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ČECH-COMPLETE MAPS

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Abstract. We introduce a new notion of "Čech-complete map", and investigate some its basic properties, invariance under perfect maps, characterizations by compactifications of Čech-complete maps and relationships between (locally) compact map, Čech-complete map and k-map.

1. Introduction

In this paper we investigate Čech-completeness on continuous maps, which we call *Čech-complete map*. This new notion of "Čech-complete map" is a continuation of generalizing the main notions and theory concerning spaces to that of maps, and is also a generalization of compact maps [10] (i.e. perfect maps) which are very important in General Topology. For the studies of paracompact maps, covering properties on maps and metrizable type maps, see [2, 3, 4], respectively. This branch of General Topology is known as General Topology of Continuous Maps or Fibrewise General Topology.

For an arbitrary topological space B let us consider the category TOP_B, the objects of which are continuous maps into the space B, and for the objects $p : X \to B$ and $q : Y \to B$, a *morphism* from p into q is a continuous map $f : X \to Y$ with the property $p = q \circ f$. This is denoted by $f : p \to q$. A morphism $f : p \to q$ is onto, closed, perfect, if respectively, such is the map $f : X \to Y$. An object $p : X \to B$ of TOP_B is called a *projection*, and X or (X, p) is called a *fibrewise space*. We also call a morphism $f : p \to q$ a *fibrewise map* when we write $f : (X, p) \to (Y, q)$ or $f : X \to Y$.

In defining properties of a projection $p : X \to B$ one does not directly involve any properties on the spaces X and B (except the existence of a

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topology). Such were the definitions given in [8, 9] for the separation axioms, compactness, weight and others. We note that this situation is a generalization of the category TOP (of topological spaces and continuous maps as morphisms), since the category TOP is isomorphic to the particular case of TOP_B in which the space B is a singleton set.

In section 2 of this paper, we refer to the notions and notations in TOP_B. In section 3, we define a Čech-complete map, and investigate some its basic properties. In section 4, we study the invariance of Čech-complete maps under perfect maps. In section 5, we investigate some characterizations of Čech-complete map by its compactifications. In section 6, we study the relationships between locally compact map, Čech-complete map and k-map.

Throughout this paper, we assume that all spaces are topological spaces, and all maps and projections are continuous. For other terminology and notations undefined in this paper, one can consult [5] about TOP, and [6, 4] about TOP_B.

2. Preliminaries

In this section, we refer to the notions and notations in Fibrewise Topology, which are used in the latter sections.

Let (B, τ) be a topological space B with a fixed topology τ. Throughout this paper, we will use the abbreviation *nbd(s)* for *neighborhood(s)*. For each $b \in B$, $N(b)$ is the set of all open nbds of b, and $\mathbf{N}$, $\mathbf{Q}$ and $\mathbf{R}$ are the sets of all natural numbers, all rational numbers and all real numbers, respectively. Note that the regularity of (B, τ) is assumed in Theorems 3.4, 5.1, 6.1 and 6.3, further the first axiom of countability of (B, τ) is assumed in Theorem 6.3.

For a projection $p : X \to B$ and each point $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset B' of B we regard $X_{B'} = p^{-1}(B')$ as a fibrewise space over B' with the projection determined by p. For a filter (base) $\mathcal{F}$ on X, $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F}\}$. For a fibrewise map $f : (X, p) \to (Y, q)$, for a filter (base) $\mathcal{F}$ on X, we define $f_*(\mathcal{F})$ as same. For a filter (base) $\mathcal{G}$ on Y, $f^*(\mathcal{G})$ is the filter generated by the family $\{f^{-1}(U) | U \in \mathcal{G}\}$.

First, we begin to define some separation axioms on maps.

Definition 2.1. *A projection $p : X \to B$ is called a T_i-*map*, $i = 0, 1, 2$ (T_2 is also called* Hausdorff*), if for all $x, x' \in X$ such that $x \neq x'$, $p(x) = p(x')$ the following condition is respectively satisfied:*

1. $i = 0$*: at least one of the points x, x' has a nbd in X not containing the other point;*
2. $i = 1$*: each of the points x, x' has a nbd in X not containing the other point;*

3. $i = 2$*: the points* x *and* x' *have disjoint nbds in* X.

Definition 2.2. *(1) A* T_0*-map* $p : X \to B$ *is called* regular *if for every point* $x \in X$ *and every closed set* F *in* X *with* $x \notin F$*, there exists a nbd* $W \in N(p(x))$ *such that* $\{x\}$ *and* $F \cap X_W$ *have disjoint nbds in* X_W
(2) A T_1*-map* $p : X \to B$ *is called* normal *if for every* $O \in \tau$*, every pair of closed disjoint sets* F_1, F_2 *of* X *and every* $b \in O$*, there exists* $W \in N(b)$ *with* $W \subset O$ *such that* $F_1 \cap X_W$ *and* $F_2 \cap X_W$ *have disjoint nbds in* X_W.

We now give the definitions of submap, compact map [10] and locally compact map [8].

Definition 2.3. (1) *The restriction of the projection* $p : X \to B$ *to a closed (resp. open, type* G_δ*, etc.) subset of the space* X *is called a* closed *(resp.* open, type G_δ*, etc.)* submap *of the map* p.

(2) *A projection* $p : X \to B$ *is called a* compact map *if it is perfect (i.e. it is closed and all its fibres* $p^{-1}(b)$ *are compact).*

(3) *A projection* $p : X \to B$ *is said to be a* locally compact map *if for each* $x \in X_b$*, where* $b \in B$*, there exists a nbd* $W \in N(b)$ *and a nbd* $U \subset X_W$ *of* x *such that* $q : X_W \cap \overline{U} \to W$ *is a compact map, where* $X_W \cap \overline{U}$ *is the closure of* U *in* X_W *and* q *is the restriction of* p *to it.*

Note that a closed submap of a (resp. locally) compact map is (resp. locally) compact, and for a (resp. locally) compact map $p : X \to B$ and every $B' \subset B$ the restriction $p|X_{B'} : X_{B'} \to B'$ is (resp. locally) compact.

Definition 2.4. (1) *For a map* $p : X \to B$*, a map* $c(p) : c_pX \to B$ *is called a* compactification *of* p *if* $c(p)$ *is compact,* X *is dense in* c_pX *and* $c(p)|X = p$.

(2) *A map* $p : X \to B$ *is called a* T_2-compactifiable map *if* p *has a compactification* $c(p) : c_pX \to B$ *which is a* T_2*-map.*

The following holds.

Proposition 2.5. (1) *For* $i = 0, 1, 2$*, every submap of a* T_i*-map is also a* T_i*-map. Every submap of a regular map is also regular.*

(2) *Compact* T_2*-map* $\Longrightarrow$ *normal map* $\Longrightarrow$ *regular map* $\Longrightarrow$ T_2*-map.*

(3) *(*[6, Section 8]*) Every normal map is* T_2*-compactifiable.*

(4) *(*[6, Section 8]*) Every locally compact* T_2*-map is* T_2*-compactifiable.*

Remark 2.6. The approach to the compactification of a map was proposed by Whyburn [11], and Pasynkov studied it in [8]. In James [6, Section 8], there is some basic study of compactifiable maps, but note that he uses a terminology "fibrewise compactification". For other study of compactifiable maps, see [1, 7].

Definition 2.7. *For the collection of fibrewise spaces* $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$*, the subspace* $X = \{t = \{t_\alpha\} \in \prod\{X_\alpha|\alpha \in \Lambda\}|\ p_\alpha t_\alpha = p_\beta t_\beta\ \forall \alpha, \beta \in \Lambda\}$ *of the*

Tychonoff product $\prod = \prod\{X_\alpha|\alpha \in \Lambda\}$ *is called the* fan product *of the spaces* X_α *with respect to the maps* p_α, $\alpha \in \Lambda$.

For the projection $pr_\alpha : \prod \to X_\alpha$ *of the product* $\prod$ *onto the factor* X_α, *the restriction* π_α *to* X *will be called the projection of the fan product onto the factor* $X_\alpha, \alpha \in \Lambda$. *From the definition of fan product we have that,* $p_\alpha \circ \pi_\alpha = p_\beta \circ \pi_\beta$ *for every* α *and* β *in* Λ. *Thus one can define a map* $p : X \to B$, *called the* product of the maps p_α, $\alpha \in \Lambda$, *by* $p = p_\alpha \circ \pi_\alpha$, $\alpha \in \Lambda$, *and* (X, p) *is called the* fibrewise product space *of* $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$.

Obviously, the projections p *and* $\pi_\alpha, \alpha \in \Lambda$, *are continuous.*

The following proposition holds.

Proposition 2.8. *Let* $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$ *be a collection of fibrewise spaces.*

1. *If each* p_α *is* T_i $(i = 0, 1, 2)$, *then the product* p *is also* T_i $(i = 0, 1, 2)$.
2. *If each* p_α *is a* surjective *regular map, then the product* p *is also a regular map*
3. *(*[6, Prop. 3.5]*) If each* p_α *is a compact map, then the product* p *is also a compact map.*
4. *If each* p_α *is a* T_2*-compactifiable map, then the product* p *is also a* T_2*-compactifiable map.*

We shall conclude this section by defining the concept of b-filters (or tied filters) which plays an important role in this paper.

Definition 2.9 ([6, Section 4]). *For a fibrewise space* (X, p), *by a* b-filter *(or* tied filter*) on* X *we mean a pair* $(b, \mathcal{F})$, *where* $b \in B$ *and* $\mathcal{F}$ *is a filter on* X *such that* b *is a limit point of the filter* $p_*(\mathcal{F})$ *on* B. *By an* adherence point *of a* b*-filter* $\mathcal{F}$ $(b \in B)$ *on* X, *we mean a point of the fibre* X_b *which is an adherence point of* $\mathcal{F}$ *as a filter on* X.

3. Definition and Basic properties of Čech-complete maps

In this section, we define a Čech-complete map and investigate some its basic properties. First, we shall begin with the following definition.

Definition 3.1. *Let* X *be a topological space, and* A *a subset of* X. *We say that the* diameter of A is less than a family $\mathcal{A} = \{A_s\}_{s \in S}$ *of subsets of the space* X, *and we shall write* $\delta(A) < \mathcal{A}$, *provided that there exists an* $s \in S$ *such that* $A \subset A_s$.

Definition 3.2. *A* T_2*-compactifiable map* $p : X \to B$ *is* Čech-complete *if for each* $b \in B$, *there exists a countable family* $\{\mathcal{A}_n\}_{n \in \mathbf{N}}$ *of open (in* X*) covers of* X_b *with the property that every* b*-filter* $\mathcal{F}$ *which contains sets of diameter less than* $\mathcal{A}_n$ *for every* $n \in \mathbf{N}$ *has an adherence point.*

Since the real line $\mathbf{R}$ with the usual topology is Čech-complete, $p:\mathbf{R}\to B$ is Čech-complete where B is a one-point space. All rational numbers $\mathbf{Q}$, as a subset of $\mathbf{R}$, is not Čech-complete, thus $p|\mathbf{Q}$ is not Čech-complete though $p|\mathbf{Q}$ is open and closed. But we have the following results.

THEOREM 3.3. *For a Čech-complete map $p : X \to B$, if F is a closed subset of X, then $p|F : F \to B$ is Čech-complete.*

PROOF. Since $p : X \to B$ is Čech-complete, for each $b \in B$ there exists a countable family $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the property in Definition 3.2. Let $\mathcal{U}_n = \mathcal{A}_n|F$ for every $n \in \mathbf{N}$, where $\mathcal{A}_n|F = \{U \cap F|U \in \mathcal{A}_n\}$, then $\{\mathcal{U}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in F) covers of F_b in F. For every b-filter $\mathcal{F}$ on the space F which contains sets of diameter less than $\mathcal{U}_n$ for every $n \in \mathbf{N}$, it generates a b-filter $\mathcal{F}_1$ which contains sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$, thus the b-filter $\mathcal{F}_1$ has an adherence point $x \in X_b$ in the space X. Since $F \in \mathcal{F}_1$ is closed in the space X and $\mathcal{F} \subset \mathcal{F}_1$, x is also an adherence point of the b-filter $\mathcal{F}$ in the space F, so $p|F$: $F \to B$ is Čech-complete. □

THEOREM 3.4. *Assume that B is regular. For a Čech-complete map $p : X \to B$, if G is a G_δ-subset of X, then $p|G$: $G \to B$ is Čech-complete.*

PROOF. First note that X is regular from the regularity of B and the fact that p is T_2-compactifiable. Let $G = \bigcap_{n\in\mathbf{N}} G_n$ with G_n open in X for every $n \in \mathbf{N}$. Since $p : X \to B$ is Čech-complete, for each $b \in B$ there exists a countable family $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the property in Definition 3.2. For every $n \in \mathbf{N}$ and $x \in G_b$, there exist $A_n(x) \in \mathcal{A}_n$ and an open nbd $V_n(x)$ of x such that $V_n(x) \subset \overline{V_n(x)} \subset G_n \cap A_n(x)$ in the space X. Let $\mathcal{V}_n = \{V_n(x) \cap G|x \in G_b\}$, then $\{\mathcal{V}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in G) covers of G_b. For every b-filter $\mathcal{F}$ on the space G which contains sets of diameter less than $\mathcal{V}_n$ for every $n \in \mathbf{N}$, there exist $F_n \in \mathcal{F}$, $x \in G_b$ and $V_n(x) \cap G \in \mathcal{V}_n$ for every $n \in \mathbf{N}$ such that

$$(*) \qquad F_n \subset V_n(x) \cap G \subset V_n(x) \subset \overline{V_n(x)} \subset G_n \cap A_n(x).$$

The b-filter $\mathcal{F}_1$ on X generated by b-filter $\mathcal{F}$ contains sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$, therefore it has an adherence point y in X. From (*) and $\mathcal{F} \subset \mathcal{F}_1$, it holds that $y \in \bigcap_{n\in\mathbf{N}} G_n = G$ and y is an adherence point of the b-filter $\mathcal{F}$ in the space G, so $p|G$: $G \to B$ is Čech-complete. □

About the product of Čech-complete maps, we have the following.

THEOREM 3.5. *Let $\{(X_n, p_n)|n \in \mathbf{N}\}$ be a countable family of fibrewise spaces and (X, p) be the fibrewise product space of $\{(X_n, p_n)|n \in \mathbf{N}\}$, where $X = \prod_B X_n$. If each p_n is a surjective Čech-complete map, then the product p is Čech-complete.*

PROOF. For every $n \in \mathbf{N}$ and $b \in B$, since p_n: $X_n \to B$ is Čech-complete, there exists a countable family $\{\mathcal{A}_{ni}\}_{i\in\mathbf{N}}$ of open (in X_n) covers of $(X_n)_b$ with the property in Definition 3.2. For every $n \in \mathbf{N}$, let $\mathcal{A}_n = \mathcal{A}_{1n} \times_B \mathcal{A}_{2(n-1)} \times_B \cdots \times_B \mathcal{A}_{n1} \times_B (\prod_B X_k)_{k>n}$, then $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in X) covers of X_b. Let $\mathcal{F}$ be a b-filter containing sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$. Then each $n \in \mathbf{N}$ there exist $A_n = A_{1n} \times_B A_{2(n-1)} \times_B \cdots \times_B A_{n1} \times_B (\prod_B X_k)_{k>n} \in \mathcal{A}_n$ and $F_n \in \mathcal{F}$ such that $F_n \subset A_n$ (and in this case, $\pi_j(F_n) \subset A_{j(n-(j-1))}$ for every $j \leq n$, where $\pi_j : \prod_B X_n \to X_j$ is the projection). Thus for each $n \in \mathbf{N}$, $(\pi_n)_*(\mathcal{F})$ is a b-filter on the space X_n which contains sets of diameter less than $\mathcal{A}_{ni}$ for every $i \in N$, therefore it has an adherence point x_n in the space X_n and $\mathcal{F}$ has an adherence point $(x_n)_{n\in\mathbf{N}}$ in the space X, Thus $p : X \to B$ is Čech-complete. □

4. Invariance under perfect maps

In this section, we study the invariance of Čech-complete maps under perfect maps. We have the following.

THEOREM 4.1. *Let a fibrewise map $f : (X, p) \to (Y, q)$ be a perfect map, and p and q be T_2-compactifiable maps. Then p is Čech-complete if and only if q is Čech-complete.*

PROOF. "Only if" part: Suppose that $p : X \to B$ is Čech-complete. For each $b \in B$ there exists a countable family $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the property in Definition 3.2. From the perfectness of $f : X \to Y$, for each $n \in \mathbf{N}$ and $y \in Y_b$ there exist finite elements $U_{n1}, U_{n2}, \ldots, U_{ni(y)}$ of $\mathcal{A}_n$ and an open nbd $V_{n,y}$ of y such that $f^{-1}(y) \subset f^{-1}(V_{n,y}) \subset \bigcup_{k\leq i(y)} U_{nk}$. For every $n \in \mathbf{N}$, let $\mathcal{V}_n = \{V_{n,y} | y \in Y_b\}$. Then $\{\mathcal{V}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in Y) covers of Y_b. For every b-filter $\mathcal{F}$ on the space Y containing sets of diameter less than $\mathcal{V}_n$ for every $n \in \mathbf{N}$, let $\mathcal{F}_1$ is the b-ultrafilter generated by the b-filter $f^*\mathcal{F}$ on the space X. Since for every $n \in \mathbf{N}$ there exists $F_n \in \mathcal{F}$ and $V_{n,y} \in \mathcal{V}_n$ such that $F_n \subset V_{n,y}$, thus $f^{-1}(F_n) \in f^*\mathcal{F} \subset \mathcal{F}_1$ and $f^{-1}(F_n) \subset \bigcup_{k\leq i(y)} U_{nk}$. Since $\mathcal{F}_1$ is a b-ultrafilter on the space X, from [6, Proposition 4.1] there exists $k_0 \leq i(y)$ such that $U_{nk_0} \in \mathcal{F}_1$ which shows that the b-filter $\mathcal{F}_1$ contains sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$. Therefore $\mathcal{F}_1$ has an adherence point x in X and the b-filter $\mathcal{F}$ has an adherence point $f(x)$ in Y, thus $q : Y \to B$ is Čech-complete.

"If" part: Suppose that $q : Y \to B$ is Čech-complete. For each $b \in B$ there exists a countable family $\{\mathcal{V}_n\}_{n\in\mathbf{N}}$ of open (in Y) covers of Y_b with the property in Definition 3.2. Let $\mathcal{A}_n = \{f^{-1}(V) | V \in \mathcal{V}_n\}$ for every $n \in \mathbf{N}$, then $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in X) covers of X_b. For every b-filter $\mathcal{F}$ on the space X containing sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$, it is easy to see that $\mathcal{F}_1 = f_*\mathcal{F}$ is a b-filter on Y which contains sets of diameter less than $\mathcal{V}_n$ for every $n \in \mathbf{N}$, thus it has an adherence point y in Y. From the perfectness of $f : X \to Y$ and [6, Proposition 4.3] the b-filter $\mathcal{F}$ has an adherence point $x \in f^{-1}(y)$, thus $p : X \to B$ is Čech-complete. □

5. Characterizations of Čech-complete maps by compactifications of maps

In this section, we investigate some characterizations of Čech-complete maps by compactifications of the maps. Further, we give an example showing there exists a projection $p : X \to B$ satisfying each fibre is Čech-complete, but p is not Čech-complete. We can prove the following theorem.

Theorem 5.1. *Suppose that B is regular. For a T_2-compactifiable map $p : X \to B$, the following are equivalent:*

(1) *p is Čech-complete.*
(2) *For every T_2-compactification $p' : X' \to B$ of p and each $b \in B$, X_b is a G_δ-subset of X'_b.*
(3) *There exists a T_2-compactification $p' : X' \to B$ of p such that X_b is a G_δ-subset of X'_b for each $b \in B$.*

Proof. (1)$\Longrightarrow$(2): Let $p' : X' \to B$ is a T_2-compactification of the map p, then X is a subset of X' and $\overline{X} = X'$. Since p is Čech-complete, there exists a countable family $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the property in Definition 3.2. For every $n \in \mathbf{N}$ and $A_\alpha \in \mathcal{A}_n = \{A_\alpha | \alpha \in \Gamma_n\}$, take an open subset U_α of X' such that $A_\alpha = X' \cap U_\alpha$. Let $G_n = \bigcup\{U_\alpha | \alpha \in \Gamma_n\}$ and $G = \bigcap_{n\in\mathbf{N}} G_n$, then it holds that $X_b = G \cap X'_b$. In fact, if $X_b \neq G \cap X'_b$, there exists a point $x \in (X'_b \cap G) - X_b$. The family $\mathcal{F} = \{F |\ F$ is a nbd of x in $X'\}$ is a b-filter with x its an adherence point in the space X'. From $\overline{X} = X'$ and $x \in X'_b \cap G$, it is easily verified that $\mathcal{F}_1 = \mathcal{F}|X$ is a b-filter on X with the property that for every $n \in \mathbf{N}$ there exists an element $F_n \in \mathcal{F}_1$ and $A_n \in \mathcal{A}_n$ such that $F_n \subset A_n$. Therefore the b-filter $\mathcal{F}_1$ has an adherence point x_0 in X. It is obvious that x_0 is an adherence point of the b-filter $\mathcal{F}$ in X' with $x \neq x_0$, which contradicts with the fact that $p' : X' \to B$ is T_2. Thus X_b is a G_δ-subset of X'_b.

(2)$\Longrightarrow$(3): Obviously.

(3)$\Longrightarrow$(1): In this proof, we use the notation $\overline{M}^{X'}$ as the closure of $M \subset X'$ in X'. Let $p' : X' \to B$ be a T_2-compactification of the map $p : X \to B$ such that for every $b \in B$, X_b is a G_δ-subset of X'_b. We can suppose $X_b = \bigcap_{n\in\mathbf{N}} G_{0n}$ where G_{0n} is open in the space X'_b for every $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, take an open subset G_n of X' such that $G_{0n} = G_n \cap X'_b$. Since $p' : X' \to B$ is a compact T_2-map and B is regular, for every $n \in \mathbf{N}$ and $x \in X_b$ there exists an open nbd V_{xn} of x such that $V_{xn} \subset \overline{V_{xn}}^{X'} \subset G_n$. Let $\mathcal{V}_n = \{V_{xn} \cap X | x \in X_b\}$, then $\{\mathcal{V}_n\}_{n\in\mathbf{N}}$ is a countable family of open (in X) covers of X_b. For every b-filter $\mathcal{F}$ which contains sets of diameter less than $\mathcal{V}_n$ for every $n \in \mathbf{N}$, there exists an element $F_n \in \mathcal{F}$ and $x_n \in X_b$ such that

$F_n \subset V_{x_n n} \cap X \in \mathcal{V}_n$, thus $F_n \subset \overline{F_n}^{X'} \subset \overline{V_{x_n n}}^{X'} \subset G_n$. Let $\mathcal{F}_1$ be the b-filter generated by $\mathcal{F}$ on the space X', then $\mathcal{F}_1$ has an adherence point x in the space X' from the compactness of $p' : X' \to B$. Since $x \in \overline{F_n}^{X'}$ for every $n \in \mathbf{N}$, it holds that $x \in (\bigcap_{n \in \mathbf{N}} G_n) \cap X'_b = X_b$. Thus the b-filter $\mathcal{F}$ has an adherence point in X, and therefore $p : X \to B$ is Čech-complete. ∎

From Definition 3.2, if $p : X \to B$ is Čech-complete, then each fibre of p is Čech-complete. But the following example shows the converse is false even if B is a compact T_2-space.

Example 5.2. There exists a map $p : X \to B$ satisfying the following:

(1) p is a T_2-compactifiable map.
(2) Each fibre of p is Čech-complete, but p is not a Čech-complete map.
(3) p is an open map. (Note that p is not a closed map.)

Construction: First, note that the space X is the same space constructed in [5, Example 1.6.19]. Let $X = \{0\} \cup (\bigcup_{m \in \mathbf{N}} X_m)$ where

$$X_m = \{\frac{1}{m}\} \cup \{\frac{1}{m} + \frac{1}{m^2 + k} \ |k \in \mathbf{N}\},$$

$A_0 = \{0\} \cup \{\frac{1}{m} \mid m \in \mathbf{N}\}$ and $Y_m = X_m - \{\frac{1}{m}\}$ for every $m \in \mathbf{N}$. We denote the n-th element of Y_m by P_{mn}. The topology on X is generated by a nbd system defined as follows: For $x \in Y_m$ for each $m \in \mathbf{N}$, $\mathcal{B}(x) = \{\{x\}\}$; for $x = \frac{1}{m}$, $\mathcal{B}(x) = \{U_n | n \in \mathbf{N}\}$, where $U_n = X_m - \{P_{mi} | i \leq n\}$; for $x = 0$,

$$\begin{aligned}\mathcal{B}(0) = \{U \mid & \text{there exists a finite set } H \subset \mathbf{N} \text{ and } F \subset X - A_0 \text{ satisfying} \\ & F \cap B_m \text{ is finite for each } m \in \mathbf{N} \text{ such that} \\ & U = X - (F \cup \bigcup_{m \in H} X_m)\}.\end{aligned}$$

This space X is perfectly normal and sequential [5, Example 1.6.19].

For each $n \in \mathbf{N}$, let $A_n = \{P_{mn} | m \in \mathbf{N}\}$, then $\{A_n\}_{n \geq 0}$ is a decomposition of X generating a quotient space $B = \{b_n | b_n = A_n \text{ and } n \geq 0\}$, which is a compact T_2-space. We denote the quotient map by $p : X \to B$.

Let $W_n = \{b_0\} \cup \{b_i | i \geq n\}$ for each $n \in \mathbf{N}$. Then it is easily verified that $\{W_n | n \in \mathbf{N}\}$ is a nbd base of b_0 in B.

(1): Since X is a perfectly normal space, p is a normal map, therefore p is T_2-compactifiable by Proposition 2.5 (3).

(2): First, it is obvious that $p^{-1}(b)$ is Čech-complete for each $b \in B$.

Next, we shall show that p is not Čech-complete. For $b_0 \in B$ and every countable family $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ of open covers of $p^{-1}(b_0) = A_0$, there exists $U_n \in \mathcal{U}_n$ such that $0 \in U_n$ and we can suppose that $U_n \in \mathcal{B}(0)$ for every $n \in \mathbf{N}$. Let $F_1 = U_1$ and $x_1 = \max F_1$. For every $n \geq 2$, let

$$F_n = U_n \cap F_{n-1} \cap (X - (\bigcup_{i<n} (X_i \cup A_i)))$$

and $x_n = \max F_n$. Let $F_0 = \{x_n | n \in \mathbf{N}\}$. Then it is easy to see that the family $\mathcal{F}$=$\{F_n | n \geq 0\}$ contains sets of diameter less than $\mathcal{U}_n$ for every $n \in \mathbf{N}$ and has the finite intersection property. Since $p(F_n) \subset W_n$ and $\{W_n | n \in \mathbf{N}\}$ is a nbd base of b_0, $\mathcal{F}$ generates a b_0-filter $\mathcal{F}'$ on X. For every $x \in A_0$, if $x = \frac{1}{m}$, then A_m is an open nbd of $\frac{1}{m}$ with $A_m \cap F_n = \emptyset$ for every $n > m$; if $x = 0$, then $U = X - F_0$ is an open nbd of 0 with $U \cap F_0 = \emptyset$. So the b_0-filter $\mathcal{F}'$ has no adherence point in X, and thus p is not Čech-complete.

Proof of (3): First, it is easily verified that the map p is open, because the image of any (three types) basic nbd in X under the map p is open in B.

Next, we shall show that p is not closed. Let $F = \{P_{nn} | n \in \mathbf{N}\}$. Then it is easy to see that F is closed in X but $p(F) = B - \{b_0\}$ is not closed in B. Thus p is not closed.

6. Relationships between Čech-complete maps and other maps

In this section, we shall prove the following implications under some conditions:

Locally compact map $\Longrightarrow$ Čech-complete map $\Longrightarrow$ k-map

where a k-map $p : X \to B$ is same as that X is a fibrewise compactly generated space over B ([6, Section 10] and Definition 6.2 in this section).

Theorem 6.1. *Suppose that B is regular. Every locally compact T_2-map is Čech-complete.*

Proof. Let $p : X \to B$ be a locally compact T_2-map. Then from [6, Section 8], the map p has the compactification $p' : X' \to B$ such that for each $b \in B$, $X'_b - X_b$ is a one-point set and X is open in X' (note that James call the space X' the *fibrewise Alexandroff compactification* of X). By this result, it is clear that p is a T_2-compactifiable map, and for each $b \in B$ X_b is an open subset of X'_b. Thus, p is Čech-complete from Theorem 5.1. □

We shall define a k-map as follows:

Definition 6.2. *(James* [6, Definitions 10.1 and 10.3]*)*

(1) *Let (X, p) be a fibrewise space. The subset H of X is* quasi-open *(resp.* quasi-closed*) if the following condition is satisfied: for each $b \in B$ and $V \in N(b)$ there exists a nbd $W \in N(b)$ with $W \subset V$ such that whenever $p|K : K \to W$ is compact then $H \cap K$ is open (resp. closed) in K.*
(2) *Let a projection $p : X \to B$ be a T_2-map. The map p is a* k-map *if every quasi-closed subset of X is closed in X or, equivalently, if every quasi-open subset of X is open in X.*

From [6, Proposition 10.2] we know that every locally compact T_2-map is a k-map. Further, we can prove the following.

Theorem 6.3. *Suppose that B is regular and satisfies the first axiom of countability. Then a Čech-complete map $p : X \to B$ is a k-map.*

Proof. First, note that we use the following notation in a space Y: For $M \subset Y$, we denote the closure of M in Y by $\overline{M}^Y$.

Suppose that H is quasi-closed but not closed in the space X. Then there exists a point $x \in \overline{H}^X - H$. Let $b = p(x)$ and $\{W_n\}_{n \in \mathbf{N}}$ be a decreasing nbd base of b in B. Since p is Čech-complete, from Theorem 5.1 there exists a T_2-compactification $p' : X' \to B$ of p such that $X_b = (\bigcap_{n\in\mathbf{N}} G_n) \cap X'_b$ with G_n open in the space X' for every $n \in \mathbf{N}$. Let $U_0 = X'$. Since p' is compact and B is regular, there exists a nbd U_n of x in X' such that $x \in \overline{U_n}^{X'} \subset U_{n-1} \cap G_n \cap p'^{-1}(W_n)$ for every $n \in \mathbf{N}$. Then $Z = \bigcap_{n\in\mathbf{N}} \overline{U_n}^{X'} \subset X$ is a closed subset of X'_b by $\bigcap_{n\in\mathbf{N}} p'^{-1}(W_n) = p'^{-1}(b)$. Therefore $p|Z : Z \to B$ is a compact map. Since H is quasi-closed, $H \cap Z$ is a compact subset of X_b. So there exists a nbd V of x in X' such that $\overline{V}^{X'} \cap (H \cap Z) = \emptyset$.

For every $n \in \mathbf{N}$, take a point $x_n \in (V \cap U_n \cap H) - \{x_k | k < n\}$ and let $F_n = \{x_i | i \geq n\}$ which is included in X. For every $n \in \mathbf{N}$, since $\overline{F_n}^{X'} \subset \overline{U_{n-1}}^{X'}$, it holds that $F'_n = \{y | y$ is an accumulation point of F_n in $X'\} \subset Z$. Thus $K_n = F_n \cup Z \subset X$ is fibrewise compact over B (i.e. $p|K_n : K_n \to B$ is compact), because $\overline{K_n}^{X'} = \overline{F_n \cup Z}^{X'} = \overline{F_n}^{X'} \cup \overline{Z}^{X'} = F_n \cup Z = K_n$, and $p' : X' \to B$ is compact. Since $F_n \subset p'^{-1}(W_n)$ for every $n \in \mathbf{N}$, $\{F_n\}_{n\in\mathbf{N}}$ generates a b-filter $\mathcal{F}$ on K_1. From the compactness of $p|K_1 : K_1 \to B$, $\mathcal{F}$ has an adherence point x_0 in the space K_1, thus $x_0 \in \overline{F_n}^{X'} \cap K_1$ for every $n \in \mathbf{N}$, therefore $x_0 \in Z$. Without loss of generality, we can suppose $x_0 \notin F_n$ for every $n \in \mathbf{N}$.

Since $F'_n \subset Z$, it is easy to see that $\overline{F_n}^X = \overline{F_n}^{X'} \cap X = (F_n \cup F'_n) \cap X = F_n \cup F'_n = \overline{F_n}^{X'}$ for every $n \in \mathbf{N}$. From $F_n \subset V \subset \overline{V}^{X'}$ and $\overline{F_n}^{X'} \subset \overline{V}^{X'} \subset X' - (H \cap Z)$, $x_0 \in \overline{F_n}^{X'} - (H \cap Z)$. For every nbd U of x_0 in X, $\emptyset \neq U \cap F_n \subset U \cap (F_n \cup (H \cap Z)) = U \cap ((F_n \cup H) \cap (F_n \cup Z)) = U \cap H \cap (F_n \cup Z) = U \cap H \cap K_n$, thus $x_0 \in \overline{H \cap K_n}^{K_n}$. But since $H \cap K_n = H \cap (F_n \cup Z) = (H \cap F_n) \cup (H \cap Z) = F_n \cup (H \cap Z)$, it holds that $x_0 \notin H \cap K_n$. Thus $H \cap K_n$ is not closed in K_n.

For every nbd $W \in N(b)$, there exists $n \in \mathbf{N}$ such that $W_n \subset W$ and $p|K_n : K_n \to W$ is compact. But from the above consideration $H \cap K_n$ is not closed in the space K_n, which contradicts with the fact that H is quasi-closed in X. □

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