Common fixed point theorems for single and set-valued maps satisfying a strict contractive condition

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Abstract. *Some new common fixed point theorems for weakly compatible single and set-valued mappings under strict contractive conditions are obtained. Our results extend, improve and complement the result of Fisher [3] and the recent one due to Ahmed [2] and others.*

Key words: *weakly compatible maps, property (E.A), single and set-valued maps, common fixed points*

AMS subject classifications: 47H10, 54H25

Received May 2, 2007 Accepted November 30, 2007

1. Introduction

Fixed and common fixed point theorems for contractive and strict contractive single and set-valued mappings are investigated by many authors. In 1986, G. Jungck introduced the notion of compatible maps to the setting of single mappings in order to generalize the concepts of commutativity and weak commutativity. Afterwards, the same author weakens the above notion by introducing the notion of weak compatibility, and recently, with B. E. Rhoades he extended the above notion to the setting of single and set-valued maps. In 2002 M. Aamri and D. El Moutawakil [1] defined a property (E.A) for self-maps which contained the class of noncompatible maps. More recently, T. Kamran [5] extended the property (E.A) for a hybrid pair of single and multivalued maps.

The aim of this paper is to give some new common fixed point theorems for single and set-valued maps under strict contractive conditions. These results unify, improve, extend and generalize the results of [2] and [3] by utilizing the property (E.A) of [5].

We begin by stating some known definitions.

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2. Preliminaries

Let $\mathcal{X}$ be a metric space with metric d. We denote by $CB(\mathcal{X})$ the class of all nonempty bounded closed subsets of $\mathcal{X}$. We define the functions $\delta(A,B)$ and $D(A,B)$ as follows:

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$
$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$

for all A, B in $CB(\mathcal{X})$. If A contains a single point a, we write $\delta(A,B) = \delta(a,B)$. Also, if B contains a single point b, it yields $\delta(A,B) = d(a,b)$.

The definition of the function $\delta(A,B)$ yields the following:

$$\begin{aligned}
&\delta(A,B) = \delta(B,A),\\
&\delta(A,B) \le \delta(A,C) + \delta(C,B),\\
&\delta(A,B) = 0 \text{ iff } A = B = \{a\},\\
&\delta(A,A) = diamA,
\end{aligned}$$

for all A, B, C in $CB(\mathcal{X})$.

Definition 2.1.([4]) *Maps $f : \mathcal{X} \to \mathcal{X}$ and $T : \mathcal{X} \to CB(\mathcal{X})$ are weakly compatible if they commute at their coincidence points, that is, if $fTx = Tfx$ whenever $fx \in Tx$.*

Definition 2.2.([1]) *Maps $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in \mathcal{X}$.*

Definition 2.3.([5]) *Maps $f : \mathcal{X} \to \mathcal{X}$ and $T : \mathcal{X} \to CB(\mathcal{X})$ are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in $\mathcal{X}$, some $t \in \mathcal{X}$ and $A \in CB(\mathcal{X})$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$.*

Now, we give our main results.

3. Main results

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a metric space, let $\mathcal{F}, \mathcal{G} : \mathcal{X} \to CB(\mathcal{X})$ and $\mathcal{I}, \mathcal{J} : \mathcal{X} \to \mathcal{X}$ be set and single-valued mappings, respectively, satisfying the conditions*

(1) $\cup\mathcal{F}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ *and* $\cup\mathcal{G}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$. *Suppose that the inequality*

(2)

$$\begin{aligned}
\delta(\mathcal{F}x, \mathcal{G}y) < {} & \alpha \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{J}y, \mathcal{G}y)\}\\
& +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{G}y) + bD(\mathcal{J}y, \mathcal{F}x)\right],
\end{aligned}$$

for all $x, y \in \mathcal{X}$, where

(3) $0 \le \alpha < 1, 0 \le a \le \frac{1}{2}, 0 \le b < \frac{1}{2}$ *holds whenever the right-hand side of* (2) *is positive. If either*

(4) *$\mathcal{F}$,$\mathcal{I}$ are weakly compatible and satisfy property (E.A); $\mathcal{G}$,$\mathcal{J}$ are weakly compatible and $\cup\mathcal{F}(\mathcal{X})$ (resp. $\mathcal{J}(\mathcal{X})$) is closed or*

(4') *$\mathcal{G}$,$\mathcal{J}$ are weakly compatible and satisfy property (E.A); $\mathcal{F}$,$\mathcal{I}$ are weakly compatible and $\cup\mathcal{G}(\mathcal{X})$ (resp. $\mathcal{I}(\mathcal{X})$) is closed.*

Then, there is a unique common fixed point z in $\mathcal{X}$ such that

$$\mathcal{F}z = \mathcal{G}z = \{z\} = \{\mathcal{I}z\} = \{\mathcal{J}z\}\,. \tag{1}$$

Proof. Suppose that $\mathcal{F}$ and $\mathcal{I}$ satisfy property (E.A). Then there exist a sequence $\{x_n\}$ in $\mathcal{X}$, $t \in \mathcal{X}$ and $A \in CB(\mathcal{X})$ such that $\lim_{n\to\infty} \mathcal{I}x_n = t \in A = \lim_{n\to\infty} \mathcal{F}x_n$. Since $\cup\mathcal{F}(\mathcal{X})$ is closed and $\cup\mathcal{F}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$, then there exists a point u in $\mathcal{X}$ such that $\mathcal{J}u = t$. Then inequality (2) gives

$$\begin{aligned}\delta(\mathcal{F}x_n, \mathcal{G}u) &< \alpha \max\{d(\mathcal{I}x_n, \mathcal{J}u), \delta(\mathcal{I}x_n, \mathcal{F}x_n), \delta(\mathcal{J}u, \mathcal{G}u)\}\\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x_n, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}x_n)\right].\end{aligned}$$

Taking the limit as n tends to infinity yields

$$\begin{aligned}\delta(\mathcal{J}u, \mathcal{G}u) &\le \alpha \max\{0, 0, \delta(\mathcal{J}u, \mathcal{G}u)\} + (1-\alpha)aD(\mathcal{J}u, \mathcal{G}u)\\ &= \alpha\delta(\mathcal{J}u, \mathcal{G}u) + (1-\alpha)aD(\mathcal{J}u, \mathcal{G}u)\\ &\le [\alpha + (1-\alpha)a]\,\delta(\mathcal{J}u, \mathcal{G}u).\end{aligned}$$

It is obvious that $[\alpha + (1-\alpha)a] < 1$. Then the above contradiction demands that $\mathcal{G}u = \{\mathcal{J}u\}$. Since $\mathcal{G}$ and $\mathcal{J}$ are weakly compatible, $\mathcal{G}u = \{\mathcal{J}u\}$ implies that $\mathcal{G}\mathcal{J}u = \mathcal{J}\mathcal{G}u$ and hence

$$\mathcal{G}\mathcal{G}u = \mathcal{G}\mathcal{J}u = \mathcal{J}\mathcal{G}u = \{\mathcal{J}\mathcal{J}u\}\,. \tag{2}$$

Again by (2) we have

$$\begin{aligned}\delta(\mathcal{F}x_n, \mathcal{G}\mathcal{G}u) &< \alpha \max\{d(\mathcal{I}x_n, \mathcal{J}\mathcal{G}u), \delta(\mathcal{I}x_n, \mathcal{F}x_n), \delta(\mathcal{J}\mathcal{G}u, \mathcal{G}\mathcal{G}u)\}\\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x_n, \mathcal{G}\mathcal{G}u) + bD(\mathcal{J}\mathcal{G}u, \mathcal{F}x_n)\right].\end{aligned}$$

By letting n tends to infinity, we obtain

$$\begin{aligned}\delta(\mathcal{J}u, \mathcal{G}\mathcal{G}u) &\le \alpha \max\{d(\mathcal{J}u, \mathcal{G}\mathcal{G}u), 0, 0\} + (1-\alpha)(a+b)D(\mathcal{J}u, \mathcal{G}\mathcal{G}u)\\ &= \alpha d(\mathcal{J}u, \mathcal{G}\mathcal{G}u) + (1-\alpha)(a+b)D(\mathcal{J}u, \mathcal{G}\mathcal{G}u)\\ &\le [\alpha + (1-\alpha)(a+b)]\,\delta(\mathcal{J}u, \mathcal{G}\mathcal{G}u)\end{aligned}$$

and since $[\alpha + (1-\alpha)(a+b)] < 1$, then we have $\mathcal{G}\mathcal{G}u = \{\mathcal{J}u\}$. Hence $\{\mathcal{J}u\} = \mathcal{G}\mathcal{G}u = \mathcal{J}\mathcal{G}u$, i.e. $\mathcal{G}u = \mathcal{G}\mathcal{G}u = \mathcal{J}\mathcal{G}u$ and $\mathcal{G}u$ is a common fixed point of $\mathcal{G}$ and $\mathcal{J}$. Since $\cup\mathcal{G}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$, then there is a point $v \in \mathcal{X}$ such that $\{\mathcal{I}v\} = \mathcal{G}u$. Moreover, the use of (2) gives

$$\begin{aligned}\delta(\mathcal{F}v, \mathcal{G}u) &< \alpha \max\{d(\mathcal{I}v, \mathcal{J}u), \delta(\mathcal{I}v, \mathcal{F}v), \delta(\mathcal{J}u, \mathcal{G}u)\}\\ &\quad +(1-\alpha)\left[aD(\mathcal{I}v, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}v)\right]\\ &= \alpha \max\{0, \delta(\mathcal{I}v, \mathcal{F}v), 0\} + (1-\alpha)bD(\mathcal{J}u, \mathcal{F}v)\\ &= \alpha\delta(\mathcal{G}u, \mathcal{F}v) + (1-\alpha)bD(\mathcal{G}u, \mathcal{F}v)\\ &\le [\alpha + (1-\alpha)b]\,\delta(\mathcal{G}u, \mathcal{F}v).\end{aligned}$$

It is easy to see that $[\alpha + (1-\alpha)b] < 1$, and therefore $\mathcal{F}v = \mathcal{G}u = \{\mathcal{I}v\}$. Since $\mathcal{F}v = \{\mathcal{I}v\}$, by the weak compatibility of $\mathcal{F}$ and $\mathcal{I}$, we get $\mathcal{F}\mathcal{I}v = \mathcal{I}\mathcal{F}v$ and hence

$$\mathcal{F}\mathcal{F}v = \mathcal{F}\mathcal{I}v = \mathcal{I}\mathcal{F}v = \{\mathcal{I}\mathcal{I}v\}. \tag{3}$$

Moreover, by (2) we can estimate

$$\begin{aligned}
\delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u) &< \alpha \max\{d(\mathcal{I}\mathcal{F}v, \mathcal{J}u), \delta(\mathcal{I}\mathcal{F}v, \mathcal{F}\mathcal{F}v), \delta(\mathcal{J}u, \mathcal{G}u)\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}\mathcal{F}v, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}\mathcal{F}v)\right] \\
&= \alpha \max\{d(\mathcal{I}\mathcal{F}v, \mathcal{J}u), 0, 0\} + (1-\alpha)(a+b)D(\mathcal{I}\mathcal{F}v, \mathcal{G}u) \\
&= \alpha d(\mathcal{F}\mathcal{F}v, \mathcal{G}u) + (1-\alpha)(a+b)D(\mathcal{F}\mathcal{F}v, \mathcal{G}u) \\
&\leq [\alpha + (1-\alpha)(a+b)]\, \delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u) < \delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u),
\end{aligned}$$

which is a contradiction, thus $\mathcal{F}\mathcal{F}v = \mathcal{G}u$, i.e., $\mathcal{F}\mathcal{G}u = \mathcal{G}u = \mathcal{I}\mathcal{G}u$ and $\mathcal{G}u$ is also a common fixed point of $\mathcal{F}$ and $\mathcal{I}$. Let $z = \mathcal{G}u$, then

$$\mathcal{F}z = \mathcal{G}z = \{z\} = \{\mathcal{I}z\} = \{\mathcal{J}z\}. \tag{4}$$

Similarly, one can obtain this conclusion by using $(4')$ in lieu of (4).

Finally, we prove that z is unique. Indeed, let z' be another common fixed point of the maps $\mathcal{I}, \mathcal{J}, \mathcal{F}$ and $\mathcal{G}$ such that $z' \neq z$. Then, by estimation (2), one may get

$$\begin{aligned}
d(z, z') &= \delta(\mathcal{F}z, \mathcal{G}z') < \alpha \max\{d(\mathcal{I}z, \mathcal{J}z'), \delta(\mathcal{I}z, \mathcal{F}z), \delta(\mathcal{J}z', \mathcal{G}z')\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}z, \mathcal{G}z') + bD(\mathcal{J}z', \mathcal{F}z)\right] \\
&= \alpha \max\{d(z, z'), 0, 0\} + (1-\alpha)(a+b)D(z, z') \\
&= \alpha d(z, z') + (1-\alpha)(a+b)D(z, z') \\
&\leq [\alpha + (1-\alpha)(a+b)]\, d(z, z') < d(z, z').
\end{aligned}$$

This contradiction implies that $z' = z$. Hence, z is the unique common fixed point of $\mathcal{I}, \mathcal{J}, \mathcal{F}$ and $\mathcal{G}$. □

If we let in *Theorem 3.1* $\mathcal{F} = \mathcal{G}$ and $\mathcal{I} = \mathcal{J}$, then we get the following result.

Corollary 3.1. *Let $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ be a self-map of a metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{X} \to CB(\mathcal{X})$ a set-valued map. Assume that $\mathcal{F}$ and $\mathcal{I}$ satisfy the conditions*

(i) $\mathcal{F}$ and $\mathcal{I}$ satisfy property $(E.A)$,

(ii) $\cup\mathcal{F}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$,

(iii) the inequality

$$\begin{aligned}
\delta(\mathcal{F}x, \mathcal{F}y) &< \alpha \max\{d(\mathcal{I}x, \mathcal{I}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{I}y, \mathcal{F}y)\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{F}y) + bD(\mathcal{I}y, \mathcal{F}x)\right],
\end{aligned}$$

for all x,$y \in \mathcal{X}$, where $0 \leq \alpha < 1$,$0 \leq a \leq \frac{1}{2}$,$0 \leq b < \frac{1}{2}$, whenever the right-hand side of inequality (iii) is positive. If $\mathcal{F}$ and $\mathcal{I}$ are weakly compatible and $\cup\mathcal{F}(\mathcal{X})$ (resp. $\mathcal{I}(\mathcal{X})$) is closed. Then, $\mathcal{F}$ and $\mathcal{I}$ have a unique common fixed point z in $\mathcal{X}$.

For three maps, we have the following result.

Corollary 3.2. *Let $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ be a self-map of a metric space $(\mathcal{X}, d)$ and $\mathcal{F}$,$\mathcal{G} : \mathcal{X} \to CB(\mathcal{X})$ two set-valued maps such that*

(i) $\cup\mathcal{F}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$ *and* $\cup\mathcal{G}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$,

(ii) *the inequality*

$$\delta(\mathcal{F}x, \mathcal{G}y) < \alpha \max\{d(\mathcal{I}x, \mathcal{I}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{I}y, \mathcal{G}y)\} \\ +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{G}y) + bD(\mathcal{I}y, \mathcal{F}x)\right],$$

holds for all $x,y \in \mathcal{X}$*, where* $0 \leq \alpha < 1, 0 \leq a \leq \frac{1}{2}, 0 \leq b < \frac{1}{2}$*, whenever the right-hand side of the above inequality is positive. Further, if either*

(iii) $\mathcal{F},\mathcal{I}$ *are weakly compatible satisfing property* $(E.A)$*;* $\mathcal{G},\mathcal{I}$ *are weakly compatible and* $\cup\mathcal{F}(\mathcal{X})$ *(resp.* $\mathcal{I}(\mathcal{X})$*) is closed or*

(iii)' $\mathcal{G},\mathcal{I}$ *are weakly compatible satisfing property* $(E.A)$*;* $\mathcal{F},\mathcal{I}$ *are weakly compatible and* $\cup\mathcal{G}(\mathcal{X})$ *(resp.* $\mathcal{I}(\mathcal{X})$*) is closed.*

Then, $\mathcal{F},\mathcal{G}$ *and* $\mathcal{I}$ *have a unique common fixed point in* $\mathcal{X}$*.*

Remarks.

(1) Truly, our result generalizes the results of M. A. Ahmed [2] and B. Fisher [3], since we have not assumed compacity nor continuity but only property (E.A) and the minimal condition of the closedness.

(2) Note that if we put $a = b = 0$ *in Theorem 3.1, we get a generalization of the result due to Fisher [3], because*

$$\delta(\mathcal{F}x, \mathcal{G}y) < \alpha \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{J}y, \mathcal{G}y)\} \\ < \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{J}y, \mathcal{G}y)\}.$$

For a set-valued map $\mathcal{F} : \mathcal{X} \to CB(\mathcal{X})$ (resp. a self-map $\mathcal{I} : \mathcal{X} \to \mathcal{X}$), we denote $F_{\mathcal{F}} = \{x \in \mathcal{X} : \mathcal{F}x = \{x\}\}$ (resp. $F_{\mathcal{I}} = \{x \in \mathcal{X} : \mathcal{I}x = x\}$).

Theorem 3.2. *Let* $\mathcal{F},\mathcal{G} : \mathcal{X} \to CB(\mathcal{X})$ *be set-valued mappings and* $\mathcal{I},\mathcal{J} : \mathcal{X} \to \mathcal{X}$ *be self-mappings on the metric space* $\mathcal{X}$*. If inequality* (2) *holds for all* $x,y \in \mathcal{X}$*, then*

$$(F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{F}} = (F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{G}}. \tag{5}$$

Proof. Let $z \in (F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{F}}$. Then, estimation (2) gives

$$\delta(\mathcal{F}z, \mathcal{G}z) < \alpha \max\{d(\mathcal{I}z, \mathcal{J}z), \delta(\mathcal{I}z, \mathcal{F}z), \delta(\mathcal{J}z, \mathcal{G}z)\} \\ +(1-\alpha)\left[aD(\mathcal{I}z, \mathcal{G}z) + bD(\mathcal{J}z, \mathcal{F}z)\right].$$

Therefore

$$\begin{aligned} \delta(z, \mathcal{G}z) &< \alpha \max\{0, 0, \delta(z, \mathcal{G}z)\} + (1-\alpha)aD(z, \mathcal{G}z) \\ &= \alpha\delta(z, \mathcal{G}z) + (1-\alpha)aD(z, \mathcal{G}z) \\ &\leq [\alpha + (1-\alpha)a]\,\delta(z, \mathcal{G}z) < \delta(z, \mathcal{G}z). \end{aligned}$$

This contradiction implies that $\mathcal{G}z = \{z\}$. Thus,

$$(F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{F}} \subset (F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{G}}. \tag{6}$$

Similarly,

$$(F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{G}} \subset (F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{F}}. \tag{7}$$

□

Theorem 3.3. *Let $\mathcal{I}$,$\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be self-mappings and $\mathcal{F}_i : \mathcal{X} \to CB(\mathcal{X})$,$i \in \mathbb{N}^* = \{1, 2, ...\}$ set-valued maps such that*

(i) $\cup\mathcal{F}_1(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ *and* $\cup\mathcal{F}_2(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$,

(ii) *the inequality*

$$\begin{aligned}\delta(\mathcal{F}_i x, \mathcal{F}_{i+1} y) < \alpha \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}_i x), \delta(\mathcal{J}y, \mathcal{F}_{i+1} y)\} \\ +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{F}_{i+1} y) + bD(\mathcal{J}y, \mathcal{F}_i x)\right]\end{aligned}$$

holds for all x,$y \in \mathcal{X}$,$\forall i \in \mathbb{N}^*$, where $0 \leq \alpha < 1$,$0 \leq a \leq \frac{1}{2}$,$0 \leq b < \frac{1}{2}$, whenever the right-hand side of (ii) is positive. Further, if either

(iii) *$\mathcal{F}_1$,$\mathcal{I}$ are weakly compatible satisfing property (E.A); $\mathcal{F}_2$,$\mathcal{J}$ are weakly compatible and $\cup\mathcal{F}_1(\mathcal{X})$ (resp. $\mathcal{J}(\mathcal{X})$) is closed or*

(iii)' *$\mathcal{F}_2$,$\mathcal{J}$ are weakly compatible satisfing property (E.A); $\mathcal{F}_1$,$\mathcal{I}$ are weakly compatible and $\cup\mathcal{F}_2(\mathcal{X})$ (resp. $\mathcal{I}(\mathcal{X})$) is closed.*

Then, there exists a unique common fixed point $z \in \mathcal{X}$ such that

$$\mathcal{F}_i z = \{\mathcal{I}z\} = \{\mathcal{J}z\} = \{z\}, \forall i \in \mathbb{N}^*. \tag{8}$$

Proof. It follows from Theorems 3.1 and 3.2. □

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