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ONE MORE VARIATION OF THE POINT-OPEN GAME

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Abstract. A topological game "Dense $G_{\delta\sigma}$-sets" (also denoted by $\mathcal{DG}$) is introduced as follows: for any $n \in \omega$ at the n-th move the player I takes a point $x_n \in X$ and II responds by taking a G_δ-set Q_n in the space X such that $x_n \in Q_n$. The play stops after ω moves and I wins if the set $\bigcup\{Q_n : n \in \omega\}$ is dense in X. Otherwise the player II is declared to be the winner. We study classes of spaces on which the player I has a winning strategy. It is evident that the I-favorable spaces constitute a generalization of the class of separable spaces. We show that there exists a neutral space for the game $\mathcal{DG}$ and prove, among other things, that Lindelöf scattered spaces and dyadic spaces are I-favorable. We characterize I-favorability for the game $\mathcal{DG}$ in the spaces $C_p(X)$; one of the applications is that, for a Lindelöf Σ-space X, the space $C_p(X)$ is I-favorable for $\mathcal{DG}$ if and only if X is ω-monolithic.

1. Introduction

We present one more version of the well known point-open game $\mathcal{PO}$ which was discovered and studied independently by F. Galvin [6] and R. Telgársky [8]. Recall that in the game $\mathcal{PO}$ at the n-th move the first player I takes a point $x_n \in X$ while the second player II replies choosing an open set $U_n \subset X$ with $x_n \in U_n$. The play is finished after ω moves and I is announced to be the winner if $\bigcup\{U_n : n \in \omega\} = X$. Otherwise II wins in the play $\{(x_n, U_n) : n \in \omega\}$.

F. Galvin [6] proved that it is independent of ZFC whether $\mathcal{PO}$ is determined on all subsets of the real line $\mathbf{R}$. Telgársky proved in [8] that if

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X is a σ-Čech-complete or pseudocompact space then $\mathcal{PO}$ is determined on X. Later in [9] he gave a ZFC example of a neutral space (i.e., a space on which neither of the players has a winning strategy) with respect to the game $\mathcal{PO}$. P. Daniels and G. Gruenhage [4] as well as S. Baldwin [3] studied the point-open game which does not end after ω moves.

Tkachuk introduced two new games θ and Ω both in the paper [11] and in the book [10] (where they were called T and TT and their main properties were formulated as exercises). The games θ and Ω differ only a little from the point-open game G. The moves in θ are exactly the same as in $\mathcal{PO}$ but the assessment of the play $\{(x_n, U_n) : n \in \omega\}$ is different: the player I wins if the set $\bigcup\{U_n : n \in \omega\}$ is dense in X. Otherwise the second player is declared to be the winner.

In the game Ω the first player still has to pick a point $x_n \in X$ at his (or her) n-th move, while the second player has more freedom — he/she also chooses an open set $U_n \subset X$ but only $x_n \in \overline{U}_n$ is required. And again I wins the play $\{(x_n, U_n) : n \in \omega\}$ if the set $\bigcup\{U_n : n \in \omega\}$ is dense in X.

It is straightforward that for any separable space X the first player has a winning strategy on X in both θ and Ω. It was proved in [11] that any product of separable spaces is I-favorable in both games θ and Ω, and if an Eberlein compact space K is I-favorable in the game Ω, then K is metrizable. The games θ and Ω do not give interesting facts for C_p-theory because every space $C_p(X)$ is I-favorable with respect to both games θ and Ω: this was also proved in [11].

In this paper we develop an idea of Telgársky to obtain a new version of the game θ. Telgársky briefly considered in [9] a game $\mathcal{PO}'$ in which at the n-th move the first player picks a point x_n and II responds with a G_δ-set $G_n \ni x_n$. A play $\{x_n, G_n : n \in \omega\}$ is won by the first player if $\bigcup_{n\in\omega} G_n = X$. Telgársky proved that this game is the same as the point-open game for the first player, i.e., I has a winning strategy on a space X in $\mathcal{PO}$ if and only if he has a winning strategy on X in $\mathcal{PO}'$. Our game $\mathcal{DG}$ which we call "Dense $G_{\delta\sigma}$-sets" has the same moves as $\mathcal{PO}'$ but the player I is the winner of a play $\{x_n, G_n : n \in \omega\}$ if the set $\bigcup_{n\in\omega} G_n$ is dense in X. This game turns out to be radically different from the game θ and has non-trivial applications for the spaces $C_p(X)$.

To provide a more intuitive notation we denote the game θ by $\mathcal{DO}$ also calling it "Dense Open Sets". We study the main categorical properties of spaces on which the first player has a winning strategy in $\mathcal{DG}$ and prove that hereditarily I-favorable spaces for $\mathcal{DG}$ coincide with hereditarily separable spaces. We show that Lindelöf scattered spaces are I-favorable for $\mathcal{DG}$ and give a characterization of existence of a winning strategy for the first player in $\mathcal{DG}$ on a space $C_p(X)$. One of the consequences of this characterization is that, for a Lindelöf Σ-space X, the space $C_p(X)$ is I-favorable for $\mathcal{DG}$ if and only if X is ω-monolithic.

2. Notation and terminology

All spaces under consideration are assumed to be Tychonoff; if X is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. If $A \subset X$ then $\tau(A, X)$ is the family of all open subsets of X which contain A; we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. A family $\mathcal{B} \subset \tau(A, X)$ is *an outer base* of A in X if for every $U \in \tau(A, X)$ there is $B \in \mathcal{B}$ such that $B \subset U$. The Stone-Čech compactification of a space X is denoted by βX. The character of X at its subspace $A \subset X$, denoted by $\chi(A, X)$, is the minimal of the cardinalities of all outer bases of A in X; let $\chi(X) = \sup\{\chi(\{x\}, X) : x \in X\}$. A space X is Čech-complete if it is a G_δ-set in βX. The space X is of pointwise countable type if for any point $x \in X$ there exists a compact $K \subset X$ such that $x \in K$ and $\chi(K, X) \leq \omega$.

Given a space X the space $C_p(X)$ is the set of all real-valued continuous functions on X endowed with the pointwise convergence topology. If X is a space and $A \subset X$, let $\pi_A(f) = f|A$ for every $f \in C_p(X)$, i.e., $\pi_A : C_p(X) \to C_p(A)$ is the restriction map. We denote by $C_p(A|X)$ the set $\pi_A(C_p(X))$ with the topology induced from $C_p(A)$. The symbol $\mathbf{R}$ stands for the set of reals with its natural topology, $\mathbf{N} = \omega\backslash\{0\}$ and $\mathbf{Q} \subset \mathbf{R}$ is the set of rationals. We denote by $\mathbf{D}$ the doubleton $\{0, 1\}$ with the discrete topology. Let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for every $n \in \omega$. The space $C_{p,n}(X)$ is called *the n-th iterated function space of X*.

If a game $\mathcal{G}$ is considered, a space X is called I-favorable with respect to $\mathcal{G}$ if the player I has a winning strategy on the space X. In the game $\mathcal{DG}$ (called "Dense $G_{\delta\sigma}$-sets") the n-th move on a space X consists in the player I picking a point $x_n \in X$ and II replying by taking a G_δ-set $Q_n \ni x_n$. In the play $\{(x_n, Q_n) : n \in \omega\}$ the first player wins if $\bigcup\{Q_n : n \in \omega\}$ is dense in X; otherwise the victory is assigned to the second player. If at the n-th move the second player picks an open set $U_n \ni x_n$ and, again, the player I wins if $\bigcup_{n\in\omega} U_n$ is dense in X, then the respective game is denoted by $\mathcal{DO}$ and called "Dense Open Sets". Observe that the game $\mathcal{DO}$ was denoted by θ in the paper [11].

A space X is ω-monolithic if $\overline{A}$ has a countable network for any countable $A \subset X$. The space X is Lindelöf Σ if it is a continuous image of a space which can be perfectly mapped onto a second countable space. For information on cardinal functions see Hodel's paper [7]. All facts of C_p-theory we use can be found in [2]. The rest of our notation is standard and follows [5].

2. Basic properties of the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets.

The class of I-favorable spaces for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets is, evidently, an extension of the class of separable spaces. We will see that in the spaces of countable pseudocharacter this property coincides with separability;

besides, checking I-favorability results in finding non-trivial and interesting properties both in general spaces and in spaces $C_p(X)$.

DEFINITION 2.1. *Given a space X, say that a game $\mathcal{DG}_{fin}$ is played on X if at the n-th move the first player takes a finite set $A_n \subset X$ (which can be empty) and the second player responds by taking a G_δ-subset P_n such that $A_n \subset P_n$. A play $\{A_n, P_n : n \in \omega\}$ is favorable for I if the set $\bigcup_{n\in\omega} P_n$ is dense in X. Otherwise the second player wins.*

THEOREM 2.2. (i) *If X is I-favorable for the point-open game of Galvin–Telgársky then X is I-favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets;*

(ii) *The game $\mathcal{DG}$ is equivalent to the game $\mathcal{DG}_{fin}$ in the sense that, on any non-empty space X, a player $J \in \{I, II\}$ has a winning strategy in $\mathcal{DG}$ if and only if J has a winning strategy in $\mathcal{DG}_{fin}$ on the space X;*

(iii) *any continuous image of a space I-favorable with respect to $\mathcal{DG}$ is I-favorable with respect to $\mathcal{DG}$;*

(iv) *if Y is I-favorable with respect to $\mathcal{DG}$ and dense in X then X is also I-favorable with respect to $\mathcal{DG}$;*

(v) *if X is I favorable for $\mathcal{DG}$ on a space X then, for any set $U \in \tau^*(X)$, the space $\overline{U}$ is also I-favorable for $\mathcal{DG}$;*

(vi) *if X_n is I-favorable with respect to $\mathcal{DG}$ for each $n \in \omega$ and $X = \bigcup\{X_n : n \in \omega\}$ then X is also I-favorable with respect to $\mathcal{DG}$.*

PROOF. (i) Given a space X which is I-favorable in the game $\mathcal{PO}$, apply Theorem 5.1 of [9] to see that in the game where I chooses at the n-th step a point x_n and II responds with a G_δ-set $Q_n \ni x_n$ the first player has a winning strategy under which the sets chosen by the second player cover the whole space X. In particular, the space X is I-favorable in the game $\mathcal{DG}$.

(ii) It is clear that any winning strategy for the first player in $\mathcal{DG}$ is also a winning strategy for I in $\mathcal{DG}_{fin}$. Analogously, any winning strategy for the player II in $\mathcal{DG}_{fin}$ is also a winning strategy for the second player in $\mathcal{DG}$. Now if s is a winning strategy of the first player for $\mathcal{DG}_{fin}$ on a non-empty space X then all moves according to s can be assumed to be non-empty so we can define a strategy σ for the player I in $\mathcal{DG}$ by taking one-by-one the points of the current finite set provided by s. When the set supplied by s is covered by our choices we take the union of the G_δ-sets chosen by II for our points and apply the strategy s to obtain one more finite set. This gives a winning strategy σ for the first player in $\mathcal{DG}$ because any play $\mathcal{P}$ according to σ can be split into a play $\mathcal{P}'$ in $\mathcal{DG}_{fin}$ such that I applies s in $\mathcal{P}'$ and the unions of G_δ-sets chosen by II in $\mathcal{P}$ and $\mathcal{P}'$ coincide.

Now if σ is a winning strategy for the second player on a non-empty space X in $\mathcal{DG}$ and a finite set $A_n \subset X$ is chosen at the n-th move in a play in $\mathcal{DG}_{fin}$ then II can consider that he/she plays in $\mathcal{DG}$ and the points of A_n are taken one-by-one by the first player (if $A_n = \emptyset$ then II responds by $P_n = \emptyset$). Applying the strategy σ for this accompanying play he/she obtains a G_δ set $P_n \supset A_n$ as the union of the sets the strategy σ gave for the points of A_n. It is straightforward that this gives a winning strategy s for the second player in $\mathcal{DG}_{fin}$.

(iii) Suppose that X is I favorable with respect to $\mathcal{DG}$ and fix a winning strategy s for the first player on the space X. If $f : X \to Y$ is a continuous onto map then let $x_0 = s(\emptyset)$ and $\sigma(\emptyset) = f(x_0)$. Proceeding inductively suppose that an initial segment $y_0, G_0, \dots, y_n, G_n$ of a play in $\mathcal{DG}$ is given on the space Y and, besides, we have $x_0, \dots, x_n \in X$ such that $f(x_i) = y_i$ for all $i \le n$ and $x_0, f^{-1}(G_0), \dots, x_n, f^{-1}(G_n)$ is an initial segment of a play in X in which I applies the strategy s.

Let $x_{n+1} = s(f^{-1}(G_0), \dots, f^{-1}(G_n))$ and $y_{n+1} = \sigma(G_0, \dots, G_n) = f(x_{n+1})$. This defines a strategy σ for the first player on the space Y and we omit the simple verification that σ is winning.

(iv) If s is a winning strategy on Y for the first player then s can be considered a strategy on X if we use s applied to the intersections of the moves of the second player with the set Y. It is evident that this gives a winning strategy on X, so X is also I-favorable for $\mathcal{DG}$.

(v) Apply (ii) to find a winning strategy s in the game $\mathcal{DG}_{fin}$ for the first player on the space X and let $W = X \setminus \overline{U}$. Consider the set $A_0 = s(\emptyset) \cap \overline{U}$ and let $\sigma(\emptyset) = A_0$. If $n \in \omega$ and moves $A_0, P_0, \dots, A_n, P_n$ are made on the space $\overline{U}$ in the game $\mathcal{DG}_{fin}$, then $Q_i = W \cup P_i$ is a G_δ-subset of X for each $i \le n$, so the set $A_{n+1} = s(Q_0, \dots, Q_n) \cap \overline{U}$ is consistently defined; let $\sigma(P_0, \dots, P_n) = A_{n+1}$. This gives us a strategy σ in $\mathcal{DG}_{fin}$ for the first player on the space $\overline{U}$ and it is easy to check that σ is a winning strategy. Applying (ii) once more we conclude that $\overline{U}$ is I-favorable for $\mathcal{DG}$.

(vi) For each $n \in \omega$ let s_n be a winning strategy for the first player on the space X_n. Choose a disjoint family $\{A_n : n \in \omega\}$ of infinite subsets of ω such that $\omega = \bigcup_{n \in \omega} A_n$. There is a unique $n \in \omega$ such that $0 \in A_n$; let $x_0 = s_n(\emptyset)$ and $s(\emptyset) = x_0$. Proceeding inductively assume that $k \in \omega$ and we are given an initial segment $x_0, G_0, \dots, x_k, G_k$ of a play in $\mathcal{DG}$ on the space X. There is a unique $m \in \omega$ such that $k + 1 \in A_m$. A part of our play say, $x_{j_1}, G_{j_1}, \dots, x_{j_l}, G_{j_l}$ is done in X_{k+1} and, by the induction hypothesis, constitutes an initial segment of $\mathcal{DG}$ in X_{k+1} if we intersect the sets G_{j_i} with X_{k+1}. Letting $x_{k+1} = s(G_0, \dots, G_k) = s_{k+1}(G_{j_1} \cap X_{k+1}, \dots, G_{j_l} \cap X_{k+1})$ we conclude the definition of a strategy s on the space X.

If $\{x_n, G_n : n \in \omega\}$ is a play on X in which I applies s, then for every $k \in \omega$ the family $\{x_n, G_n \cap X_k : n \in A_k\}$ is a play on X_k in which I applies s_k so $\bigcup\{G_n \cap X_k : n \in A_k\}$ is dense in X_k. An immediate consequence is that $\bigcup_{n \in \omega} G_n$ is dense in X so s is a winning strategy of the first player on X. ∎

It was proved in [11] that if a space X is I-favorable with respect to the game of dense open sets on a space X then X is weakly Lindelöf. The I-favorable spaces for the game $\mathcal{DG}$ have an analogous stronger property.

PROPOSITION 2.3. *If a space X is I-favorable with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets and $\mathcal{Q}$ is a cover of X with G_δ-sets then there exists a countable $\mathcal{Q}' \subset \mathcal{Q}$ such that $\bigcup \mathcal{Q}'$ is dense in X.*

PROOF. If no countable subfamily of $\mathcal{Q}$ has a dense union in X then pick, for any point $x \in X$, a set $Q_x \in \mathcal{Q}$ with $x \in Q_x$. This gives a strategy for the second player on X defined as follows: if $n \in \omega$ and moves $x_0, G_0, \dots, x_{n-1}, G_{n-1}, x_n$ are made then the second player responds with $G_n = s(x_0, \dots, x_n) = Q_{x_n}$. It is clear that s is a winning strategy for II on X so the space X is not I-favorable which is a contradiction. □

COROLLARY 2.4. *If $\psi(X) = \omega$ and the space X is I-favorable with respect to $\mathcal{DG}$ then X is separable.*

PROOF. Observe that $\{\{x\} : x \in X\}$ is a cover of X with G_δ-sets and apply Proposition 2.3. □

COROLLARY 2.5. *If a space X is of pointwise countable type (in particular, if X is Čech-complete) and I-favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets then it has a dense σ-compact subspace.*

For a normal weakly Lindelöf space X it was proved by Bell, Ginsburg and Woods (see [7, Theorem 4.13]) that $|X| \le 2^{\chi(X)}$. Since it is an open question whether this inequality holds for all Tychonoff weakly Lindelöf spaces X, it is natural to prove it for new classes of weakly Lindelöf spaces. In our context this is the class of I-favorable spaces with respect to the game $\mathcal{DO}$ of dense open sets; we will show that in this class the inequality of Bell, Ginsburg and Woods still holds. To do so, we will need the following generalization of Theorem 2.11(i) of [11].

THEOREM 2.6. *If X is a I-favorable with respect to the game $\mathcal{DO}$ of dense open sets then $d(X) \le \chi(X)$. Consequently, if X is I favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets then $d(X) \le \chi(X)$.*

PROOF. Let $\kappa = \chi(X)$ and fix, for any $x \in X$, a local base $\mathcal{B}(x)$ of the space X at the point x such that $|\mathcal{B}(x)| \le \kappa$. Take a winning strategy s of the first player on the space X in $\mathcal{DO}$ and let $A_0 = \{s(\emptyset)\}$; proceeding inductively assume that we have sets $A_0 \subset \dots \subset A_n$ and $|A_n| \le \kappa$.

The cardinality of the set

$$A'_{n+1} = \{s(G_0, \dots, G_n) : G_i \in \bigcup\{\mathcal{B}(x) : x \in A_n\}$$

and there are $x_0, \dots, x_n \in A_n$ such that $x_0, G_0, \dots, x_n, G_n$ is an initial segment of a play in which I applies the strategy $s\}$ does not exceed κ. Therefore the set $A_{n+1} = A_n \cup A'_{n+1}$ also has cardinality at most κ and hence our inductive procedure can be continued to give us an increasing sequence $\{A_n : n \in \omega\}$ of subsets of X of cardinality $\leq \kappa$.

It suffices to show that $A = \bigcup_{n\in\omega} A_n$ is dense in X so assume that it is not. Fix a non-empty open set $O \subset X\backslash A$. It is easy to see that A has the following property:

(1) if $x_0, \dots, x_n \in A$ while $x_0, G_0, \dots, x_n, G_n$ is an initial segment of play in which I applies s and $G_0, \dots, G_n \in \mathcal{B} = \bigcup\{\mathcal{B}(x) : x \in A\}$ then $s(G_0, \dots, G_n) \in A$.

By regularity of X we can find a set $O_1 \in \tau^*(X)$ such that $\overline{O_1} \subset O$. Since $x_0 = s(\emptyset) \in A \subset X\backslash O$, we can find $U_0 \in \mathcal{B}(x_0)$ for which $U_0 \cap \overline{O}_1 = \emptyset$. Proceeding inductively suppose that $x_0, G_0, \dots, x_n, G_n$ is an initial segment of a play in which I applies s while $G_0, \dots, G_n \in \mathcal{B}$ and $G_i \cap \overline{O}_1 = \emptyset$ for all $i \leq n$. The property (1) shows that the point $x_{n+1} = s(G_0, \dots, G_n)$ belongs to A and hence $x_{n+1} \notin \overline{O}_1$. Therefore there exists $G_{n+1} \in \mathcal{B}(x_{n+1}) \subset \mathcal{B}$ with $G_{n+1} \cap \overline{O}_1 = \emptyset$. This shows that our inductive procedure can be continued to obtain a play $\{x_n, G_n : n \in \omega\}$ in which the first player applies s while $G_n \cap \overline{O}_1 = \emptyset$ for all $n \in \omega$. This implies that the set $\bigcup_{n\in\omega} G_n$ is not dense in X, i.e., the strategy s is not winning which is a contradiction. □

Corollary 2.7. *If a space X is I-favorable in the game $\mathcal{DO}$ of dense open sets then* $|X| \leq 2^{\chi(X)}$.

Proof. By Theorem 2.6 we have $c(X) \leq d(X) \leq \chi(X)$ so we can apply a theorem of Hajnal–Juhasz (see [7, Theorem 4.9]) to see that $|X| \leq 2^{c(X)\chi(X)} = 2^{\chi(X)}$. □

Example 2.8. There exists a space X which is I-favorable with respect to $\mathcal{DG}$ while $c(X) > \omega$ and $l(X) > \omega$.

Proof. Let X_0 be the one-point compactification of a discrete space of cardinality ω_1. If $x_0 = s(\emptyset)$ is the unique non-isolated point of X_0 and the second player takes a G_δ-set $G_0 \ni x_0$ then $X_0\backslash G_0$ is countable; take an enumeration $\{x_n : n \in \mathbf{N}\}$ of the set $X_0\backslash G_0$. If the first player chooses x_n at the n-th move for all $n \in \mathbf{N}$ then he/she wins no matter what the second player does. Therefore the space X_0 is I-favorable.

Now if $X_1 = \mathbf{D}^{\omega_1}\backslash\{p\}$, where $p \in \mathbf{D}^{\omega_1}$ is an arbitrary point, then X_1 is a separable non-Lindelöf space. Consequently, X_1 is I-favorable for $\mathcal{DG}$ and hence so is $X = X_0 \oplus X_1$ by Theorem 2.2. □

It was proved in [11] that it is consistent with ZFC that there exist non-separable spaces which are hereditarily I-favorable with respect to the game $\mathcal{DO}$ of dense open sets. It turns out that the hereditary version of I-favorability for $\mathcal{DG}$ coincides with hereditary separability.

THEOREM 2.9. *A space X is hereditarily I-favorable in the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if X is hereditarily separable.*

PROOF. Since sufficiency is clear, assume that X is hereditarily I-favorable for $\mathcal{DG}$. A discrete I-favorable space for $\mathcal{DG}$ is countable by Corollary 2.4 so $s(X) = \omega$. If X is not hereditarily separable then there is an uncountable left-separated space $Y \subset X$. Any left-separated space of countable spread is hereditarily Lindelöf (see [1, Theorem 1.2.9]) so $hl(Y) = \omega$ and hence $\psi(Y) = \omega$. Applying Corollary 2.4 to the space Y we conclude that $d(Y) = \omega$ and hence Y is countable which is a contradiction. □

PROPOSITION 2.10. *Any Lindelöf scattered space is I-favorable with respect to the game of dense $G_{\delta\sigma}$-sets.*

PROOF. Given a Lindelöf scattered space X apply Theorem 9.3 of the paper [8] to see that X is I-favorable in the game of Galvin–Telgársky. Now it follows from Theorem 2.2(i) that X is also I-favorable for $\mathcal{DG}$. □

EXAMPLE 2.11. The space ω_1 of all countable ordinals with its interval topology is countably compact and scattered; however, it follows from Theorem 2.6 that ω_1 is not I-favorable even with respect to the game $\mathcal{DO}$ of dense open sets because $d(\omega_1) = \omega_1 > \chi(\omega_1) = \omega$.

EXAMPLE 2.12. There exists a Lindelöf P-space which is neutral with respect to the game $\mathcal{DG}$.

PROOF. It was proved in [11] that there exists a neutral Lindelöf P-space X for the game $\mathcal{DO}$; since in P-spaces the games $\mathcal{DO}$ and $\mathcal{DG}$ coincide, the space X is also neutral for $\mathcal{DG}$. □

DEFINITION 2.13. *Given a space X and $Y \subset C_p(X)$ say that Y is strongly separating if for any $f, g \in Y$ we have $f + g \in Y$ and, for every $J \in \tau^*(\mathbf{R})$, if $F \subset X$ is a closed set and $x \in X \backslash F$ then there exists a function $f \in Y$ such that $f(F) \subset \{0\}$ and $f(x) \in J$.*

LEMMA 2.14. *If Y is a strongly separating subset of $C_p(X)$ then for every closed $F \subset X$, if $K \subset X \backslash F$ is a finite set and a set $J_x \in \tau^*(\mathbf{R})$ is chosen for each $x \in K$ then there exists a function $f \in Y$ such that $f(F) \subset \{0\}$ and $f(x) \in J_x$ for all $x \in K$.*

PROOF. For every $x \in K$ we can find a function $f_x \in Y$ such that $f_x(x) \in J_x$ and $f_x(F \cup (K \backslash \{x\})) \subset \{0\}$. It is clear that $f = \sum\{f_x : x \in K\}$ is the required function. □

THEOREM 2.15. *Suppose that X is a space and Y is a strongly separating subset of $C_p(X)$. Then Y is I-favorable with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if, for any countable set $A \subset X$, the space $\pi_{\overline{A}}(Y)$ is separable.*

PROOF. For any $A \subset X$ and $f \in Y$ let $[f, A] = \{g \in Y : g|A = f|A\}$. Assume first that Y is I-favorable with respect to $\mathcal{DG}$ and fix a countable set $A \subset X$. The space $\pi_{\overline{A}}(Y)$ is also I-favorable with respect to $\mathcal{DG}$ being a continuous image of the space Y; besides, $\psi(\pi_{\overline{A}}(Y)) \le \psi(C_p(\overline{A})) = \omega$ so the space $\pi_{\overline{A}}(Y)$ has to be separable by Corollary 2.4. This proves necessity.

Now assume that the space $\pi_{\overline{A}}(Y)$ is separable for any countable $A \subset X$ and fix any function $u \in Y$. Let $s(\emptyset) = f_0 = u$ and suppose that the second player takes a G_δ-set $G_0 \ni f_0$. It is easy to find a countable set $A_0 \subset X$ such that $[f_0, A_0] \subset G_0$; by our assumption there exists a countable set $E_0 \subset Y$ such that $\pi_{\overline{A}_0}(E_0)$ is dense in $\pi_{\overline{A}_0}(Y)$. Choose an infinite set $M_0 \subset \omega\backslash\{0\}$ such that $\omega\backslash M_0$ is also infinite and let $\{g_l : l \in M_0\}$ be an enumeration of the set E_0 in which every $g \in E_0$ occurs infinitely many times.

Proceeding inductively, assume that $k \in \omega$ and we have an initial segment $f_0, G_0, \dots, f_k, G_k$ of a play in $\mathcal{DG}$ on the space Y and countable sets A_i, E_i, M_i for all $i \le k$ with the following properties:

(2) $A_0 \subset \ldots \subset A_k \subset X$ and $[f_i, A_i] \subset G_i$ for all $i \le k$;
(3) $E_0, \dots, E_k \subset Y$ and $\pi_{\overline{A}_i}(E_i)$ is dense in $\pi_{\overline{A}_i}(Y)$ for each $i \le k$;
(4) the family $\mathcal{M}_i = \{M_j : j \le i\}$ is disjoint, $\bigcup\mathcal{M}_i \subset \omega\backslash\{0, \dots, i\}$ and the set $\omega\backslash(\bigcup\mathcal{M}_i)$ is infinite for every $i \le k$;
(5) for every $i \le k$, an enumeration $\{g_l : l \in M_i\}$ of the set E_i is taken in which every $g \in E_i$ occurs infinitely many times;
(6) if $i \le k$ and $i \in M_j$ for some $j < i$ then $f_i = g_i$.

If the number $k + 1$ does not belong to $\bigcup\mathcal{M}_k$, the first player's move is to take the function $f_{k+1} = u$ and let $s(G_0, \dots, G_k) = f_{k+1}$. If $k + 1 \in \bigcup\mathcal{M}_k$ then there exists a unique $i \le k$ such that $k + 1 \in M_i$; the first player's move in this case is $f_{k+1} = g_{k+1}$ and $s(G_0, \dots, G_k) = f_{k+1}$. If the second player responds with a G_δ-set $G_{k+1} \ni f_{k+1}$ then it is easy to see that there exists a countable set $A_{k+1} \subset X$ for which $A_k \subset A_{k+1}$ and $[f_{k+1}, A_{k+1}] \subset G_{k+1}$. Our assumption about the space Y makes it possible to find a countable set $E_{k+1} \subset Y$ such that $\pi_{\overline{A}_{k+1}}(E_{k+1})$ is dense in $\pi_{\overline{A}_{k+1}}(Y)$. Choose an infinite set $M_{k+1} \subset \omega\backslash((\bigcup\mathcal{M}_k) \cup \{0, \dots, k+1\})$ in such a way that the set $\omega\backslash((\bigcup\mathcal{M}_k) \cup M_{k+1})$ is still infinite and take an enumeration $\{g_l : l \in M_{k+1}\}$ of the set E_{k+1} in which every $g \in E_{k+1}$ occurs infinitely many times.

It is straightforward that the properties (2)–(6) are still fulfilled for all $i \le k+1$ so we completed our definition of a strategy s for the first player in the game of dense $G_{\delta\sigma}$-sets on Y. Suppose that $\{f_i, G_i : i \in \omega\}$ is a play in which I applies the strategy s. We also have the family $\{A_i, E_i, M_i : i \in \omega\}$ with the properties (2)–(6). To prove that $G = \bigcup_{i\in\omega} G_i$ is dense in Y fix a function $g \in Y$, a finite set $K \subset X$ and $\varepsilon > 0$. Let $O = \{f \in Y : |f(x) - g(x)| < \varepsilon$ for all $x \in K\}$; we must prove that $G \cap O \ne \emptyset$.

For $A = \bigcup_{i\in\omega} \overline{A}_i$ there exists $k \in \omega$ such that $K' = K \cap A = K \cap \overline{A}_k$. The set $O' = \{f \in \pi_{\overline{A}_k}(Y) : |f(x) - g(x)| < \varepsilon$ for every $x \in K'\}$ is non-empty and open in $\pi_{\overline{A}_k}(Y)$ so the properties (3) and (5) guarantee that there is $l \in M_k$ such that $l \geq k$ and $|g_l(x) - g(x)| < \varepsilon$ for all $x \in K'$. Since $f_l = g_l$ by the property (6), we also have $|f_l(x) - g(x)| < \varepsilon$ for all $x \in K'$. By our choice of k, the set $K'' = K \backslash K'$ does not meet $\overline{A}_l$; since Y is strongly separating, we can apply Lemma 2.14 to find a function $h \in Y$ such that $h(\overline{A}_l) \subset \{0\}$ and $|h(x) + f_l(x) - g(x)| < \varepsilon$ for all $x \in K''$. Now, if $w = h + f_l$ then $w|\overline{A}_l = f_l|\overline{A}_l$ so $w \in [f_l, \overline{A}_l] \subset G_l$; since also $w \in O$, it follows from $w \in O \cap G_l$ that $O \cap G \neq \emptyset$ and hence the set G is dense in Y, i.e., s is, indeed, a winning strategy for the first player in the game $\mathcal{DG}$ on the space Y. ∎

Corollary 2.16. *A space $C_p(X)$ is I-favorable in the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if, for every countable $A \subset X$, the set $C_p(\overline{A}|X)$ is separable.*

Corollary 2.17. *If X is normal, then $C_p(X)$ is I-favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if $iw(\overline{A}) = \omega$ for any countable $A \subset X$.*

Proof. Normality of X implies that $C_p(\overline{A}|X) = C_p(\overline{A})$ for each $A \subset X$; besides, $d(C_p(\overline{A}|X)) = d(C_p(\overline{A})) = iw(\overline{A})$ so $C_p(\overline{A}|X)$ is separable if and only if $iw(\overline{A}) = \omega$; Corollary 2.16 does the rest. ∎

Corollary 2.18. *If $l^*(\overline{A}) = iw(\overline{A}) = \omega$ for any countable $A \subset X$ (in particular, if X is an ω-monolithic space) then $C_p(X)$ is I-favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets.*

Proof. If $A \subset X$ is countable then $C_p(\overline{A})$ has countable tightness; this, together with separability of $C_p(\overline{A})$ implies that any dense subspace of $C_p(\overline{A})$ is separable. The set $C_p(\overline{A}|X)$ being dense in $C_p(\overline{A})$ we conclude that it is separable so $C_p(X)$ is I-favorable in $\mathcal{DG}$ by Corollary 2.16. ∎

Corollary 2.19. *If X is a scattered Lindelöf space then $C_p(X)$ is I-favorable for the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if $\overline{A}$ is countable for any countable set $A \subset X$.*

Proof. Sufficiency is a trivial consequence of Corollary 2.18. If $C_p(X)$ is I-favorable for $\mathcal{DG}$ and $A \subset X$ is countable then $\overline{A}$ has a weaker second countable topology by Corollary 2.17. Since every continuous second countable image of a Lindelöf scattered space is countable, we conclude that $\overline{A}$ is countable. ∎

Corollary 2.20. *If $C_p(X)$ is ω-stable then it is I-favorable for $\mathcal{DG}$.*

Corollary 2.21. *If X is ω-stable (in particular, if X is a Lindelöf Σ-space) then $C_{p,2n+2}(X)$ is I-favorable for $\mathcal{DG}$ for all $n \in \omega$.*

Corollary 2.22. *If X is ω-monolithic (in particular, if X is metrizable) then $C_{p,2n+1}(X)$ is I-favorable for $\mathcal{DG}$ for all $n \in \omega$.*

Theorem 2.23. *If X is a Lindelöf Σ-space then $C_p(X)$ is I-favorable with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets if and only if X is ω-monolithic.*

Proof. Sufficiency is a consequence of Corollary 2.18. Now, if $C_p(X)$ is I-favorable for $\mathcal{DG}$ and $A \subset X$ is countable then it follows from normality of X and Corollary 2.17 that $iw(\overline{A}) = \omega$. However, if a Lindelöf Σ-space has countable i-weight then it has a countable network (see [2, Corollary II.6.27]) so $nw(\overline{A}) = \omega$ and hence X is ω-monolithic. □

Theorem 2.24. *Any product of separable spaces is I-favorable with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets.*

Proof. If Y is a product of separable spaces then it is easy to see that there exists a cardinal κ such that Y contains a dense continuous image of ω^κ. By Theorem 2.2 it suffices to show that the space ω^X is I-favorable for $\mathcal{DG}$ for every set X. It is immediate that ω^X is a continuous image of the space $\mathbf{Q}^X$. If we consider X with the discrete topology then $C_p(X) = \mathbf{R}^X$ and $\mathbf{Q}^X$ is a strongly separating subset of $C_p(X)$. Since $\overline{A} = A$ for every countable $A \subset X$, the space $\pi_{\overline{A}}(\mathbf{Q}^X) = \pi_A(\mathbf{Q}^X) = \mathbf{Q}^A$ is even second countable so we can apply Theorem 2.15 to see that $\mathbf{Q}^X$ is I-favorable for $\mathcal{DG}$. □

Corollary 2.25. *Any dyadic compact space is I-favorable with respect to $\mathcal{DG}$.*

Proposition 2.26. *Suppose that X_t is a space and a set $Y_t \subset C_p(X_t)$ is strongly separating for each $t \in T$. If Y_t is I-favorable with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets for every $t \in T$ then $Y = \prod_{t\in T} Y_t$ is I-favorable with respect to $\mathcal{DG}$. In particular, if every $C_p(X_t)$ is I-favorable for $\mathcal{DG}$ then $\prod\{C_p(X_t) : t \in T\}$ is I-favorable with respect to $\mathcal{DG}$.*

Proof. If $X = \bigoplus\{X_t : t \in T\}$ then $C_p(X)$ is canonically homeomorphic to $\prod\{C_p(X_t) : t \in T\}$ and it is easy to see that, applying this canonical homeomorphism, we can identify Y with a strongly separating subset of $C_p(X)$. If $A \subset X$ is a countable set then let $A_t = A \cap X_t$ for each $t \in T$; there is a countable $S \subset T$ such that $A = \bigcup\{A_t : t \in S\}$ and hence $\overline{A} = \bigoplus\{\overline{A}_t : t \in S\}$. It is straightforward that $\pi_{\overline{A}}(Y)$ is homeomorphic to $\prod\{\pi_{\overline{A}_t}(Y_t) : t \in S\}$; since every $\pi_{\overline{A}_t}(Y_t)$ is separable by Theorem 2.15, the space $\pi_{\overline{A}}(Y)$ is also separable so we can apply Theorem 2.15 again to conclude that Y is I-favorable for $\mathcal{DG}$. □

3. Open problems

This paper is the very first step on the way of the study of the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets. We hope that the obtained results show that this game provides an interesting insight into generalizations of separable spaces. The open problems presented below outline possible new ways to develop the subject.

PROBLEM 3.1. Suppose that $iw(\overline{A}) = \omega$ for any countable $A \subset X$. Must $C_p(X)$ be I-favorable for $\mathcal{DG}$?

PROBLEM 3.2. Suppose that X and Y are I-favorable spaces with respect to the game $\mathcal{DG}$ of dense $G_{\delta\sigma}$-sets. Must the space $X \times Y$ be also I-favorable with respect to $\mathcal{DG}$?

PROBLEM 3.3. Suppose that G is a pseudocompact topological group. Must G be I-favorable with respect to $\mathcal{DG}$?

PROBLEM 3.4. Is true that any product of spaces I-favorable with respect to $\mathcal{DG}$ is I-favorable with respect to $\mathcal{DG}$?

PROBLEM 3.5. Suppose that a space X is I-favorable with respect to $\mathcal{DG}$. Must $C_p(C_p(X))$ be I-favorable with respect to $\mathcal{DG}$?

PROBLEM 3.6. Is it true that $d(X) \leq \psi(X)$ for any space X which is I-favorable with respect to $\mathcal{DG}$?

PROBLEM 3.7. Is there a compact space which is neutral for the game $\mathcal{DG}$?

PROBLEM 3.8. Is it true that $L_p(K)$ is I-favorable with respect to $\mathcal{DG}$ for any compact space K?

PROBLEM 3.9. Find a characterization for an Eberlein compact space K to be I-favorable for $\mathcal{DG}$. For example, is it true that K is I-favorable for $\mathcal{DG}$ if and only if K is *metrizably scattered*, i.e., any non-empty subspace of K has a non-empty open metrizable subspace?

PROBLEM 3.10. Suppose that K and L are Eberlein compact spaces which are I-favorable for $\mathcal{DG}$. Must $X \times Y$ be I-favorable for $\mathcal{DG}$?

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