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VORONOVSKAJA-TYPE THEOREM FOR CERTAIN GBS OPERATORS

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Abstract. In this paper we will demonstrate a Voronovskaja-type theorem and approximation theorem for GBS operator associated to a linear positive operator.

1. Introduction

In this section, we recall some notions and results which we will use in this article.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$ (see [7,28]).

In 1932, E. Voronovskaja in the paper [31], proved the result contained in the following theorem.

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THEOREM 1.1. *Let $f \in C([0,1])$ be a two times derivable function in the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m\,[(B_m f)(x) - f(x)] = \frac{x(1-x)}{2}\, f''(x) \tag{1.3}$$

holds.

Let $p \in \mathbb{N}_0$. For $m \in \mathbb{N}$, F. Schurer (see [26]) introduced and studied in 1962 the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, named Bernstein-Schurer operators, defined for any function $f \in C([0,1+p])$ by

$$\left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.4}$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x) \tag{1.5}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m+p\}$.

For $m \in \mathbb{N}$, let the operators $M_n : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.6}$$

for any $x \in [0,1]$.
These operators were introduced in 1967 by J. L. Durrmeyer in [11] and were studied in 1981 by M. M. Derriennic in [9].

For $m \in \mathbb{N}$, let the operators $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt, \tag{1.7}$$

for any $x \in [0,1]$.

The operators K_m, $m \in \mathbb{N}$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [14]).

For the following construction see [18].

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if } \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if } \beta \in \mathbb{Z}. \end{cases} \tag{1.8}$$

For the real number β, we have that

$$m + \beta \geq \gamma_\beta \tag{1.9}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\big\{1+\beta, \{\beta\}\big\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.10}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha - \beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.11}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.12}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.8)-(1.11), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.13}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$.

These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [27]. In [27], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [6] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.14}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \,|\, f : [0,\infty) \to \mathbb{R},\ f$ bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$ consider the operators $S_m : C_2\left([0,\infty)\right) \to C\left([0,\infty)\right)$ defined for any function $f \in C_2\left([0,\infty)\right)$ by

$$\left(S_m f\right)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.15}$$

for any $x \in [0,\infty)$, where $C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$.

The operators $(S_m)_{m\geq 1}$ are named Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [16].

They were intensively studied by J. Favard in 1944 in [12] and O. Szász in 1950 in [29].

Let for $m \in \mathbb{N}$ the operators $V_m : C_2([0,\infty)) \to C([0,\infty))$ be defined for any function $f \in C_2([0,\infty))$ by

$$(1.16)\qquad (V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right),$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [4].

W. Meyer-König and K. Zeller have introduced in [15] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [8], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(1.17)\qquad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are named the Meyer-König and Zeller operators.

Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In the paper [13], M. Ismail and C. P. May consider the operators $(R_m)_{m\geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(1.18)\qquad (R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right)$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following functions sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{ f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I \}$, $C(I) = \{ f \mid f : I \to \mathbb{R},\ f \text{ continuous on } I \}$ and $C_B(I) = B(I) \cap C(I)$.

If $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$(1.19)\qquad \omega(f;\delta) = \sup\left\{ |f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta \right\}.$$

Let $I, J \subset \mathbb{R}$ intervals, $E(I \times J)$, $F(I \times J)$ which are subsets of the set of real functions defined on $I \times J$ and $L : E(I \times J) \to F(I \times J)$ be a linear positive operator.

The operator $UL : E(I \times J) \to F(I \times J)$ defined for any function $f \in E(I \times J)$, any $(x, y) \in I \times J$ by

$$(ULf)(x,y) = (L(f(x,*) + f(\cdot\,,y) - f(\cdot\,,*))\,(x,y) \tag{1.20}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [3]).

If $f \in E(I \times J)$ and $(x, y) \in I \times J$, let the functions $f_x = f(x, *)$, $f^y = f(\cdot\,, y) : I \times J \to \mathbb{R}$, $f_x(s, t) = f(x, t)$, $f^y(s, t) = f(s, y)$ for any $(s, t) \in I \times J$. Then, we can consider that f_x, f^y are functions of real variable, $f_x : J \to \mathbb{R}$, $f_x(t) = f(x, t)$ for any $t \in J$ and $f^y : I \to \mathbb{R}$, $f^y(s) = f(s, y)$ for any $s \in I$.

2. Preliminaries

For the following construction and result see [19] and [21], where $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$.

Let I, J be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$.

Definition 2.1. For $m \in \mathbb{N}$ define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{2.1}$$

for any $f \in E(I)$ and $x \in J$.

Proposition 2.2. *The L_m, $m \in \mathbb{N}$ operators are linear and positive on $E(I \cap J)$.*

Definition 2.3. For $m \in \mathbb{N}$, let $L_m : E(I) \to F(J)$ be an operator defined in (2.1). For $i \in \mathbb{N}_0$, define $T^*_{m,i}$ by

$$\left(T^*_{m,i} L_m\right)(x) = m^i \left(L_m \psi^i_x\right)(x) = m^i \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(\psi^i_x), \tag{2.2}$$

for any $x \in I \cap J$, where for $x \in I$, $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ for any $t \in I$.

In what follows $s \in \mathbb{N}_0$ is even and we suppose that the operators $(L_m)_{m \geq 1}$ verify the conditions: there exists, the smallest α_s, $\alpha_{s+2} \in [0, \infty)$ so that

$$\lim_{m \to \infty} \frac{\left(T^*_{m,j} L_m\right)(x)}{m^{\alpha j}} = B_j\,(x) \in \mathbb{R}, \tag{2.3}$$

$x \in I \cap J$, $j \in \{s, s + 2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.4}$$

Theorem 2.4. *Let $f : I \to \mathbb{R}$ be a function.*

If $x \in I \cap J$ and f is a s times differentiable function in x with $f^{(s)}$ continuous in x, then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right] = 0. \tag{2.5}$$

Assume that f is s times differentiable function on I, with $f^{(s)}$ continuous on I and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K, so that for any $m \geq m(s)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} \leq k_j\,, \tag{2.6}$$

where $j \in \{s, s+2\}$. Then the convergence given in (2.5) *is uniform on K and*

$$\begin{aligned} m^{s-\alpha_s}&\left|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right| \\ &\leq \frac{1}{s!}\left(k_s + k_{s+2}\right)\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right), \end{aligned} \tag{2.7}$$

for any $x \in K$ and $m \geq m(s)$.

In the following we consider that

$$\left(T^*_{m,0}L_m\right)(x) = 1 \tag{2.8}$$

for any $x \in I \cap J$ and any $m \in \mathbb{N}$.

Corollary 2.5. *Let $f : I \to \mathbb{R}$ be a function. Assume that f is s times differentiable in $x \in I \cap J$ and $f^{(s)}$ is continuous in x. Then*

$$\lim_{m\to\infty}(L_m f)(x) = f(x) \tag{2.9}$$

if $s = 0$ and

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right] = \frac{f^{(s)}(x)}{s!}\,B_s(x) \tag{2.10}$$

if $s \geq 2$.

If f is s times differentiable function on $I \cap J$, with $f^{(s)}$ continuous on $I \cap J$ and (2.6) *takes place for an interval $K \subset I \cap J$ then the convergence from* (2.9) *and* (2.10) *is uniform on K.*

From (2.3) and (2.8) it results that

$$\alpha_0 = 0 \tag{2.11}$$

and then

$$k_0 = 1. \tag{2.12}$$

Corollary 2.6. *Let $f : I \to \mathbb{R}$ be a function.*
If $x \in I \cap J$ and f is continuous in x, then

$$\lim_{m\to\infty} (L_m f)(x) = f(x). \tag{2.13}$$

Assume that f is continuous on I and there exists an interval $K \subset I \cap J$ such that there exists $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K, so that for any $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2} L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{2.14}$$

Then the convergence given in (2.13) *is uniform on K and*

$$|(L_m f)(x) - f(x)| \leq (1 + k_2)\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{2.15}$$

for any $x \in K$ and $m \in \mathbb{N}$, $m \geq m(0)$.

Proof. It results from Theorem 2.4. □

For $m \in \mathbb{N}$, let the linear positive functionals $A^*_{m,k} : E(I \times I) \to \mathbb{R}$ with the property: if $(x, y) \in I \times I$, then

$$A^*_{m,k}(f) = A_{m,k}(F), \tag{2.16}$$

$$A^*_{m,k}(f_x) = A_{m,k}(f_x) \tag{2.17}$$

and

$$A^*_{m,k}(f^y) = A_{m,k}(f^y), \tag{2.18}$$

for any $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, any $f \in E(I \times I)$, where we note $F : I \to \mathbb{R}$, $F(t) = f(t, t)$ for any $t \in I$.

Now, with the help of $(L_m)_{m\geq 1}$ operators, we construct a sequence of bivariate operators. In the following let $\delta \in [0, 1]$.

Definition 2.7. The operators $L^\delta_m : E(I \times I) \to F(J \times J)$, $m \in \mathbb{N}$, defined for any function $f \in E(I \times I)$ and any $(x, y) \in J \times J$ by

$$\left(L^\delta_m f\right)(x, y) = \sum_{k=0}^{p_m} \left(\delta\varphi_{m,k}(x) + (1 - \delta)\varphi_{m,k}(y)\right) A^*_{m,k}(f) \tag{2.19}$$

are named the bivariate operators of δL-type.

Theorem 2.8. *The L^δ_m, $m \in \mathbb{N}$ operators are linear and positive on $E((I \times I) \cap (J \times J))$.*

Proof. The proof follows immediately. □

3. Main results

Theorem 3.1. *Let $f : I \times I \to \mathbb{R}$ be a function.*

If f is continuous in (x, x), $(y, y) \in (I \times I) \cap (J \times J)$, then

$$\lim_{m\to\infty} \left(L_m^\delta f\right)(x, y) = \delta f(x, x) + (1 - \delta)f(y, y). \tag{3.1}$$

Assume that f is continuous on $I \times I$ and there exists an interval $K \subset I \cap J$ such that there exists $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K, so that for any $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2}L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{3.2}$$

Then the convergence given in (3.1) *is uniform on $K \times K$ and*

$$\left|\left(L_m^\delta f\right)(x, y) - (\delta f(x, x) + (1-\delta)f(y, y))\right| \leq (1+k_2)\omega\left(F; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{3.3}$$

for any $(x, y) \in K \times K$ and $m \in \mathbb{N}$, $m \geq m(0)$.

Proof. If $m \in \mathbb{N}$, then

$$\left(L_m^\delta f\right)(x, y) = \delta \sum_{k=0}^{p_m} \varphi_{m,k}(x)A^*_{m,k}(f) + (1 - \delta) \sum_{k=0}^{p_m} \varphi_{m,k}(y)A^*_{m,k}(f)$$

and taking (2.16) into account, we obtain

$$\left(L_m^\delta f\right)(x, y) = \delta(L_mF)(x) + (1 - \delta)(L_mF)(y). \tag{3.4}$$

We have

$$\begin{aligned}
&\left|\left(L_m^\delta f\right)(x) - [\delta f(x, x) + (1 - \delta)f(y, y)]\right| \\
&= |\delta[(L_mF)(x) - F(x)] + (1 - \delta)[(L_mF)(y) - F(y)]| \\
&\leq \delta\,|(L_mF)(x) - F(x)| + (1 - \delta)\,|(L_mF)(y) - F(y)|
\end{aligned}$$

and apply the Corollary 2.6. ∎

Remark 3.2. In general, the sequence $\left(L_m^\delta f\right)_{m\geq1}$, where $f : I \times I \to \mathbb{R}$ is doesn't converge to the function f.

Lemma 3.3. *Let the GBS operators $\left(UL_m^\delta\right)_{m\geq1}$ associated to the $\left(L_m^\delta\right)_{m\geq1}$ operators. If $m \in \mathbb{N}$, $UL_m^\delta : E(I \times I) \to F(J \times J)$ have the form*

$$\begin{aligned}
(UL_m^\delta f)(x, y) =&\,\delta\left[(L_m\, f_x)(x) + (L_mf^y)(x)\right] \\
&+ (1 - \delta)\left[(L_m\, f_x)(y) + (L_mf^y)(y)\right] - (L_m^\delta f)(x, y) \\
=&\,\delta\left[(L_m\, f_x)(x) + (L_mf^y)(x) - (L_mF)(x)\right] \\
&+ (1 - \delta)\left[(L_m\, f_x)(y) + (L_mf^y)(y) - (L_mF)(y)\right]
\end{aligned} \tag{3.5}$$

where $(x, y) \in J \times J$ and $f \in E(I \times I)$.

PROOF. It results from definition of GBS operator, (2.1), (2.16) - (2.18) and (3.4). ∎

THEOREM 3.4. *Let $f : I \times I \to \mathbb{R}$ be a function.*

If $(x, y) \in (I \times I) \cap (J \times J)$, the functions f_x, f^y and F are s times differentiable in x and y, the functions $\dfrac{\partial^s f_x}{\partial \tau^s}$, $\dfrac{\partial^s f^y}{\partial t^s}$ and $F^{(s)}$ are continuous in x and y, then

$$\begin{aligned}(3.6)\qquad &\lim_{m\to\infty} m^{s-\alpha_s}\Bigg\{ \left(UL_m^{\delta} f\right)(x,y) \\ &-\sum_{i=0}^{s}\frac{1}{m^i i!}\Bigg[\delta\left(\frac{\partial^i f}{\partial \tau^i}(x,x)+\frac{\partial^i f}{\partial t^i}(x,y)-F^{(i)}(x)\right)\left(T^*_{m,i}L_m\right)(x) \\ &+(1-\delta)\left(\frac{\partial^i f}{\partial \tau^i}(x,y)+\frac{\partial^i f}{\partial t^i}(y,y)-F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg\}=0.\end{aligned}$$

Assume that the functions f_x, f^y and F are s times differentiable on I for any $x, y \in I$, with $\dfrac{\partial^s f^y}{\partial \tau^s}$, $\dfrac{\partial^s f_x}{\partial t^s}$ and $F^{(s)}$ continuous on I for any $x, y \in I$ and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K, so that for any $m \geq m(s)$ and any $x \in K$ we have

$$(3.7)\qquad \frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} \leq k_j$$

where $j \in \{s, s+s\}$. Then the convergence given in (3.6) *is uniform on $K \times K$ and*

$$\begin{aligned}(3.8)\quad m^{s-\alpha_s}\Bigg| &\left(UL_m^{\delta} f\right)(x,y) \\ &-\sum_{i=0}^{s}\frac{1}{m^i i!}\Bigg[\delta\left(\frac{\partial^i f}{\partial \tau^i}(x,x)+\frac{\partial^i f}{\partial t^i}(x,y)-F^{(i)}(x)\right)\left(T^*_{m,i}L_m\right)(x) \\ &+(1-\delta)\left(\frac{\partial^i f}{\partial \tau^i}(x,y)+\frac{\partial^i f}{\partial t^i}(y,y)-F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg| \\ \leq &\frac{1}{s!}(k_s+k_{s+2})\Bigg[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+ \\ &+\omega\left(F^{(s)};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\end{aligned}$$

for any $x, y \in K$ and any $m \geq m(s)$.

PROOF. We use the (2.5) relation from Theorem 3.1 for the functions f_x, f^y and F and we obtain (3.8) relation. If we note by S the left member of (3.8) relation and taking (2.7) relation into account, we can write

$$
\begin{aligned}
S =& m^{s-\alpha_s}\left|\delta\left\{\left[(L_m f_x)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial\tau^i}(x,x)\left(T^*_{m,i}L_m\right)(x)\right]\right.\right.\\
&+\left[(L_m f^y)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right]\\
&+\left.\left[\sum_{i=0}^{s}\frac{1}{m^i i!}F^{(i)}(x)\left(T^*_{m,i}L_m\right)(x)-(L_m F)(x)\right]\right\}\\
&+(1-\delta)\left\{\left[(L_m f_x)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial\tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\right]\right.\\
&+\left[(L_m f^y)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(y,y)\left(T^*_{m,i}L_m\right)(y)\right]\\
&+\left.\left.\left[\sum_{i=0}^{s}\frac{1}{m^i i!}F^{(i)}(y)\left(T^*_{m,i}L_m\right)(y)-(L_m F)(y)\right]\right\}\right|\\
\le&\delta\left[m^{s-\alpha_s}\left|(L_m f_x)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial\tau^i}(x,x)\left(T^*_{m,i}L_m\right)(x)\right|\right.\\
&+m^{s-\alpha_s}\left|(L_m f^y)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right|\\
&+\left.m^{s-\alpha_s}\left|(L_m F)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}F^{(i)}(x)\left(T^*_{m,i}L_m\right)(x)\right|\right]\\
&+(1-\delta)\left[m^{s-\alpha_s}\left|(L_m f_x)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial\tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\right|\right.\\
&+\left|(L_m f^y)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(y,y)\left(T^*_{m,i}L_m\right)(y)\right|\\
&+\left|(L_m F)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}F^{(i)}(y)\left(T^*_{m,i}L_m\right)(y)\right|\\
\le&\delta\left\{\frac{1}{s!}(k_s+k_{s+2})\left[\omega\left(\frac{\partial^s f_x}{\partial\tau^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\right.\right.
\end{aligned}
$$

$$+\,\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(F^{(s)};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\Bigg\}$$

$$+\,(1-\delta)\Bigg\{\frac{1}{s!}\left(k_s+k_{s+2}\right)\Bigg[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)$$

$$+\,\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(F^{(s)};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\Bigg\},$$

from where we obtain (3.8) relation. From (3.8) the uniform convergence for (3.6) results. □

THEOREM 3.5. *Let $f : I\times I\to\mathbb{R}$ be a function.*

If $(x,y)\in(I\times I)\cap(J\times J)$, the functions f_x, f^y and F are s times differentiable in x and y, the functions $\dfrac{\partial^s f_x}{\partial\tau^s}$, $\dfrac{\partial^s f^y}{\partial t^s}$ and $F^{(s)}$ are continuous in x and y, then

$$\lim_{m\to\infty}\left(UL_m^{\delta}f\right)(x,y)=f(x,y) \tag{3.9}$$

if $s=0$, and

$$\lim_{m\to\infty} m^{s-\alpha_s}\Bigg\{\left(UL_m^{\delta}f\right)(x,y)-\sum_{i=0}^{s-1}\frac{1}{m^i i!}\Bigg[\delta\bigg(\frac{\partial^i f}{\partial\tau^i}(x,x)+\frac{\partial^i f}{\partial t^i}(x,y)-F^{(i)}(x)\bigg)\left(T^*_{m,i}L_m\right)(x)$$
$$+(1-\delta)\left(\frac{\partial^i f}{\partial\tau^i}(x,y)+\frac{\partial^i f}{\partial t^i}(y,y)-F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg\}$$
$$=\frac{1}{s!}\Bigg[\delta\left(\frac{\partial^s f}{\partial\tau^s}(x,x)+\frac{\partial^s f}{\partial t^s}(x,y)-F^{(s)}(x)\right)B_s(x)$$
$$+(1-\delta)\left(\frac{\partial^s f}{\partial\tau^s}(x,y)+\frac{\partial^s f}{\partial t^s}(y,y)-F^{(s)}(y)\right)B_s(y)\Bigg]. \tag{3.10}$$

Assume that the functions f_x, f^y and F are s times differentiable on I for any $x,y\in I$, with $\dfrac{\partial^s f_x}{\partial\tau^s}$, $\dfrac{\partial^s f^y}{\partial t^s}$, $F^{(s)}$ continuous on I for any $x,y\in I$ and there exists an interval $K\subset I\cap J$ such that there exist $m(s)\in\mathbb{N}$ and $k_j\in\mathbb{R}$ depending on K so that for any $m\geq m(s)$ and any $x\in K$ we have

$$\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}}\leq k_j \tag{3.11}$$

where $j\in\{s,s+2\}$. Then the convergence given in (3.9) *and* (3.10) *is uniform on $K\times K$.*

PROOF. It results from Theorem 3.4 and Corollary 2.5. ∎

COROLLARY 3.6. *Let $f: I\times I$ be a function.*

If $(x,y)\in(I\times I)\cap(J\times J)$, the functions f_x, f^y and F are continuous in x and y, then

$$\lim_{m\to\infty}\left(UL_m^{\delta}f\right)(x,y)=f(x,y). \tag{3.12}$$

Assume that the functions f_x, f^y and F are continuous on I for any $x,y\in I$ and there exists an interval $K\subset I\cap J$ such that there exist $m(0)\in\mathbb{N}$ and $k_2\in\mathbb{R}$ depending on K, so that for any $m\in\mathbb{N}$, $m\geq m(0)$ and any $x\in K$ we have

$$\frac{\left(T_{m,2}^{*}L_m\right)(x)}{m^{\alpha_2}}\leq k_2. \tag{3.13}$$

Then the convergence given in (3.12) *is uniform on $K\times K$ and*

$$\begin{aligned}\left|\left(UL_m^{\delta}f\right)(x,y)-f(x,y)\right|\leq(1+k_2)\Bigg[&\omega\left(f_x;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\\&+\omega\left(f^y;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(F;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\end{aligned} \tag{3.14}$$

for any $x,y\in K$ and any $m\geq m(s)$.

PROOF. It results from Theorem 3.4 for $s=0$. ∎

COROLLARY 3.7. *Let $f: I\times I\to\mathbb{R}$ be a function.*

If $(x,y)\in(I\times I)\cap(J\times J)$, the functions f_x, f^y and F are two times differentiable in x and y, the functions $\dfrac{\partial^2 f_x}{\partial\tau^2}$, $\dfrac{\partial^2 f^y}{\partial t^2}$ and F'' are continuous in x and y, then

$$\begin{aligned}\lim_{m\to\infty}m^{2-\alpha_2}\Bigg\{&\left(UL_m^{\delta}f\right)(x,y)-f(x,y)\\&-\frac{1}{m}\Bigg[\delta\left(\frac{\partial f}{\partial\tau}(x,x)+\frac{\partial f}{\partial t}(x,y)-F'(x)\right)\left(T_{m,1}^{*}L_m\right)(x)\\&+(1-\delta)\left(\frac{\partial f}{\partial\tau}(x,y)+\frac{\partial f}{\partial t}(y,y)-F'(y)\right)\left(T_{m,1}^{*}L_m\right)(y)\Bigg]\Bigg\}\\=&\frac{1}{2}\Bigg[\delta\left(\frac{\partial^2 f}{\partial\tau^2}(x,x)+\frac{\partial^2 f}{\partial t^2}(x,y)-F''(s)\right)B_s(x)\\&+(1-\delta)\left(\frac{\partial^2 f}{\partial\tau^2}(x,y)+\frac{\partial^2 f}{\partial t^2}(y,y)-F''(y)\right)B_s(y)\Bigg].\end{aligned} \tag{3.15}$$

Assume that the functions f_x*,* f^y *and* F *are two times differentiable on* I *for any* $x, y \in I$*, with* $\dfrac{\partial^2 f_x}{\partial \tau^2}$*,* $\dfrac{\partial^2 f^y}{\partial t^2}$ *and* F'' *continuous on* I *for any* $x, y \in I$ *and there exists an interval* $K \subset I \cap J$ *such that there exist* $m(2) \in \mathbb{N}$ *and* $k_j \in \mathbb{R}$ *depending on* K *so that for any* $m \geq m(2)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,j} L_m\right)}{m^{\alpha_j}} \leq k_j \tag{3.16}$$

where $j \in \{2, 4\}$*. Then the convergence given in* (3.15) *is uniform on* $K \times K$*.*

Proof. It results from Theorem 3.5 for $s = 2$. □

In the following, by particularization and applying Theorem 3.5, Corollary 3.6 and Corollary 3.7, we can obtain Voronovskaja's type theorem and approximation theorem for some known operators. Because every application is a simple substitute in the theorems of this section, we won't replace anything.

In Applications 3.1-3.4, let $p_m = m$, $\varphi_{m,k} = p_{m,k}$, where $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$ and $K = [0, 1]$.

Application 3.1. If $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$ and $f \in C([0, 1])$ then we obtain the Bernstein operators. We have $k_2 = \dfrac{5}{4}$, $k_4 = \dfrac{19}{16}$, $\left(T^*_{m,1} B_m\right)(x) = 0$, $x \in [0, 1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [19]).

If $\delta = \dfrac{1}{2}$ we obtain the GBS operators $\left(UB_m^{\frac{1}{2}}\right)_{m \geq 1}$ associated to the $\left(B_m^{\frac{1}{2}}\right)_{m \geq 0}$ operators, studied in the paper [2]. These operators do not satisfy the assumptions of Theorem A from paper [3]. There exists no satisfactory choice of δ_1 and δ_2 in Corollary 5 to express the degree of approximation of $\left(UB_m^{\frac{1}{2}}\right)_{m \geq 1}$ operators (see [3]).

Application 3.2. If $I = J = [0, 1]$, $E(I) = L_1([0, 1])$, $F(J) = C([0, 1])$, $A_{m,k}(f) = (m + 1) \displaystyle\int_0^1 p_{m,k}(t) f(t) dt$, where $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$ and $f \in L_1([0, 1])$, then we obtain the Durrmeyer operators. In this case $k_2 = \dfrac{3}{2}$, $k_4 = \dfrac{7}{4}$, $\left(T^*_{m,1} M_m\right)(x) = \dfrac{m(1 - 2x)}{m + 2}$, $x \in [0, 1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 3$ (see [19]).

Application 3.3. If $I = J = [0,1]$, $E(I) = L_1([0,1])$, $F(J) = C([0,1])$, $A_{m,k}(f) = (m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in L_1([0,1])$, then we obtain the Kantorovich operators. We have $k_2 = 1$, $k_4 = \frac{3}{2}$, $\left(T^*_{m,1}K_m\right)(x) = \frac{m}{2(m+1)}(1-2x)$, $x \in [0,1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 3$ (see [19]).

Application 3.4. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$. If $I = [0, \mu^{(\alpha,\beta)}]$, $J = [0,1]$, $E(I) = C([0,\mu^{(\alpha,\beta)}])$, $F(J) = C([0,1])$, $A_{m,k} = f\left(\frac{k+\alpha}{m+\beta}\right)$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,\mu^{(\alpha,\beta)}])$, then we obtain the Stancu operators.

Application 3.5. Let $p \in \mathbb{N}_0$. If $I = [0,1+p]$, $J = [0,1]$, $E(I) = C([0,1+p])$, $F(J) = C([0,1])$, $K = [0,1]$, $\varphi_{m,k} = \widetilde{p}_{m,k} = p_{m+p,k}$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$, $p_m = m+p$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,1+p])$, then we obtain the Schurer operators.

In Applications 3.6–3.8 and Application 3.10 let $K = [0,b]$, $b > 0$.

Application 3.6. If $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = \binom{m}{k} x^k \frac{1}{(1+x)^m}$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m+1-k}\right)$, $p_m = m$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,\infty))$, then we obtain the Bleimann-Butzer-Hahn operators. In this case $k_2 = 4b(1+b)^2$ (see [22] or [25]).

In Applications 3.7–3.10 let $p_m = \infty$ for any $m \in \mathbb{N}$.

Application 3.7. If $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$, then we obtain the Mirakjan-Favard-Szász operators. We have $k_2 = b$, $k_4 = 3b^2 + b$, $\left(T^*_{m,1}S_m\right)(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [21]).

Application 3.8. If $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = (1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$, then we obtain the Baskakov operators. In this case $k_2 = b(1+b)$, $k_4 = 9b^4 + 18b^3 + 10b^2 + b$, $\left(T^*_{m,1}V_m\right)(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [21]).

Application 3.9. If $I = J = K = [0,1]$, $E(I) = E(J) = C([0,1])$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m+k}\right)$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,1])$, then we obtain the Meyer-König and Zeller operators. We have $k_2 = 2$ (see [21]).

Application 3.10. If $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-\frac{(m+k)x}{1+x}} \frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,\infty))$, then we obtain the Ismail-May operators. In this case $k_2 = b(1+b)^2$, $k_4 = b^2(1+b)^4 + 1$, $\left(T^*_{m,1}R_m\right)(x) = A_{m,1}(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $m(0) = 1$ and $m(2) = m_2$ (see [24]).

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