0))$ is infinite-dimensional in this case. Using methods from [2, 4, 24], we get that irreducible $A(L(-\frac{1}{2}(l+1),0))$-modules from the category $\mathcal{O}$ are in one-to-one correspondence with zeros of the certain set of polynomials $\mathcal{P}_0$. By determining a basis for the vector space $\mathcal{P}_0$, we obtain the classification of irreducible $A(L(-\frac{1}{2}(l+1),0))$-modules from the category $\mathcal{O}$. Using results from [28], we obtain the classification of irreducible weak $L(-\frac{1}{2}(l+1),0)$-modules from the category $\mathcal{O}$. Using this classification and results from [17], we show that every weak $L(-\frac{1}{2}(l+1),0)$-module from the category $\mathcal{O}$ is completely reducible.

In the case when l is odd, the lowest half-integer admissible level for affine Lie algebra of type $A_l^{(1)}$ is $-\frac{1}{2}l$, and the maximal submodule of the generalized Verma module is generated by a singular vector of conformal weight 4. It is more complicated to determine the formula for singular vector in that case, and to use the method for classification from [2, 4, 24].

In this paper $\mathbb{Z}_+$ denotes the set of nonnegative integers.

2. Vertex operator algebras associated to affine Lie algebras

In this section we review certain results on vertex operator algebras and corresponding modules. Specially, we recall some results on vertex operator algebras associated to affine Lie algebras.

2.1. *Vertex operator algebras and modules.* Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [6, 10, 11]). An *ideal* in a vertex operator algebra V is a subspace I of V satisfying $Y(a,z)I \subseteq I[[z,z^{-1}]]$ for any $a \in V$. Given an ideal I in V, such that $\mathbf{1} \notin I$, $\omega \notin I$, the quotient V/I admits a natural vertex operator algebra structure.

Let (M, Y_M) be a weak module for a vertex operator algebra V (cf. [22]). A $\mathbb{Z}_+$*-graded weak* V-module ([12]) is a weak V-module M together with a $\mathbb{Z}_+$-gradation $M = \oplus_{n=0}^{\infty} M(n)$ such that

$$a_m M(n) \subseteq M(n+r-m-1) \quad \text{for } a \in V_{(r)}, m, n, r \in \mathbb{Z},$$

where $M(n) = 0$ for $n < 0$ by definition.

A weak V-module M is called a V*-module* if $L(0)$ acts semisimply on M with the decomposition into $L(0)$-eigenspaces $M = \oplus_{\alpha \in \mathbb{C}} M_{(\alpha)}$ such that for any $\alpha \in \mathbb{C}$, $\dim M_{(\alpha)} < \infty$ and $M_{(\alpha+n)} = 0$ for $n \in \mathbb{Z}$ sufficiently small.

2.2. *Zhu's $A(V)$ theory.* Let V be a vertex operator algebra. Following [28], we define bilinear maps $* : V \times V \to V$ and $\circ : V \times V \to V$ as follows. For any homogeneous $a \in V$ and for any $b \in V$, let

$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, a}}{z^2} Y(a,z)b,$$
$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, a}}{z} Y(a,z)b$$

and extend to $V \times V \to V$ by linearity. Denote by $O(V)$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. For $a \in V$, denote by $[a]$ the image of a under the projection of V onto $A(V)$. The multiplication $*$ induces the multiplication on $A(V)$ and $A(V)$ has a structure of an associative algebra.

Proposition 2.1 ([12, Proposition 1.4.2]). *Let I be an ideal of V. Assume $\mathbf{1} \notin I$, $\omega \notin I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/A(I)$, where $A(I)$ is the image of I in $A(V)$.*

For any homogeneous $a \in V$ we define $o(a) = a_{\mathrm{wt}a-1}$ and extend this map linearly to V.

PROPOSITION 2.2 ([28, Theorem 2.1.2, Theorem 2.2.1]). (a) *Let* $M = \oplus_{n=0}^{\infty} M(n)$ *be a* $\mathbb{Z}_+$*-graded weak* V*-module. Then* $M(0)$ *is an* $A(V)$*-module defined as follows:*

$$[a].v = o(a)v,$$

for any $a \in V$ *and* $v \in M(0)$.

(b) *Let* U *be an* $A(V)$*-module. Then there exists a* $\mathbb{Z}_+$*-graded weak* V*-module* M *such that the* $A(V)$*-modules* $M(0)$ *and* U *are isomorphic.*

PROPOSITION 2.3 ([28, Theorem 2.2.2]). *The equivalence classes of the irreducible* $A(V)$*-modules and the equivalence classes of the irreducible* $\mathbb{Z}_+$*-graded weak* V*-modules are in one-to-one correspondence.*

2.3. *Modules for affine Lie algebras.* Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_+ \subset \Delta$ the set of positive roots, θ the highest root and $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form, normalized by the condition $(\theta, \theta) = 2$.

The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ equipped with the usual bracket operation and the canonical central element c (cf. [14]). Let $h^\vee$ be the dual Coxeter number of $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\hat{\mathfrak{g}}$. Denote by $\hat{\Delta}$ the set of roots of $\hat{\mathfrak{g}}$, by $\hat{\Delta}^{\mathrm{re}}$ (resp. $\hat{\Delta}^{\mathrm{re}}_+$) the set of real (resp. positive real) roots of $\hat{\mathfrak{g}}$ and by $\alpha^\vee$ denote the coroot of a real root $\alpha \in \hat{\Delta}^{\mathrm{re}}$.

For every weight $\lambda \in \hat{\mathfrak{h}}^*$, denote by $M(\lambda)$ the Verma module for $\hat{\mathfrak{g}}$ with highest weight λ, and by $L(\lambda)$ the irreducible $\hat{\mathfrak{g}}$-module with highest weight λ.

Let U be a $\mathfrak{g}$-module, and let $k \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially on U and c as scalar k. Considering U as a $\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$-module, we have the induced $\hat{\mathfrak{g}}$-module (so called *generalized Verma module*)

$$N(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+)} U.$$

For a fixed $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$-module with highest weight μ. We shall use the notation $N(k, \mu)$ to denote the $\hat{\mathfrak{g}}$-module $N(k, V(\mu))$. Denote by $J(k, \mu)$ the maximal proper submodule of $N(k, \mu)$ and $L(k, \mu) = N(k, \mu)/J(k, \mu)$.

2.4. *Admissible modules for affine Lie algebras.* Let $\hat{\Delta}^{\vee\mathrm{re}}$ (resp. $\hat{\Delta}^{\vee\mathrm{re}}_+$) $\subset \hat{\mathfrak{h}}$ be the set of real (resp. positive real) coroots of $\hat{\mathfrak{g}}$. Fix $\lambda \in \hat{\mathfrak{h}}^*$. Let $\hat{\Delta}^{\vee\mathrm{re}}_\lambda = \{\alpha \in \hat{\Delta}^{\vee\mathrm{re}} \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, $\hat{\Delta}^{\vee\mathrm{re}}_{\lambda+} = \hat{\Delta}^{\vee\mathrm{re}}_\lambda \cap \hat{\Delta}^{\vee\mathrm{re}}_+$, $\hat{\Pi}^\vee$ the set of simple coroots in $\hat{\Delta}^{\vee\mathrm{re}}$ and $\hat{\Pi}^\vee_\lambda = \{\alpha \in \hat{\Delta}^{\vee\mathrm{re}}_{\lambda+} \mid \alpha$ not equal to a sum of several coroots from $\hat{\Delta}^{\vee\mathrm{re}}_{\lambda+}\}$. Define ρ in the usual way, and denote by $w.\lambda$ the "shifted" action of an element w of the Weyl group of $\hat{\mathfrak{g}}$.

Recall that a weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* (cf. [16, 17, 27]) if the following properties are satisfied:

$$\langle \lambda + \rho, \alpha \rangle \notin -\mathbb{Z}_+ \text{ for all } \alpha \in \hat{\Delta}_+^{\vee\mathrm{re}},$$
$$\mathbb{Q}\hat{\Delta}_\lambda^{\vee\mathrm{re}} = \mathbb{Q}\hat{\Pi}^\vee.$$

The irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ is called admissible if the weight $\lambda \in \hat{\mathfrak{h}}^*$ is admissible.

We shall use the following results from [16] and [17]:

Proposition 2.4 ([16, Corollary 2.1]). *Let λ be an admissible weight. Then*

$$L(\lambda) = \frac{M(\lambda)}{\sum_{\alpha \in \hat{\Pi}_\lambda^\vee} U(\hat{\mathfrak{g}}) v^\alpha} \ ,$$

where $v^\alpha \in M(\lambda)$ is a singular vector of weight $r_\alpha.\lambda$, the highest weight vector of $M(r_\alpha.\lambda) = U(\hat{\mathfrak{g}})v^\alpha \subset M(\lambda)$.

Proposition 2.5 ([17, Theorem 4.1]). *Let M be a $\hat{\mathfrak{g}}$-module from the category $\mathcal{O}$ such that for any irreducible subquotient $L(\nu)$ the weight ν is admissible. Then M is completely reducible.*

2.5. *Vertex operator algebras $N(k,0)$ and $L(k,0)$, for $k \neq -h^\vee$.* Since $V(0)$ is the one-dimensional trivial $\mathfrak{g}$-module, it can be identified with $\mathbb{C}$. Denote by $\mathbf{1} = 1 \otimes 1 \in N(k,0)$. We note that $N(k,0)$ is spanned by the elements of the form $x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}$, where $x_1,\ldots,x_m \in \mathfrak{g}$ and $n_1,\ldots,n_m \in \mathbb{Z}_+$, with $x(n)$ denoting the representation image of $x \otimes t^n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Vertex operator map $Y(\cdot,z) : N(k,0) \to (\mathrm{End}\ N(k,0))[[z,z^{-1}]]$ is uniquely determined by defining $Y(\mathbf{1},z)$ to be the identity operator on $N(k,0)$ and

$$Y(x(-1)\mathbf{1},z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n-1},$$

for $x \in \mathfrak{g}$. In the case $k \neq -h^\vee$, $N(k,0)$ has a conformal vector

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} x^i(-1)^2 \mathbf{1}, \tag{2.1}$$

where $\{x^i\}_{i=1,\ldots,\dim\mathfrak{g}}$ is an arbitrary orthonormal basis of $\mathfrak{g}$ with respect to the form $(\cdot,\cdot)$. We have the following result from [12] (see also [9, 15, 21, 22, 24]):

Proposition 2.6. *If $k \neq -h^\vee$, the quadruple $(N(k,0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra.*

The associative algebra $A(N(k,0))$ is identified in next proposition:

Proposition 2.7 ([12, Theorem 3.1.1]). *The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathfrak{g})$. The isomorphism is given by $F : A(N(k,0)) \to U(\mathfrak{g})$*

$$F([x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}]) = (-1)^{n_1+\cdots+n_m} x_m \cdots x_1,$$

for any $x_1, \dots, x_m \in \mathfrak{g}$ *and any* $n_1, \dots, n_m \in \mathbb{Z}_+$.

Since every $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is also an ideal in the vertex operator algebra $N(k,0)$, it follows that $L(k,0)$ is a vertex operator algebra, for any $k \neq -h^\vee$. The associative algebra $A(L(k,0))$ is identified in the next proposition, in the case when the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by one singular vector.

PROPOSITION 2.8. *Assume that the maximal* $\hat{\mathfrak{g}}$*-submodule of* $N(k,0)$ *is generated by a singular vector, i.e.* $J(k,0) = U(\hat{\mathfrak{g}})v$. *Then*

$$A(L(k,0)) \cong \frac{U(\mathfrak{g})}{I},$$

where I *is the two-sided ideal of* $U(\mathfrak{g})$ *generated by* $v' = F([v])$.
Let U *be a* $\mathfrak{g}$*-module. Then* U *is an* $A(L(k,0))$*-module if and only if* $IU = 0$.

2.6. *Modules for associative algebra* $A(L(k,0))$. In this subsection we present the method from [2, 4, 24] for classification of irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations. We assume that the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by a singular vector v.

Denote by $_L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X, f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$. Let R be a $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector $v' = F([v])$ under the adjoint action. Clearly, R is an irreducible highest weight $U(\mathfrak{g})$-module with the highest weight vector v'. Let R_0 be the zero-weight subspace of R. The next proposition follows from [2, Proposition 2.4.1], [4, Lemma 3.4.3]:

PROPOSITION 2.9. *Let* $V(\mu)$ *be an irreducible highest weight* $U(\mathfrak{g})$*-module with the highest weight vector* v_μ, *for* $\mu \in \mathfrak{h}^*$. *The following statements are equivalent:*
(1) $V(\mu)$ *is an* $A(L(k,0))$*-module,*
(2) $RV(\mu) = 0$,
(3) $R_0 v_\mu = 0$.

Let $r \in R_0$. Clearly there exists the unique polynomial $p_r \in S(\mathfrak{h})$ such that

$$rv_\mu = p_r(\mu)v_\mu.$$

Set $\mathcal{P}_0 = \{\, p_r \mid r \in R_0\}$. We have:

COROLLARY 2.10. *There is one-to-one correspondence between*
(1) *irreducible* $A(L(k,0))$*-modules from the category* $\mathcal{O}$,
(2) *weights* $\mu \in \mathfrak{h}^*$ *such that* $p(\mu) = 0$ *for all* $p \in \mathcal{P}_0$.

3. Simple Lie algebra of type A_l

Let $\Delta = \{\epsilon_i - \epsilon_j \mid i,j = 1,\dots,l+1,\ i \neq j\}$ be the root system of type A_l. Fix the set of positive roots $\Delta_+ = \{\epsilon_i - \epsilon_j \mid i < j\}$. Then the simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\dots$, $\alpha_l = \epsilon_l - \epsilon_{l+1}$. The highest root is $\theta = \epsilon_1 - \epsilon_{l+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_l$.

Let $\mathfrak{g}$ be the simple Lie algebra associated to the root system of type A_l. Let e_i, f_i, h_i, $i = 1,\dots,l$ be the Chevalley generators of $\mathfrak{g}$. Fix the root vectors:

$$
\begin{aligned}
e_{\epsilon_i-\epsilon_j} &= [e_{j-1},[e_{j-2},[\dots[e_{i+1},e_i]\dots]]], \qquad i<j,\\
f_{\epsilon_i-\epsilon_j} &= [f_i,[f_{i+1},[\dots[f_{j-2},f_{j-1}]\dots]]], \qquad i<j.
\end{aligned}
$$

Denote by $h_\alpha = \alpha^\vee = [e_\alpha, f_\alpha]$ coroots, for any positive root $\alpha \in \Delta_+$. It is clear that $h_{\alpha_i} = h_i$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding triangular decomposition of $\mathfrak{g}$. Denote by $\omega_1,\dots,\omega_l \in \mathfrak{h}^*$ the fundamental weights of $\mathfrak{g}$, defined by $\omega_i(\alpha_j^\vee) = \delta_{ij}$ for all $i,j = 1,\dots,l$.

4. Vertex operator algebra $L(-\frac{1}{2}(l+1),0)$ associated to affine Lie algebra of type $A_l^{(1)}$, for even l

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to simple Lie algebra $\mathfrak{g}$ of type A_l. We want to show that the maximal $\hat{\mathfrak{g}}$-submodule of $N(-\frac{1}{2}(l+1),0)$ is generated by a singular vector, for even l. We need two lemmas to prove that.

Denote by λ the weight $-\frac{1}{2}(l+1)\Lambda_0$. Then $N(-\frac{1}{2}(l+1),0)$ is a quotient of $M(\lambda)$ and $L(-\frac{1}{2}(l+1),0) \cong L(\lambda)$.

Lemma 4.1. *The weight $\lambda = -\frac{1}{2}(l+1)\Lambda_0$ is admissible and*

$$\hat{\Pi}_\lambda^\vee = \{(2\delta-\theta)^\vee, \alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_l^\vee\}.$$

Proof. Clearly

$$
\begin{aligned}
&\langle \lambda+\rho, \alpha_i^\vee\rangle = 1 \quad \text{for } i = 1,\dots,l,\\
&\langle \lambda+\rho, \alpha_0^\vee\rangle = -\frac{1}{2}(l-1),
\end{aligned}
$$

which implies

$$\langle \lambda+\rho, (2\delta-\theta)^\vee\rangle = \langle \lambda+\rho, 2\alpha_0^\vee + \alpha_1^\vee + \ldots + \alpha_l^\vee\rangle = 1.$$

The claim of lemma now follows easily. ∎

Lemma 4.2. *Vector*

$$
\begin{aligned}
v \;=\; & \sum_{i=1}^{l} \frac{l-2i+1}{l+1} h_i(-1)e_\theta(-1)\mathbf{1} - \sum_{i=1}^{l-1} e_{\epsilon_1-\epsilon_{i+1}}(-1)e_{\epsilon_{i+1}-\epsilon_{l+1}}(-1)\mathbf{1}\\
& -\frac{1}{2}(l-1)e_\theta(-2)\mathbf{1}
\end{aligned}
$$

is a singular vector in $N(-\frac{1}{2}(l+1),0)$.

PROOF. It can be directly verified that

$$e_j(0).v=0,\ j=1,\dots,l,$$
$$f_\theta(1).v=0.$$

$\square$

THEOREM 4.3. *The maximal* $\hat{\mathfrak{g}}$*-submodule of* $N(-\frac{1}{2}(l+1),0)$ *is* $J(-\frac{1}{2}(l+1),0)=U(\hat{\mathfrak{g}})v$*, where*

$$v = \sum_{i=1}^{l}\frac{l-2i+1}{l+1}h_i(-1)e_\theta(-1)\mathbf{1}-\sum_{i=1}^{l-1}e_{\epsilon_1-\epsilon_{i+1}}(-1)e_{\epsilon_{i+1}-\epsilon_{l+1}}(-1)\mathbf{1} -\frac{1}{2}(l-1)e_\theta(-2)\mathbf{1}.$$

PROOF. It follows from Theorem 2.4 and Lemma 4.1 that the maximal submodule of the Verma module $M(\lambda)$ is generated by $l+1$ singular vectors with weights

$$r_{2\delta-\theta}.\lambda, r_{\alpha_1}.\lambda,\dots,r_{\alpha_l}.\lambda.$$

It follows from Lemma 4.2 that v is a singular vector of weight $\lambda-2\delta+\theta=r_{2\delta-\theta}.\lambda$. Other singular vectors have weights

$$r_{\alpha_i}.\lambda=\lambda-\langle\lambda+\rho,\alpha_i^\vee\rangle\alpha_i=\lambda-\alpha_i,\ i=1,\dots,l,$$

so the images of these vectors under the projection of $M(\lambda)$ onto $N(-\frac{1}{2}(l+1),0)$ are 0. Therefore, the maximal submodule of $N(-\frac{1}{2}(l+1),0)$ is generated by the vector v, i.e. $J(-\frac{1}{2}(l+1),0)=U(\hat{\mathfrak{g}})v$. $\square$

It follows that

$$L(-\frac{1}{2}(l+1),0)\cong\frac{N(-\frac{1}{2}(l+1),0)}{U(\hat{\mathfrak{g}})v}.$$

Using Theorem 4.3 and Proposition 2.8 we can determine the associative algebra $A(L(-\frac{1}{2}(l+1),0))$:

PROPOSITION 4.4. *The associative algebra* $A(L(-\frac{1}{2}(l+1),0))$ *is isomorphic to the algebra* $U(\mathfrak{g})/I$*, where* I *is the two-sided ideal of* $U(\mathfrak{g})$ *generated by*

$$v'=\sum_{i=1}^{l}\frac{l-2i+1}{l+1}h_ie_\theta-\sum_{i=1}^{l-1}e_{\epsilon_{i+1}-\epsilon_{l+1}}e_{\epsilon_1-\epsilon_{i+1}}+\frac{1}{2}(l-1)e_\theta.$$

PROOF. The maximal submodule $J(-\frac{1}{2}(l+1),0)$ od $N(-\frac{1}{2}(l+1),0)$ is generated by the singular vector v. It follows from Proposition 2.8 that

$$A(L(-\frac{1}{2}(l+1),0))\cong\frac{U(\mathfrak{g})}{I},$$

where I is the two-sided ideal in $U(\mathfrak{g})$ generated by $v' = F([v])$. Proposition 2.7 now implies that

$$\begin{aligned} v' = F([v]) &= \sum_{i=1}^{l} \frac{l-2i+1}{l+1} e_\theta h_i - \sum_{i=1}^{l-1} e_{\epsilon_{i+1}-\epsilon_{l+1}} e_{\epsilon_1-\epsilon_{i+1}} + \frac{1}{2}(l-1)e_\theta \\ &= \sum_{i=1}^{l} \frac{l-2i+1}{l+1} h_i e_\theta - \sum_{i=1}^{l-1} e_{\epsilon_{i+1}-\epsilon_{l+1}} e_{\epsilon_1-\epsilon_{i+1}} + \frac{1}{2}(l-1)e_\theta, \end{aligned}$$

which implies the claim of proposition. □

5. Classification of irreducible weak $L(-\frac{1}{2}(l+1),0)$-modules from category $\mathcal{O}$

In this section we classify irreducible weak $L(-\frac{1}{2}(l+1),0)$-modules that are in category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-modules, using methods from [2, 4, 24] presented in Subsection 2.6. First, we determine a basis for the vector space $\mathcal{P}_0$ defined in that subsection. Recall that ${}_L$ denotes the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X, f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$.

Lemma 5.1. *Let*

$$p_i(h) = h_i \left(\sum_{j=1}^{i-1} \frac{-2j}{l+1} h_j + \frac{l-2i+1}{l+1} h_i + \sum_{j=i+1}^{l} \frac{2l-2j+2}{l+1} h_j + \frac{1}{2}(l+1) - i \right),$$

for $i = 1, \ldots, l$. *Then* $p_1, \ldots, p_l \in \mathcal{P}_0$.

Proof. We claim that

$$(-1)^i (f_i f_{i-1} \ldots f_1 f_{i+1} \ldots f_l)_L\, v' \in p_i(h) + U(\mathfrak{g})\mathfrak{n}_+, \text{ for } i = 1, \ldots, l. \tag{5.1}$$

One can easily verify that for $i \in \{1, \ldots, n\}$ the following relations hold:

$$\begin{aligned} &(f_i f_{i-1} \ldots f_1 f_{i+1} \ldots f_l)_L\, e_\theta = (-1)^i h_i, \\ &(f_i f_{i-1} \ldots f_1)_L\, e_{\epsilon_1-\epsilon_{j+1}} = (-1)^i f_{\epsilon_{j+1}-\epsilon_{i+1}}, \quad j < i, \\ &(f_{i+1} \ldots f_l)_L\, e_{\epsilon_{j+1}-\epsilon_{l+1}} = e_{\epsilon_{j+1}-\epsilon_{i+1}}, \quad j < i, \\ &(f_{i-1} f_{i-2} \ldots f_1)_L\, e_{\epsilon_1-\epsilon_{j+1}} = (-1)^{i-1} f_{\epsilon_{j+1}-\epsilon_i}, \quad j < i-1, \\ &(f_i f_{i+1} \ldots f_l)_L\, e_{\epsilon_{j+1}-\epsilon_{l+1}} = e_{\epsilon_{j+1}-\epsilon_i}, \quad j < i-1. \end{aligned}$$

Using Proposition 4.4, we obtain

$$\begin{aligned}(f_i f_{i-1}\dots f_1 f_{i+1}\dots f_l)_L\, v' \in (-1)^i \sum_{j=1}^{l} \frac{l-2j+1}{l+1} h_j h_i \\ - (-1)^i \sum_{j=1}^{i-2} (h_i h_j + e_{\epsilon_{j+1}-\epsilon_{i+1}} f_{\epsilon_{j+1}-\epsilon_{i+1}} - e_{\epsilon_{j+1}-\epsilon_i} f_{\epsilon_{j+1}-\epsilon_i}) \\ - (-1)^i (h_i h_{i-1} + e_i f_i) \\ - (-1)^{i+1} \sum_{j=i+1}^{l} h_j h_i + \frac{1}{2}(l-1)(-1)^i h_i + U(\mathfrak{g})\mathfrak{n}_+.\end{aligned}$$

Since

$$\begin{aligned}e_{\epsilon_{j+1}-\epsilon_{i+1}} f_{\epsilon_{j+1}-\epsilon_{i+1}} - e_{\epsilon_{j+1}-\epsilon_i} f_{\epsilon_{j+1}-\epsilon_i} \\ \in h_{\epsilon_{j+1}-\epsilon_{i+1}} - h_{\epsilon_{j+1}-\epsilon_i} + U(\mathfrak{g})\mathfrak{n}_+ = h_i + U(\mathfrak{g})\mathfrak{n}_+,\end{aligned}$$

it follows that

$$\begin{aligned}(-1)^i (f_i f_{i-1}\dots f_1 f_{i+1}\dots f_l)_L\, v' \in \sum_{j=1}^{l} \frac{l-2j+1}{l+1} h_j h_i \\ - \sum_{j=1}^{i-1} h_j h_i + \sum_{j=i+1}^{l} h_j h_i - (i-1) h_i + \frac{1}{2}(l-1) h_i + U(\mathfrak{g})\mathfrak{n}_+,\end{aligned}$$

which implies relation (5.1). ∎

Lemma 5.2.

$$\mathcal{P}_0 = span_{\mathbb{C}}\{p_1,\dots,p_l\}.$$

Proof. Lemma 5.1 implies that $p_1,\dots,p_l$ are linearly independent polynomials in the set $\mathcal{P}_0$. It follows from the definition of set $\mathcal{P}_0$ that $\dim \mathcal{P}_0 = \dim R_0$, where R is the highest weight $U(\mathfrak{g})$-module with highest weight θ, and R_0 the zero-weight subspace of R. Since R is isomorphic to the adjoint module for $\mathfrak{g}$, it follows that $\dim R_0 = l$. Thus, polynomials $p_1,\dots,p_l$ form a basis for $\mathcal{P}_0$. ∎

Proposition 5.3. *For every subset $S = \{i_1,\dots,i_k\} \subseteq \{1,2,\dots,l\}$, $i_1 < \dots < i_k$, we define weights*

$$\mu_S = \sum_{j=1}^{k} \left(\sum_{s=j+1}^{k} (-1)^{s-j} i_s + \sum_{s=1}^{j-1} (-1)^{j-s+1} i_s + (-1)^{k-j+1} \frac{l+1}{2} \right) \omega_{i_j},$$

where $\omega_1,\dots,\omega_l$ are fundamental weights for $\mathfrak{g}$. Then the set

$$\{V(\mu_S) \mid S \subseteq \{1,2,\dots,l\}\}$$

provides the complete list of irreducible $A(L(-\frac{1}{2}(l+1),0))$-modules from the category $\mathcal{O}$.

PROOF. It follows from Corollary 2.10 and Lemma 5.2 that highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(L(-\frac{1}{2}(l+1),0))$-modules $V(\mu)$ are in one-to-one correspondence with solutions of the system of polynomial equations

$$h_i\Big(-\sum_{j=1}^{i-1} jh_j + \frac{l-2i+1}{2}h_i + \sum_{j=i+1}^{l}(l-j+1)h_j + \frac{1}{4}(l+1)^2 - \frac{1}{2}i(l+1)\Big) = 0, \tag{5.2}$$

for $i = 1,\ldots,l$.

Let $i,j \in \{1,2,\ldots,l\}$, $i<j$. If we multiply the i-th equation of system (5.3) by h_j, and the j-th equation by h_i and then subtract these equations, we obtain

$$h_ih_j\,(h_i + 2h_{i+1} + 2h_{i+2} + \ldots + 2h_{j-1} + h_j + j - i) = 0. \tag{5.3}$$

Let $S = \{i_1,\ldots,i_k\}$, $i_1 < \ldots < i_k$ be the subset of $\{1,2,\ldots,l\}$ such that $h_i = 0$ for $i \notin S$ and $h_i \neq 0$ for $i \in S$. From relation (5.3) and relation (5.3) for $i = i_k$, we get the system

$$\begin{aligned}
&h_{i_1} + h_{i_2} + i_2 - i_1 = 0\\
&h_{i_2} + h_{i_3} + i_3 - i_2 = 0\\
&\qquad\vdots\\
&h_{i_{k-1}} + h_{i_k} + i_k - i_{k-1} = 0\\
&-i_1h_{i_1} - i_2h_{i_2} - \ldots - i_{k-1}h_{i_{k-1}}\\
&\qquad + \frac{l-2i_k+1}{2}h_{i_k} + \frac{1}{4}(l+1)^2 - \frac{1}{2}i_k(l+1) = 0.
\end{aligned} \tag{5.4}$$

If we multiply the first equation of system (5.4) by i_1, the second equation by $i_2 - i_1$, the third equation by $i_3 - i_2 + i_1$, $\ldots$, the $(k-1)$-th equation by $i_{k-1} - i_{k-2} + \ldots + (-1)^k i_1$ and then sum these equations and the k-th equation, we obtain:

$$\Big(\frac{l+1}{2} - i_k + i_{k-1} - \ldots + (-1)^k i_1\Big) \times \Big(h_{i_k} + \frac{l+1}{2} - i_{k-1} + i_{k-2} - \ldots + (-1)^{k-1} i_1\Big) = 0.$$

Since l is even, we have

$$\frac{l+1}{2} - i_k + i_{k-1} - \ldots + (-1)^k i_1 \neq 0,$$

which implies

$$h_{i_k} = i_{k-1} - i_{k-2} + \ldots + (-1)^k i_1 - \frac{l+1}{2}.$$

Using the first $k-1$ equations of system (5.4) one can easily obtain that

$$h_{i_j} = \sum_{s=j+1}^{k} (-1)^{s-j} i_s + \sum_{s=1}^{j-1} (-1)^{j-s+1} i_s + (-1)^{k-j+1} \frac{l+1}{2}, \ j = 1, \ldots, k$$

is a solution of this system. Thus, $V(\mu_S)$ is an irreducible $A(L(-\frac{1}{2}(l+1), 0))$-module which implies the claim of proposition. ∎

It follows from Zhu's theory that:

THEOREM 5.4. *The set*

$$\{L(-\frac{1}{2}(l+1), \mu_S) \mid S \subseteq \{1, 2, \ldots, l\}\}$$

provides the complete list of irreducible weak $L(-\frac{1}{2}(l+1), 0)$-modules from the category $\mathcal{O}$.

Theorem 5.4 implies that there are 2^l irreducible weak $L(-\frac{1}{2}(l+1), 0)$-modules from category $\mathcal{O}$. The weight μ_S is a dominant integral weight for $\mathfrak{g}$ if and only if $S = \emptyset$, i.e. if and only if $\mu_S = 0$. It follows that

COROLLARY 5.5. *$L(-\frac{1}{2}(l+1), 0)$ is the only irreducible $L(-\frac{1}{2}(l+1), 0)$-module.*

6. COMPLETE REDUCIBILITY OF WEAK $L(-\frac{1}{2}(l+1), 0)$-MODULES FROM CATEGORY $\mathcal{O}$

In this section we show that every weak $L(-\frac{1}{2}(l+1), 0)$-module from category $\mathcal{O}$ is completely reducible. We introduce the notation $\lambda_S = -\frac{1}{2}(l+1)\Lambda_0 + \mu_S$, for every $S \subseteq \{1, 2, \ldots, l\}$. The following lemma is crucial for proving complete reducibility.

LEMMA 6.1. *The weight $\lambda_S \in \hat{\mathfrak{h}}^*$ is admissible, for every $S \subseteq \{1, 2, \ldots, l\}$.*

PROOF. We have to show

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+ \text{ for all } \tilde{\alpha} \in \hat{\Delta}_+^{\mathrm{re}}, \tag{6.1}$$

$$\mathbb{Q}\hat{\Delta}_{\lambda_S}^{\vee \mathrm{re}} = \mathbb{Q}\hat{\Pi}^\vee. \tag{6.2}$$

First, let us prove relation (6.1). Any positive real root $\tilde{\alpha} \in \hat{\Delta}_+^{\mathrm{re}}$ of $\hat{\mathfrak{g}}$ is of the form $\tilde{\alpha} = \alpha + m\delta$, for $m > 0$ and $\alpha \in \Delta$ or $m = 0$ and $\alpha \in \Delta_+$. Positive roots of $\mathfrak{g}$ are $\epsilon_i - \epsilon_j$, $i < j$, and negative roots are $-(\epsilon_i - \epsilon_j)$, $i < j$.

Clearly, $(\bar{\rho}, \epsilon_i - \epsilon_j) = j - i$. Let $s, t \in \{1, \ldots, k\}$ be the indices such that $S \cap \{i, i+1, \ldots, j-1\} = \{i_s, \ldots, i_t\}$. Clearly, $i_s \geq i$ and $i_t \leq j-1$. Furthermore, $(\mu_S, \epsilon_i - \epsilon_j) = h_{i_s} + \ldots + h_{i_t}$.

We obtain

$$\langle \lambda_S+\rho,\tilde{\alpha}^\vee\rangle = \frac{1}{2}m(l+1)+(\bar{\rho},\alpha)+(\mu_S,\alpha), \tag{6.3}$$

where $\bar{\rho}$ is the sum of fundamental weights of $\mathfrak{g}$. Let $S=\{i_1,\ldots,i_k\}\subseteq\{1,2,\ldots,l\}$, $i_1<\ldots<i_k$. Proposition 5.3 implies that $\mu_S=\sum_{j=1}^k h_{i_j}\omega_{i_j}$, where

$$h_{i_j}=\sum_{s=j+1}^k(-1)^{s-j}i_s+\sum_{s=1}^{j-1}(-1)^{j-s+1}i_s+(-1)^{k-j+1}\frac{l+1}{2},\ j=1,\ldots,k.$$

First consider the case $\alpha=\epsilon_i-\epsilon_j$, $i<j$ and $m\geq 0$.

If $t-s+1$ is even, then using relations from system (5.4) we get

$$(\mu_S,\epsilon_i-\epsilon_j)=-(i_{s+1}-i_s)-\ldots-(i_t-i_{t-1})\geq -(i_t-i_s),$$

and relation (6.3) implies

$$\begin{aligned}\langle \lambda_S+\rho,\tilde{\alpha}^\vee\rangle &\geq (\bar{\rho},\epsilon_i-\epsilon_j)+(\mu_S,\epsilon_i-\epsilon_j)\geq (j-i)-(i_t-i_s)\\ &=(j-i_t)+(i_s-i)>0.\end{aligned}$$

Suppose now that $t-s+1$ is odd. Then $(\mu_S,\epsilon_i-\epsilon_j)\notin\mathbb{Z}$, and if $m=0$, then $\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle\notin\mathbb{Z}$. Let $m\geq 1$. Then

$$\begin{aligned}(\mu_S,\epsilon_i-\epsilon_j)&=h_{i_s}+\ldots\ +h_{i_{t-1}}+h_{i_t}\\ &=-(i_{s+1}-i_s)-\ldots-(i_{t-1}-i_{t-2})+h_{i_t}\\ &\geq -(i_{t-1}-i_s)+h_{i_t}.\end{aligned}$$

We have

$$h_{i_t}=\sum_{s=t+1}^k(-1)^{s-t}i_s+\sum_{s=1}^{t-1}(-1)^{t-s+1}i_s+(-1)^{k-t+1}\frac{l+1}{2}.$$

Clearly, $\sum_{s=1}^{t-1}(-1)^{t-s+1}i_s\geq 0$. If $k-t$ is even, then

$$h_{i_t}\geq (i_{t+2}-i_{t+1})+\ldots+(i_k-i_{k-1})-\frac{1}{2}(l+1)\geq -\frac{1}{2}(l+1),$$

which implies

$$(\mu_S,\epsilon_i-\epsilon_j)\geq -(i_{t-1}-i_s)-\frac{1}{2}(l+1).$$

It follows that

$$\begin{aligned}\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle&\geq\frac{1}{2}(l+1)+(\bar{\rho},\epsilon_i-\epsilon_j)+(\mu_S,\epsilon_i-\epsilon_j)\\ &\geq\frac{1}{2}(l+1)+(j-i)-(i_{t-1}-i_s)-\frac{1}{2}(l+1)\\ &=(j-i_{t-1})+(i_s-i)>0.\end{aligned}$$

If $k-t$ is odd, then

$$h_{i_t} \geq (i_{t+2}-i_{t+1}) + \ldots + (i_{k-1}-i_{k-2}) - i_k + \frac{1}{2}(l+1) \geq \frac{1}{2}(l+1) - i_k,$$

which implies

$$(\mu_S, \epsilon_i - \epsilon_j) \geq -(i_{t-1}-i_s) + \frac{1}{2}(l+1) - i_k.$$

We obtain

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geq \frac{1}{2}(l+1) + (\bar{\rho}, \epsilon_i - \epsilon_j) + (\mu_S, \epsilon_i - \epsilon_j) \\ &\geq \frac{1}{2}(l+1) + (j-i) - (i_{t-1}-i_s) + \frac{1}{2}(l+1) - i_k \\ &= (l-i_k) + (j-i_{t-1}) + (i_s - i) + 1 > 0.\end{aligned}$$

Thus, we have proved that, if $\alpha = \epsilon_i - \epsilon_j$, $i < j$ and $m \geq 0$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$.

Now, let us consider the case $\alpha = -(\epsilon_i - \epsilon_j)$, $i < j$ and $m \geq 1$.

Then

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle = \frac{1}{2}m(l+1) - (\bar{\rho}, \epsilon_i - \epsilon_j) - (\mu_S, \epsilon_i - \epsilon_j).$$

If $t-s+1$ is even, then $(\mu_S, \epsilon_i - \epsilon_j)$ is an integer and $(\mu_S, \epsilon_i - \epsilon_j) \leq 0$, so if m is odd, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin \mathbb{Z}$. If m is even, then $m \geq 2$, and we get

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \geq (l+1) - (j-i) = (l-j) + i + 1 > 0.$$

If $t-s+1$ is odd, then

$$(\mu_S, \epsilon_i - \epsilon_j) = h_{i_s} + \ldots + h_{i_{t-1}} + h_{i_t} = -(i_{s+1}-i_s) - \ldots - (i_{t-1}-i_{t-2}) + h_{i_t},$$

which implies

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \geq \frac{1}{2}(l+1) - (j-i) + (i_{s+1}-i_s) + \ldots + (i_{t-1}-i_{t-2}) - h_{i_t}.$$

If $k-t$ is even, then

$$h_{i_t} = i_{t-1} - i_{t-2} + \ldots + (-1)^{t-1} i_2 + (-1)^t i_1 + i_k - i_{k-1} + \ldots + i_{t+2} - i_{t+1} - \frac{1}{2}(l+1),$$

which implies

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \geq{}& \frac{1}{2}(l+1) - (j-i) + (i_{s-1} - i_{s-2} + \ldots + (-1)^{t+1} i_1) \\ &- i_k + (i_{k-1}-i_{k-2}) + \ldots + (i_{t+3}-i_{t+2}) + i_{t+1} + \frac{1}{2}(l+1).\end{aligned}$$

Clearly

$$i_{s-1} - i_{s-2} + \ldots + (-1)^{t+1} i_1 \geq 0,$$

from which we get

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \geq (l-i_k) + i + (i_{t+1}-j) + (i_{k-1}-i_{k-2}) + \ldots + (i_{t+3}-i_{t+2}) + 1 > 0.$$

If $k-t$ is odd, then

$$h_{i_t} = i_{t-1} - i_{t-2} + \ldots + (-1)^t i_1 - i_k + i_{k-1} - \ldots + i_{t+2} - i_{t+1} + \frac{1}{2}(l+1),$$

which implies

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geq \frac{1}{2}(l+1) - (j-i) + (i_{s-1} - i_{s-2} + \ldots + (-1)^{t+1} i_1) \\ &\quad + i_k - i_{k-1} + \ldots + i_{t+3} - i_{t+2} + i_{t+1} - \frac{1}{2}(l+1) \\ &\geq (i_{t+1} - j) + i + (i_k - i_{k-1}) + \ldots + (i_{t+3} - i_{t+2}) > 0.\end{aligned}$$

We have proved that, if $\alpha = -(\epsilon_i - \epsilon_j)$, $i<j$ and $m \geq 1$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$.

Thus, we have verified the relation (6.1). Moreover, one can easily check that coroots

$$\begin{aligned}&(\delta - \alpha_{i_j})^\vee, \ j = 1, \ldots, k, \\ &\alpha_{i_j}^\vee + \alpha_{i_j+1}^\vee + \ldots + \alpha_{i_{j+1}}^\vee, \ j = 1, \ldots, k-1, \\ &\alpha_i^\vee, \ i \notin S, \ i \in \{1, 2, \ldots, l\}\end{aligned}$$

are elements of the set $\hat{\Delta}_{\lambda_S}^{\vee\mathrm{re}}$ which implies $\mathbb{Q}\hat{\Delta}_{\lambda_S}^{\vee\mathrm{re}} = \mathbb{Q}\hat{\Pi}^\vee$, and relation (6.2) is also proved. $\square$

THEOREM 6.2. *Let M be a weak $L(-\frac{1}{2}(l+1), 0)$-module from the category $\mathcal{O}$. Then M is completely reducible.*

PROOF. Let $L(\lambda)$ be some irreducible subquotient of M. Then $L(\lambda)$ is an irreducible weak $L(-\frac{1}{2}(l+1), 0)$-module, and Theorem 5.4 implies that there exists $S \subseteq \{1, 2, \ldots, l\}$ such that $\lambda = -\frac{1}{2}(l+1)\Lambda_0 + \mu_S$. It follows from Lemma 6.1 that such λ is admissible. Theorem 2.5 now implies that M is completely reducible. $\square$

REMARK 6.3. Using the fact that $L(-\frac{1}{2}(l+1), 0)$ is the only irreducible $L(-\frac{1}{2}(l+1), 0)$-module (Corollary 5.5), one can show (as in [25, Lemma 26]) that every $L(-\frac{1}{2}(l+1), 0)$-module is in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-module. It follows now from Theorem 6.2 that every $L(-\frac{1}{2}(l+1), 0)$-module is completely reducible, which implies that it is a direct sum of copies of $L(-\frac{1}{2}(l+1), 0)$.

ACKNOWLEDGEMENTS.

This paper (together with [25]) is a part of author's Ph.D. Thesis. The author would like to thank Professor Dražen Adamović and Professor Mirko Primc for their long time support and for numerous helpful advice on this and related work.

References

[1] D. Adamović, *Some rational vertex algebras,* Glas. Mat. Ser. III **(29)(49)** (1994), 25-40.

[2] D. Adamović, Representations of vertex algebras associated to symplectic affine Lie algebra at half-integer levels (Croatian), Ph.D. Thesis, University of Zagreb, 1996.

[3] D. Adamović, *A construction of admissible $A_1^{(1)}$-modules of level $-\frac{4}{3}$,* J. Pure Appl. Algebra **196** (2005), 119-134.

[4] D. Adamović and A. Milas, *Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$,* Math. Res. Lett. **2** (1995), 563-575.

[5] D. Adamović and O. Perše, *Representations of certain non-rational vertex operator algebras of affine type,* J. Algebra **319** (2008), 2434-2450.

[6] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster,* Proc. Natl. Acad. Sci. USA **83** (1986), 3068-3071.

[7] C. Dong, H. Li and G. Mason, *Vertex operator algebras associated to admissible representations of* $\widehat{\mathrm{sl}}_2$, Comm. Math. Phys. **184** (1997), 65-93.

[8] B. Feigin and F. Malikov, *Fusion algebra at a rational level and cohomology of nilpotent subalgebras of* $\widehat{\mathrm{sl}}(2)$, Lett. Math. Phys. **31** (1994), 315-325.

[9] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2001.

[10] I. Frenkel, Y.-Z. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules,* Mem. Amer. Math. Soc. **104**, 1993.

[11] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[12] I. Frenkel and Y.-C. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras,* Duke Math. J. **66** (1992), 123-168.

[13] A. Ch. Ganchev, V. B. Petkova and G. M. T. Watts, *A note on decoupling conditions for generic level $\widehat{\mathrm{sl}}(3)_k$ and fusion rules,* Nuclear Phys. B **571** (2000), 457-478.

[14] V. G. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[15] V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, AMS, Vol. 10 (1998).

[16] V. Kac and M. Wakimoto, *Modular invariant representations of infinite dimensional Lie algebras and superalgebras,* Proc. Natl. Acad. Sci. USA **85** (1988), 4956-4960.

[17] V. Kac and M. Wakimoto, *Classification of modular invariant representations of affine algebras,* in: Infinite Dimensional Lie algebras and groups, Advanced Series in Math. Phys. **7**, World Scientific, Teaneck NJ, 1989.

[18] V. Kac and W. Wang, *Vertex operator superalgebras and their representations,* in: Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992), 161–191, Contemp. Math. **175**, AMS, Providence RI, 1994.

[19] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras, I, II,* J. Amer. Math. Soc. **6** (1993), 905-947, 949-1011.

[20] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras, III, IV,* J. Amer. Math. Soc. **7** (1994), 335-381, 383-453.

[21] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, Progress in Math., Vol. 227, Birkhauser, Boston, 2004.

[22] H.-S. Li, *Local systems of vertex operators, vertex superalgebras and modules,* J. Pure Appl. Algebra **109** (1996), 143-195.

[23] F. Malikov, B. Feigin, and D. Fuchs, *Singular vectors in Verma modules over Kac-Moody algebras (Russian),* Funktsional. Anal. i Prilozhen. **20** (1986), 25-37.

[24] A. Meurman and M. Primc, *Annihilating fields of standard modules of* $\tilde{sl(2,\mathbb{C})}$ *and combinatorial identities,* Mem. Amer. Math. Soc. **137**, AMS, Providence RI, 1999.
[25] O. Perše, *Vertex operator algebras associated to type B affine Lie algebras on admissible half-integer levels,* J. Algebra **307** (2007), 215-248.
[26] O. Perše, *Vertex operator algebra analogue of embedding of* B_4 *into* F_4*,* J. Pure Appl. Algebra **211** (2007), 702-720.
[27] M. Wakimoto, Lectures on infinite-dimensional Lie algebra, World Scientific, River Edge NJ, 2001.
[28] Y.-C. Zhu, *Modular invariance of characters of vertex operator algebras,* J. Amer. Math. Soc. **9** (1996), 237-302.

O. Perše
Department of Mathematics
University of Zagreb
Bijenička 30
10 000 Zagreb
Croatia
E-mail: perse@math.hr

Received: 9.6.2007.