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ORBITS TENDING STRONGLY TO INFINITY UNDER PAIRS OF OPERATORS ON REFLEXIVE BANACH SPACES

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Abstract. In [4] we gave some sufficient conditions under which, for given pair of bounded linear operators T and S on an infinite-dimensional complex Hilbert space H, there is a dense set of vectors in H with orbits under T and S tending strongly to infinity. In this paper we are going to extend these results for pairs of operators on an infinite-dimensional complex and reflexive Banach space.

1. Introduction

Throughout this paper X will denote an infinite-dimensional reflexive Banach space over the field of complex numbers $\mathbb{C}$ and $B(X)$ the algebra of all bounded linear operators on X. For $T \in B(X)$, with $r(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ we will denote the spectral radius, the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Recall that $\sigma_p(T)$ consists of all eigenvalues for T and $\sigma_a(T)$ consists of all $\lambda \in \sigma(T)$ for which there is a sequence of unit vectors $(x_n)_{n\geq 1}$ in X such that $\|Tx_n - \lambda x_n\| \to 0$, as $n \to +\infty$; any such sequence is called a sequence of *almost eigenvectors* for λ. It is known that, unlike the point spectrum, which can be empty, the approximate point spectrum is nonempty for every $T \in B(X)$; it contains both the boundary $\partial\sigma(T)$ and $\sigma_p(T)$ [3, Prop. VII.6.7].

Orbit of $x \in X$ under $T \in B(X)$ is the sequence $\mathrm{Orb}(T,x)=\{T^n x : n \geq 0\}$. We are interested in the operators $T \in B(X)$ for which there is $x \in X$ whose orbit $\mathrm{Orb}(T,x)$ tends strongly to infinity, i.e. $\|T^n x\| \to +\infty$, as $n \to +\infty$.

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Obviously, if $\sigma_p(T)$ contains a point λ with $|\lambda| > 1$, then for every corresponding nonzero vector x in the eigenspace $\operatorname{Ker}(T-\lambda)$, the orbit will tend strongly to infinity: $\|T^n x\| = |\lambda|^n \|x\| \to +\infty$, as $n \to +\infty$. In general, $\operatorname{Ker}(T-\lambda)$ is not dense in X (relative to the norm topology). In order to produce a *dense* set of vectors in X whose orbits under T tend strongly to infinity we have to look at the points in the approximate point spectrum which are not eigenvalues.

In [5] we gave a complete proof of the following result, originally stated by B. Beauzamy [2, Thm. 2.A.5]: if $T \in B(X)$ and the circle $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$ contains a point in $\sigma(T)$ which is not an eigenvalue for T, then for every positive sequence $(\alpha_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$, in every open ball in X with radius strictly larger then $\sum_{n\geq 1} \alpha_n$, there is $x \in X$ satisfying $\|T^n x\| \geq \alpha_n r(T)^n/2$ for all $n \geq 1$. (For some additional results with similar estimates for the orbits we refer to [6] and [7]). Note that, if $r(T) > 1$, then the space will contain a dense set of vectors $x \in X$ with orbits under T tending strongly to infinity.

The proof of the previous result that we gave in [5] suggests that it will remain true if $r(T)$ is replaced with $|\lambda|$, for any $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$. Thus we have

Theorem 1.1. *If $T \in B(X)$ and $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$, then for every positive sequence $(\alpha_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$, in every open ball in X with radius $2\sum_{n\geq 1} \alpha_n$, there is $x \in X$ satisfying*

$$\|T^n x\| \geq \frac{1}{2}\alpha_n |\lambda|^n, \text{ for all } n \geq 1.$$

We are going to extend this result for suitable pairs of operators in $B(X)$.

2. Preliminary results

The first result in this section describes the behavior of the sequences of almost eigenvectors, relative to the weak topology, for those points in the approximate point spectrum that are not eigenvalues. Although this follows from [2, Prop. II.1.13], for completeness we give bellow its proof.

Proposition 2.1. *If $A \in B(X)$ and $\lambda \in \sigma_a(A)\backslash\sigma_p(A)$, then every corresponding sequence of almost eigenvectors for λ tends weakly to* 0.

Proof. Let $(u_n)_{n\geq 1}$ be any sequence of almost eigenvectors for $\lambda \in \sigma_a(A)\backslash\sigma_p(A)$.

Since in the case of reflexive Banach space $\operatorname{ball} X = \{x \in X : \|x\| \leq 1\}$ is weakly compact, there is a subsequence $(u_{n_k})_{k\geq 1}$ of $(u_n)_{n\geq 1}$ and $u \in \operatorname{ball} X$ so that $u_{n_k} \to u$ (weakly). Then $Au_{n_k} - \lambda u_{n_k} \to Au - \lambda u$ (weakly) and, by $\|Au_n - \lambda u_n\| \to 0$, as $n \to +\infty$, $Au_{n_k} - \lambda u_{n_k} \to 0$ (weakly). But, in the case of Banach spaces the weak topology separates the points; hence $Au - \lambda u = 0$.

If we assume that $u \neq 0$, then $Au - \lambda u = 0$ will imply that $\lambda \in \sigma_p(A)$, which contradicts our assumption. So, $u = 0$. Thus, we obtain that $u = 0$ is the only accumulation point for $(u_n)_{n\geq 1}$ relative to weak topology. This, together with the weakly compactness of ball X, gives $u_n \to 0$ (weakly). ∎

We also need the following modified version of [2, Lemma III.2.A.6] and [4, Lemma 2.2].

LEMMA 2.2. *If $(u_n)_{n\geq 1}$ is a sequence in X (not necessarily reflexive) which tends weakly to* 0 *and $A \in B(X)$, then for every $u \in X$ and $\delta > 0$*

(*a*) $\limsup_{n\to+\infty} \|A(u + \delta u_n)\| \geq \|Au\|$;

(*b*) *if $\|Au_n\| \to \alpha$, as $n \to +\infty$, then*

$$\limsup_{n\to+\infty} \|A(u + \delta u_n)\| \geq \max\left\{\frac{1}{2}\alpha\delta, \|Au\|\right\}.$$

PROOF. If $(u_n)_{n\geq 1}$ tends weakly to 0, then for every bounded linear functional x^* on X

$$|x^*(Au)| = \lim_{n\to+\infty} |x^*(A(u + \delta u_n))| \leq \limsup_{n\to+\infty} \|x^*\| \cdot \|A(u + \delta u_n)\|$$

and consequently

$$\|Au\| = \sup\{|x^*(Au)| : x^* \in X^*, \|x^*\| \leq 1\} \leq \limsup_{n\to+\infty} \|A(u + \delta u_n)\|,$$

where X^* denotes the dual space of X. This proves (*a*).

If $u \in X$ is such that $\|Au\| \geq \alpha\delta/2$, then the assertion in (*b*) follows from (*a*). If $\|Au\| < \alpha\delta/2$, then for all $n \geq 1$

$$\|A(\delta u_n)\| - \frac{\alpha\delta}{2} \leq \|A(u + \delta u_n)\| + \|Au\| - \frac{\alpha\delta}{2} \leq \|A(u + \delta u_n)\|,$$

which implies that

$$\frac{\alpha\delta}{2} = \lim_{n\to+\infty} \left(\|A(\delta u_n)\| - \frac{\alpha\delta}{2}\right) \leq \limsup_{n\to+\infty} \|A(u + \delta u_n)\|.$$

∎

3. Main results

THEOREM 3.1. *If $T, S \in B(X)$, $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$ and $\mu \in \sigma_a(S)\backslash\sigma_p(S)$, then for any two positive sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$ and $\sum_{n\geq 1} \beta_n < +\infty$, in every open ball in X with radius $2\sum_{n\geq 1}(\alpha_n + \beta_n)$ there is $z \in X$ satisfying simultaneously*

$$\|T^n z\| \geq \frac{1}{2}\alpha_n |\lambda|^n \text{ and } \|S^n z\| \geq \frac{1}{2}\beta_n |\mu|^n, \text{ for all } n \geq 1.$$

PROOF. By Proposition 2.1 there are sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in X such that

(a) $\|x_n\| = 1 = \|y_n\|$ for all $n \geq 1$;
(b) $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ both tend weakly to 0;
(c) $\|Tx_n - \lambda x_n\| \to 0$ and $\|Sy_n - \mu y_n\| \to 0$, as $n \to +\infty$.

Since λ, as an element of $\sigma_a(T)\backslash\sigma_p(T) \subseteq \sigma(T)$, is with $|\lambda| \leq r(T) \leq \|T\|$, for each $k \geq 1$

$$\begin{aligned}\left\|T^k x_n - \lambda^k x_n\right\| &= \left\|(T^{k-1} + \lambda T^{k-2} + \ldots + \lambda^{k-2}T + \lambda^{k-1})(Tx_n - \lambda x_n)\right\| \\ &\leq k\,\|T\|^{k-1}\,\|Tx_n - \lambda x_n\|\,.\end{aligned}$$

This implies that $\left\|T^k x_n - \lambda^k x_n\right\| \to 0$, as $n \to +\infty$, for all $k \geq 1$. In the same way we obtain that $\left\|S^k y_n - \mu^k y_n\right\| \to 0$, for all $k \geq 1$, and hence, by (a),

$$\left\|T^k x_n\right\| \to |\lambda|^k \text{ and } \left\|S^k y_n\right\| \to |\mu|^k\,, \text{ when } n \to +\infty, \text{ for all } k \geq 1. \tag{3.1}$$

Fix $x \in X$, $0 < \varepsilon < 1/2$ and any two sequence $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$ and $\sum_{n\geq 1} \beta_n < +\infty$.

We are going to prove that there are positive integers $0 < n_1 < m_1 < \cdots < n_l < m_l < \ldots$ such that for every $l \geq 1$ the vector

$$z_l = x + (1+\varepsilon)(\alpha_1 x_{n_1} + \beta_1 y_{m_1} + \cdots + \alpha_l x_{n_l} + \beta_l y_{m_l}),$$

satisfies

$$\left\|T^j z_l\right\| > \frac{1}{2}\alpha_j\,|\lambda|^j \text{ and } \left\|S^j z_l\right\| > \frac{1}{2}\beta_j\,|\mu|^j\,, \text{ for all } 1 \leq j \leq l.$$

The proof will be made with the use of mathematical induction. First we prove that there exists $z_1 \in X$ such that $\|Tz_1\| > \frac{1}{2}\alpha_1|\lambda|$ and $\|Sz_1\| > \frac{1}{2}\beta_1|\mu|$. Since the sequence $(x_n)_{n\geq 1}$ tends weakly to 0, by Lemma 2.2.(b) we have

$$\limsup_{n\to+\infty} \|T(x + (1+\varepsilon)\alpha_1 x_n)\| \geq \max\left\{\frac{1}{2}(1+\varepsilon)\alpha_1\,|\lambda|\,, \|Tx\|\right\} > \frac{1}{2}\,\alpha_1\,|\lambda|\,.$$

This allows us to find a positive integer n_1 so that $z_1' = x + (1+\varepsilon)\alpha_1 x_{n_1}$ satisfies $\|Tz_1'\| > \frac{1}{2}\alpha_1\,|\lambda|\,.$

In the same way we obtain that

$$\limsup_{n\to+\infty} \|S(z_1' + (1+\varepsilon)\beta_1 y_n)\| \geq \max\left\{\frac{1}{2}(1+\varepsilon)\beta_1\,|\mu|\,, \|Sz_1'\|\right\} > \frac{1}{2}\beta_1\,|\mu|\,.$$

So, we can find integers $n_1 < m(1) < m(2) < \ldots$ such that

$$\left\|S(z_1' + (1+\varepsilon)\beta_1 y_{m(i)})\right\| > \frac{1}{2}\beta_1\,|\mu|\,, \text{ for all } i \geq 1.$$

Clearly, $(y_{m(i)})_{i\geq 1}$ tends weakly to 0. Hence, by Lemma 2.2.(a)

$$\limsup_{i\to+\infty} \left\|T(z_1' + (1+\varepsilon)\beta_1 y_{m(i)})\right\| \geq \|Tz_1'\| > \frac{1}{2}\alpha_1\,|\lambda|$$

and consequently, there is $i_0 \geq 1$ so that $\left\|T(z_1' + (1+\varepsilon)\beta_1 y_{m(i_0)})\right\| > \frac{1}{2}\alpha_1\,|\lambda|$.

Let $m_1 = m(i_0) > n_1$ and $z_1 = z_1' + (1+\varepsilon)\beta_1 y_{m_1} = x + (1+\varepsilon)(\alpha_1 x_{n_1} + \beta_1 y_{m_1})$. Then

$$\|Tz_1\| > \frac{1}{2}\alpha_1 |\lambda| \text{ and } \|Sz_1\| > \frac{1}{2}\beta_1 |\mu| .$$

Suppose that we have found integers $0 < n_1 < m_1 < \ldots < n_{l-1} < m_{l-1}$ for some $l \geq 2$ such that each

$$z_k = x + (1+\varepsilon)(\alpha_1 x_{n_1} + \beta_1 y_{m_1} + \ldots + \alpha_k x_{n_k} + \beta_k y_{m_k}),\ 1 \leq k \leq l-1$$

satisfies both (3.2) and (3.3) bellow

$$\left\|T^j z_k\right\| > \frac{1}{2}\alpha_j |\lambda|^j , \text{ for all } 1 \leq j \leq k \leq l-1, \tag{3.2}$$

$$\left\|S^j z_k\right\| > \frac{1}{2}\beta_j |\mu|^j , \text{ for all } 1 \leq j \leq k \leq l-1. \tag{3.3}$$

Now we start with the sequence $(z_{l-1} + (1+\varepsilon)\alpha_l x_n)_{n\geq 1}$. First we are going to show that there are strictly increasing sequences of positive integers $(N_j(n))_{n\geq 1}$, $j = 1, \ldots, l-1, l, \ldots, 2l-2$ such that

(P.1) $(N_{j+1}(n))_{n\geq 1}$ is a subsequence of $(N_j(n))_{n\geq 1}$, for every $1 \leq j \leq 2l-3$,

(P.2) $\left\|T^j(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_j(n)})\right\| > \frac{1}{2}\alpha_j |\lambda|^j$, for all $1 \leq j \leq l-1$ and $n \geq 1$,

(P.3) $\left\|S^j(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_{j+l-1}(n)})\right\| > \frac{1}{2}\beta_j |\mu|^j$, for all $1 \leq j \leq l-1$ and $n \geq 1$.

By Lemma 2.2.(a) and (3.2) , for $j = 1$ and $k = l-1$, we have

$$\begin{aligned} \sup_{n\geq m} \|T(z_{l-1} + (1+\varepsilon)\alpha_l x_n)\| &\geq \limsup_{n\to+\infty} \|T(z_{l-1} + (1+\varepsilon)\alpha_l x_n)\| \\ &\geq \|Tz_{l-1}\| > \frac{1}{2}\alpha_1 |\lambda| , \end{aligned}$$

for all $m \geq 1$. This allows us to find a strictly increasing sequence of positive integers $(N_1(n))_{n\geq 1}$ such that

$$\left\|T(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_1(n)})\right\| > \frac{1}{2}\alpha_1 |\lambda|, \text{ for all } n \geq 1.$$

Now, suppose that for some $s \leq l-2$ we have found the sequences $(N_j(n))_{n\geq 1}$, $j = 1, \ldots, s$, with the desired properties, (that is (P.1) and (P.2) hold for all $j = 1, \ldots, s$).

Since $(x_{N_s(n)})_{n\geq 1}$ tends weakly to 0 (as a subsequence of $(x_n)_{n\geq 1}$), by Lemma 2.2.(a) and (3.2), for $j = s+1$ and $k = l-1$, we have

$$\begin{aligned}\sup_{n\geq m} \left\| T^{s+1}(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_s(n)}) \right\| \\ &\geq \limsup_{n\to+\infty} \left\| T^{s+1}(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_s(n)}) \right\| \\ &\geq \left\| T^{s+1} z_{l-1} \right\| > \frac{1}{2}\alpha_{s+1} \left|\lambda\right|^{s+1},\end{aligned}$$

for all $m \geq 1$. So, we can find a strictly increasing sequence of positive integers $(N_{s+1}(n))_{n\geq 1}$ such that $(N_{s+1}(n))_{n\geq 1}$ is a subsequence of $(N_s(n))_{n\geq 1}$ and

$$\left\| T^{s+1}(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_{s+1}(n)}) \right\| > \frac{1}{2}\alpha_{s+1} \left|\lambda\right|^{s+1}, \text{ for all } n \geq 1.$$

Thus, inductively we find the sequences $(N_j(n))_{n\geq 1}$, $j = 1, \ldots, l-1$. To find the sequences $(N_j(n))_{n\geq 1}$, $j = l, \ldots, 2l-2$ we only need to repeat the previous discussion with $(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_{l-1}(n)})_{n\geq 1}$, S and (3.3), instead of $(z_{l-1} + (1+\varepsilon)\alpha_l x_n)_{n\geq 1}$, T and (3.2), respectively.

The sequence $(x_{N_{2l-2}(n)})_{n\geq 1}$ tends weakly to 0 and $\left\| T^l x_{N_{2l-2}(n)} \right\| \to \left|\lambda\right|^l$, as $n \to +\infty$. Hence, by Lemma 2.2.(b),

$$\begin{aligned}\limsup_{n\to+\infty} \left\| T^l(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_{2l-2}(n)}) \right\| &\geq \max\left\{ \frac{1}{2}(1+\varepsilon)\alpha_l \left|\lambda\right|^l, \left\| T^l z_{l-1} \right\| \right\} \\ &> \frac{1}{2}\alpha_l \left|\lambda\right|^l,\end{aligned}$$

and consequently, we can find a positive integer i'_0 such that $N_{2l-2}(i'_0) > m_{l-1}$ and

$$\left\| T^l(z_{l-1} + (1+\varepsilon)\alpha_l x_{N_{2l-2}(i'_0)}) \right\| > \frac{1}{2}\alpha_l \left|\lambda\right|^l. \tag{3.4}$$

Put $n_l = N_{2l-l}(i'_0) > m_{l-1}$ and $z'_l = z_{l-1} + (1+\varepsilon)\alpha_l x_{n_l}$. Then, by (P.1), (P.2), (P.3) and (3.4)

$$\left\| T^j z'_l \right\| > \frac{1}{2}\alpha_j \left|\lambda\right|^j, \text{ for all } 1 \leq j \leq l, \tag{3.5}$$

$$\left\| S^j z'_l \right\| > \frac{1}{2}\beta_j \left|\mu\right|^j, \text{ for all } 1 \leq j \leq l-1. \tag{3.6}$$

Starting with sequence $(z'_l + (1+\varepsilon)\beta_l y_n)_{n\geq 1}$ and applying Lemma 2.2.(a), (3.5) and (3.6), in the same way as we applied Lemma 2.2.(a), (3.2) and (3.3) for $(z_{l-1} + (1+\varepsilon)\alpha_l x_n)_{n\geq 1}$, we can find strictly increasing sequences of positive integers $(M_j(n))_{n\geq 1}$, $j = 1, \ldots, l-1, l, \ldots, 2l-2, 2l-1$ such that

(P.4) $(M_{j+1}(n))_{n\geq 1}$ is a subsequence of $(M_j(n))_{n\geq 1}$, for every $1 \leq j \leq 2l-2$.

(P.5) $\left\|S^j(z_l' + (1+\varepsilon)\beta_l y_{M_j(n)})\right\| > \dfrac{1}{2}\beta_j \left|\mu\right|^j$, for all $1 \leq j \leq l-1$ and $n \geq 1$;

(P.6) $\left\|T^j(z_l' + (1+\varepsilon)\beta_l y_{M_{j+l-1}(n)})\right\| > \dfrac{1}{2}\alpha_j \left|\lambda\right|^j$, for all $1 \leq j \leq l$ and $n \geq 1$.

The sequence $(y_{M_{2l-1}(n)})_{n\geq 1}$ tends weakly to 0 and $\left\|S^l y_{M_{2l-1}(n)}\right\| \to |\mu|^l$, as $n \to +\infty$. Hence, by Lemma 2.2.(b)

$$\begin{aligned}\limsup_{n\to+\infty} \left\|S^l(z_l' + (1+\varepsilon)\beta_l y_{M_{2l-1}(n)})\right\| &\geq \max\left\{\frac{1}{2}(1+\varepsilon)\beta_l \left|\mu\right|^l, \left\|S^l z_l'\right\|\right\} \\ &> \frac{1}{2}\alpha_l \left|\lambda\right|^l,\end{aligned}$$

and consequently we can find a positive integer i_0'' such that $M_{2l-1}(i_0'') > n_l$ and

$$\left\|S^l(z_l' + (1+\varepsilon)\beta_l y_{M_{2l-1}(i_0'')})\right\| > \frac{1}{2}\beta_l \left|\mu\right|^l. \tag{3.7}$$

Put $m_l = M_{2l-1}(i_0') > n_l$ and

$$z_l = z_l' + (1+\varepsilon)\beta_l y_{m_l} = x + (1+\varepsilon)(\alpha_1 x_{n_1} + \beta_1 y_{m_1} + \ldots + \alpha_l x_{n_l} + \beta_l y_{m_l}). \tag{3.8}$$

Then, by (P.4), (P.5), (P.6) and (3.7),

$$\left\|T^j z_l\right\| > \frac{1}{2}\alpha_j \left|\lambda\right|^j, \text{ for all } 1 \leq j \leq l, \tag{3.9}$$

$$\left\|S^j z_l\right\| > \frac{1}{2}\beta_j \left|\mu\right|^j, \text{ for all } 1 \leq j \leq l. \tag{3.10}$$

Thus, by induction, we obtain that there are integers $0 < n_1 < m_1 < \ldots < n_l < m_l < \ldots$ such that the sequence $(z_l)_{l\geq 1}$, given with (3.8), satisfies both (3.9) and (3.10), for all $l \geq 1$. But then, since $\sum_{n\geq 1} \alpha_n < +\infty$ and $\sum_{n\geq 1} \beta_n < +\infty$, for any l and k with $l > k$, by (a) we have

$$\|z_l - z_k\| = (1+\varepsilon)\left\|\sum_{i=k+1}^{l} (\alpha_i x_{n_i} + \beta_i y_{m_i})\right\| \leq (1+\varepsilon)\sum_{i=k+1}^{l} (\alpha_i + \beta_i) \to 0,$$

as $k, l \to +\infty$. This means that $(z_l)_{l\geq 1}$ is a Cauchy sequence in the Banach space X. Hence, there is $z \in X$ so that

$$z = \lim_{l\to+\infty} z_l = x + (1+\varepsilon)\sum_{i=1}^{+\infty} (\alpha_i x_{n_i} + \beta_i y_{m_i}),$$

and this is the vector with the desired properties. Namely

1. since $0 < \varepsilon < 1/2$

$$\begin{aligned} \|z-x\| &= \lim_{l\to+\infty} \|z_l - x\| = (1+\varepsilon) \lim_{l\to+\infty} \left\| \sum_{i=1}^{l} (\alpha_i x_{n_i} + \beta_i y_{m_i}) \right\| \\ &< (1+2\varepsilon) \lim_{l\to+\infty} \sum_{i=1}^{l} (\alpha_i + \beta_i) = 2 \sum_{n\geq 1} (\alpha_n + \beta_n), \end{aligned}$$

and, for all $n \geq 1$ (by (3.9) and (3.10))

2. $\|T^n z\| = \lim_{l\to+\infty} \|T^n z_l\| \geq \frac{1}{2}\alpha_n |\lambda|^n$ and $\|S^n z\| = \lim_{l\to+\infty} \|S^n z_l\| \geq \frac{1}{2}\beta_n |\mu|^n$.

□

Corollary 3.2. *If $\sigma_a(T)\backslash\sigma_p(T)$ and $\sigma_a(S)\backslash\sigma_p(S)$ both have a nonempty intersection with the domain $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$, then there is a dense set of vectors $z \in X$ with both $\mathrm{Orb}(T, z)$ and $\mathrm{Orb}(S, z)$ tending strongly to infinity.*

4. On orbits tending strongly to infinity under T and $f(T)$

4.1. *The general case.* Let Ω be a nonempty subset of the complex plane whose boundary consists of finite number of rectifiable Jordan curves, oriented in the positive sense and $\mathrm{Hol}(\Omega)$ the set of all holomorphic functions on some open neighborhood of the closure of Ω.

Theorem 4.1. *If $T \in B(X)$, $\sigma(T) \subset \Omega$ and $f \in \mathrm{Hol}(\Omega)$, then*

(a) $\sigma(f(T)) = f(\sigma(T))$;

(b) $\sigma_a(f(T)) = f(\sigma_a(T))$;

(c) $\sigma_p(f(T)) \subseteq f(\sigma_p(T))$ *and, if f is non-constant function on each of the components of Ω, then $\sigma_p(f(T)) = f(\sigma_p(T))$.*

For the proof of these results we refer the reader to [3, Thm.VII.4.6], [1, Thm.2.48] and [8, Thm.10.33]. (For the basics of the Riesz's functional calculus see also [2].)

If, in addition to the hypotheses of Theorem 4.1.(c), we assume that f is injective, then $f(\sigma_a(T)\backslash\sigma_p(T)) = f(\sigma_a(T))\backslash f(\sigma_p(T)) = \sigma_a(f(T))\backslash\sigma_p(f(T))$.

Now, applying Theorem 3.1 on T and $S = f(T)$ we have the following result.

Theorem 4.2. *If $T \in B(X)$, $\lambda \in \sigma_a(T)\backslash\sigma_p(T)$, $\sigma(T) \subset \Omega$ and $f \in \mathrm{Hol}(\Omega)$ is injective and non-constant function on each of the components of Ω, then for any two positive sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$ and $\sum_{n\geq 1} \beta_n < +\infty$, in every open ball in X with radius $2\sum_{n\geq 1} (\alpha_n + \beta_n)$ there is $z \in X$ satisfying simultaneously*

$$\|T^n z\| \geq \frac{1}{2}\alpha_n |\lambda|^n \text{ and } \|S^n z\| \geq \frac{1}{2}\beta_n |f(\lambda)|^n, \text{ for all } n \geq 1.$$

And, by Corollary 3.2, we have:

Corollary 4.3. *If* $T \in B(X)$, $\sigma(T) \subset \Omega$, $f \in \mathrm{Hol}(\Omega)$ *is injective and non-constant function on each of the components of* Ω *and* $\sigma_a(T) \backslash \sigma_p(T)$ *contains points* λ *and* μ *with* $|\lambda| > 1$ *and* $|f(\mu)| > 1$, *then there is a dense set of vectors* $z \in X$ *such that both* $\mathrm{Orb}(T, z)$ *and* $\mathrm{Orb}(f(T), z)$ *tend strongly to infinity.*

4.2. *The case of invertible operator.* If $T \in B(X)$ is an invertible operator, then

$$\sigma(T) \subseteq \left\{\lambda \in \mathbb{C} : [r(T^{-1})]^{-1} \leq |\lambda| \leq r(T)\right\}.$$

Since $f(\lambda) = \lambda^{-1}$ is holomorphic, non-constant and injective function on $\Omega = \{\lambda \in \mathbb{C} : m \leq |\lambda| \leq M\} \supset \sigma(T)$, where $0 < m < [r(T^{-1})]^{-1}$ and $M > r(T)$, by Theorem 4.2 we obtain:

Theorem 4.4. *If* $T \in B(X)$ *is invertible operator and* $\lambda \in \sigma_a(T) \backslash \sigma_p(T)$, *then for any two positive sequences* $(\alpha_n)_{n \geq 1}$ *and* $(\beta_n)_{n \geq 1}$ *with* $\sum_{n \geq 1} \alpha_n < +\infty$ *and* $\sum_{n \geq 1} \beta_n < +\infty$, *in every open ball in* X *with radius* $2 \sum_{n \geq 1} (\alpha_n + \beta_n)$ *there is* $z \in X$ *satisfying simultaneously*

$$\|T^n z\| \geq \frac{1}{2} \alpha_n |\lambda|^n \text{ and } \|T^{-n} z\| \geq \frac{1}{2} \beta_n |\lambda|^{-n}, \text{ for all } n \geq 1.$$

By Corollary 4.3 we also have the following result.

Corollary 4.5. *If* T *is invertible operator and* $\sigma_a(T) \backslash \sigma_p(T)$ *has a nonempty intersection with both* $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ *and* $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, *then there is a dense set of vectors* $z \in X$ *such that both* $\mathrm{Orb}(T, z)$ *and* $\mathrm{Orb}(T^{-1}, z)$ *tend strongly to infinity.*

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