An extension of Knopp's core theorem for complex bounded sequences

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Abstract. *In this paper, using the idea used by Choudhary, we extend previously known results on the core for complex bounded sequences.*

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1. Introduction

Let m, c be linear spaces of complex bounded and convergent sequences $x = \{x_n\}$, respectively, normed by $\|x\| = \sup |x_n|$.

We define functionals l and L on linear space of a real bounded sequence by

$$l(x) = \text{liminf}\, x_n\, ; L(x) = \text{limsup}\, x_n.$$

Let $A = (a_{nk})$ be an infinite matrix and write

$$(Ax)_n := \sum_k a_{nk} x_k$$

if the series converges for each $n \in \mathbb{N}$. By Ax we denote the sequence $\{(Ax)_n\}$. If $\lim Ax = \lim x$ for each $x \in c$, we say that A is regular [2] , [9] and write $A \in (c, c; p)$. Silverman Toeplitz theorem gives the necessary and sufficent conditions for regularity of the matrix A [2] , [9].

Matrix $A = (a_{nk})$ is called normal if it is a lower semi-triangular matrix with non-zero diagonal entries [2].

For brevity we shall denote the Knopp core of x by $K - core\, \{x\}$; recall [2] , [4] that

$$K - core\, \{x\} := \bigcap_{n=1}^{\infty} C_n(x)$$

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where $C_n(x)$ is the least closed convex hull of $\{x_k\}_{k\geq n}$. If x is a real bounded sequence, then $K - core\{x\}$ will be a closed interval $[l(x), L(x)]$.

The famous Knopp's core theorem (see [2], [3], [5], [6], [8], [11]) determines a class of regular matrices for which $L(Ax) \leq L(x)$ for all real bounded sequences x; that is $K - core\{Ax\} \subseteq K - core\{x\}$.

Let $\mathbb{C}$ denote the set of complex numbers. In Shcherbakoff [10] it is shown that for every bounded x,

$$K - core\{x\} := \bigcap_{z\in\mathbb{C}} B_x(z),$$

where

$$B_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z| \right\} .$$

Shcherbakoff [10] generalized the notion of the core of a bounded complex sequence by introducing the idea of the generalized $\alpha - core$ of a bounded complex sequence x as

$$K^{(\alpha)} - core\{x\} := \bigcap_{z\in\mathbb{C}} B_x^{\alpha}(z)$$

where

$$B_x^{\alpha}(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \alpha \limsup_k |x_k - z| , \ \alpha \geq 1 \right\}.$$

when $\alpha = 1$, $K^{(\alpha)} - core\{x\}$ reduces the usual Knopp core.

In [7] Natarajan has proved the following theorem.

Theorem A. *When $K = \mathbb{R}$ or $\mathbb{C}$, an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$ $n, k = 0, 1, 2, ...$ is such that*

$$K - core\{Ax\} \subseteq K^{(\alpha)} - core\{x\} , \ \alpha \geq 1 \tag{1}$$

for any bounded sequence x if and only if A is regular and satisfies

$$\limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \leq \alpha.$$

This result for the case $\alpha = 1$ yields a simple proof of Knopp's core theorem.

In this paper, using the idea used by Choudhary [1], we generalizes inclusion (1).

2. Main results

Before giving the main result we first state a result due to Choudhary [1] that we need for our purposes.

Lemma 1. *Let n be fixed. In order that, whenever Bx is bounded, $(Ax)_n$ should be defined, it is necessary and sufficent that*

(*i*) $c_{nk} = \sum_{v=k}^{\infty} a_{nv} b_{vk}^{-1}$ *exist for all k;*

(*ii*) $\sum\limits_{k=0}^{\infty} |c_{nk}| < \infty$ *(for all n);*

(*iii*) $\sum\limits_{k=0}^{j} \left| \sum\limits_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \; (j \to \infty)$

should hold for the n considered. If these conditions are satisfied, then for bounded Bx,

$$(Ax)_n = (Cy)_n \tag{2}$$

where $y := Bx$.

Whenever B is normal, B has a reciprocal. Denote its reciprocal by $B^{-1} = (b_{nk}^{-1})$. Note that if B is a normal matrix, then the space $m_B := \{x : Bx \in m\}$ is isometrically isomorphic to m. Hence given a sequence $y \in m_B$, then there exists a unique sequence $x \in m_B$ so that $y := Bx$.

Now we are ready to state our first result:

Theorem 1. *Let* $B = (b_{nk})$ *be a normal matrix and A any matrix. In order that, whenever Bx is bounded, Ax should exist and be bounded and that*

$$K - core\,\{Ax\} \subseteq K^{(\alpha)} - core\,\{Bx\}\,,\; \alpha \geq 1 \tag{3}$$

it is necessary and sufficient that

(*i*) $C = AB^{-1}$ *exists;*

(*ii*) C *is regular*;

(*iii*) $\limsup\limits_{n\to\infty} \left(\sum\limits_{k=0}^{\infty} |c_{nk}| \right) \leq \alpha$

(*iv*) *for any fixed n*

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \quad (j \to \infty)$$

Proof. Assume that (3) holds. Write $y := Bx$. Let $(Ax)_n$ be exist for each n whenever y is bounded. Then by Lemma 1, (*i*) and (*iv*) of Theorem hold. Morever, for every bounded y we have (2). Hence, by (3) we get

$$K - core\,\{Cy\} \subseteq K^{(\alpha)} - core\,\{y\}\,,\; \alpha \geq 1$$

for every bounded y. Now it follows from Theorem A that (*ii*) and (*iii*) hold.

Sufficiency. Observe that conditions (*i*) and (*iv*) imply the conditions of Lemma 1. So (2) holds and Cy is bounded whenever $y \in m$. Now from Theorem A, (*ii*) and (*iii*) imply that

$$K - core\,\{Cy\} \subseteq K^{(\alpha)} - core\,\{y\}\,,\; \alpha \geq 1$$

provided y is bounded. Writing $y = Bx$ we immediately get (3), whence the result. □

Recall that the matrix A is called row-finite if every row contains only a finite number of non-zero elements. In this case (*iii*) of Theorem 1 is zero for sufficiently large j; hence (*iii*) is evidently satisfied. So, Theorem 1 reduces to the following

Theorem 2. *Let $B = (b_{nk})$ be a normal matrix. Then for a row-finite matrix A,*

$$K - core\,\{Ax\} \subseteq K^{(\alpha)} - core\,\{x\}\,,\ \alpha \geq 1,\ (\text{for all } x \in m_B)$$

if and only if (i) and (iii) hold.

If we interchange the roles of matrices A and B in Theorem 1, we immediately get the following

Theorem 3. *Let $B = (b_{nk})$ and $A = (a_{nk})$ be normal matrices. Then for all $x \in m_B \cap m_A$ we have that*

$$K - core\,\{Ax\} = K^{(\alpha)} - core\,\{Bx\})$$

if and only if

a) $C = AB^{-1}$ *and* $D = BA^{-1}$ *exist;*

b) C *and* D *are regular;*

c) $\limsup\limits_{n\to\infty} \left(\sum\limits_{k=0}^{\infty} |c_{nk}| \right) \leq \alpha$

$\limsup\limits_{n\to\infty} \left(\sum\limits_{k=0}^{\infty} |d_{nk}| \right) \leq \alpha$

d) *for any fixed n*

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 (j \to \infty),$$

and

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} b_{nv} a_{vk}^{-1} \right| \to 0 (j \to \infty).$$

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