New summation formula for ${}_3F_2\left(\frac{1}{2}\right)$ and a Kummer–type II transformation of ${}_2F_2(x)$

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Abstract. *The aim of this research note is a new summation formula for the series ${}_3F_2\left(\frac{1}{2}\right)$. The summation formula is then applied to establish a Kummer–type II transformation for the series ${}_2F_2(x)$.*

Key words: *hypergeometric ${}_3F_2$, Kummer–type transformations, summation formula*

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1. Introduction and results required

We start with the Kummer–type I transformation [4] for the series ${}_1F_1$, viz.

$$e^{-x}\,{}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\middle|\, x\right] = {}_1F_1\left[\begin{matrix} b-a \\ b \end{matrix}\middle|\, -x\right]. \tag{1}$$

Recently, Paris [2] generalized (1) in the form

$$e^{-x}\,{}_2F_2\left[\begin{matrix} a,\ 1+d \\ b,\ d \end{matrix}\middle|\, x\right] = {}_2F_2\left[\begin{matrix} b-a-1,\ f+1 \\ b,\ f \end{matrix}\middle|\, -x\right] \tag{2}$$

where

$$f := \frac{d(a-b+1)}{a-d}. \tag{3}$$

The well–known Kummer type II transformation [3] is

$$e^{-x/2}\,{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\middle|\, x\right] = {}_0F_1\left[\begin{matrix} \overline{} \\ a+1/2 \end{matrix}\middle|\, \frac{x^2}{16}\right]. \tag{4}$$

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Motivated by the Kummer–type I transformation (2) for the hypergeometric function ${}_2F_2$, the authors' aim is to present a Kummer–type II transformation for the generalized hypergeometric function ${}_2F_2$. For this, first we establish a new summation formula for the value ${}_3F_2\big(\frac{1}{2}\big)$ and then apply it to get the desired results.

The following summation formula obtained earlier by Lavoie *et al.* [1] closely related to Gauss' second summation theorem will be required in our present investigation:

$$
{}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b)+1 \end{matrix}\,\middle|\,\frac{1}{2}\right] = \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{1}{2}(a+b)+1\big)\Gamma\big(\frac{1}{2}(a-b)\big)}{\Gamma\big(\frac{1}{2}(a-b)+1\big)} \times \left\{\frac{1}{\Gamma\big(\frac{a}{2}\big)\Gamma\big(\frac{1}{2}(b+1)\big)} - \frac{1}{\Gamma\big(\frac{b}{2}\big)\Gamma\big(\frac{1}{2}(a+1)\big)}\right\} \tag{5}
$$

where the Euler Gamma–function is defined as

$$
\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx \qquad \big(\Re\{s\} > 0\big)\,. \tag{6}
$$

Its connection to the Beta–function is

$$
B(s,r) = \frac{\Gamma(s)\Gamma(r)}{\Gamma(s+r)} \qquad \big(\min\big\{\Re\{s\}, \Re\{r\}\big\} > 0\big)\,. \tag{7}
$$

Finally, we assume all necessary constraints upon parameters of the considered hypergeometric functions, Gamma–, and Beta–functions such that ensure their convergence.

2. New summation formula

Theorem 1.

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c+1 \\ \frac{1}{2}(a+b)+1,\ c \end{matrix}\,\middle|\,\frac{1}{2}\right] = \frac{a+b}{c(a-b)}\Gamma\big(\tfrac{1}{2}\big)\Gamma\big(\tfrac{1}{2}(a+b)\big) \times \left\{\frac{c-b}{\Gamma\big(\frac{a}{2}\big)\Gamma\big(\frac{1}{2}(b+1)\big)} + \frac{a-c}{\Gamma\big(\frac{b}{2}\big)\Gamma\big(\frac{1}{2}(a+1)\big)}\right\}. \tag{8}
$$

Proof. Let us show the validity of (8). In this goal let us prove

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c+1 \\ \frac{1}{2}(a+b)+1,\ c \end{matrix}\,\middle|\,\frac{1}{2}\right] = {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b)+1 \end{matrix}\,\middle|\,\frac{1}{2}\right] + \frac{ab}{c(a+b+2)}\,{}_2F_1\left[\begin{matrix} a+1,\ b+1 \\ \frac{1}{2}(a+b)+2 \end{matrix}\,\middle|\,\frac{1}{2}\right]. \tag{9}
$$

Consider the series definition of ${}_3F_2$

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c+1 \\ \tfrac{1}{2}(a+b)+1,\ c \end{matrix}\,\middle|\,\frac{1}{2}\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c+1)_n}{\left(\frac{1}{2}(a+b)+1\right)_n (c)_n} \frac{(1/2)^n}{n!}\,, \tag{10} $$

where $(\tau)_n = \tau(\tau+1)\cdots(\tau+n-1)$ is the Pochhammer symbol (shifted factorial). It is easy to see that

$$ \frac{(a)_n(b)_n(c+1)_n}{\left(\frac{1}{2}(a+b)+1\right)_n (c)_n} = \frac{(a)_n(b)_n}{\left(\frac{1}{2}(a+b)+1\right)_n} + \frac{ab\,n}{c\left(\frac{1}{2}(a+b)+1\right)} \cdot \frac{(a+1)_{n-1}(b+1)_{n-1}}{\left(\frac{1}{2}(a+b)+2\right)_{n-1}}\,. $$

So, (10) becomes

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} a,\ b,\ c+1 \\ \tfrac{1}{2}(a+b)+1,\ c \end{matrix}\,\middle|\,\frac{1}{2}\right] &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{\left(\frac{1}{2}(a+b)+1\right)_n} \frac{(1/2)^n}{n!} \\ &\quad + \frac{ab}{c(a+b+2)} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{\left(\frac{1}{2}(a+b)+2\right)_{n-1}} \frac{(1/2)^{n-1}}{(n-1)!}\,. \end{aligned} $$

This expression is equivalent to the asserted formula (9).

The two ${}_2F_1$ on the right–hand side of (9) can be evaluated with the help of (5) and after a little simplification, we easily arrive at the right–hand side of (8). □

3. Kummer–type II transformation

As the application of the new summation formula (8), a Kummer–type II transformation will be derived in this section.

Theorem 2. *The following Kummer–type* II *transformation formula holds true*

$$ e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 1+d \\ 2a+1,\ d \end{matrix}\,\middle|\,x\right] = {}_0F_1\left[\begin{matrix} \overline{} \\ a+\frac{1}{2} \end{matrix}\,\middle|\,\frac{x^2}{16}\right] - \frac{x(1-2a/d)}{2(2a+1)}\,{}_0F_1\left[\begin{matrix} \overline{} \\ a+\frac{3}{2} \end{matrix}\,\middle|\,\frac{x^2}{16}\right]. \tag{11} $$

Proof. In order to prove (11), we proceed as follows. Let

$$ e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 1+d \\ 2a+1,\ d \end{matrix}\,\middle|\,x\right] = \sum_{n=0}^{\infty} a_n x^n\,. \tag{12} $$

Now, in the product

$$ e^{-x/2}\,{}_2F_2\left[\begin{matrix} a,\ 1+d \\ 2a+1,\ d \end{matrix}\,\middle|\,x\right] $$

it is not hard to see that the coefficient of x^n is obtained as

$$ a_n = \frac{(a)_n(d+1)_n}{(2a+1)_n\,n!}\,{}_3F_2\left[\begin{matrix} -n,\ 2a-n,\ 1-d-n \\ 1-a-n,\ -d-n \end{matrix}\,\middle|\,\frac{1}{2}\right]. \tag{13} $$

Changing n to $2n$ and using the result (8), the coefficient (13) of the even–powered terms in the series (12) becomes

$$a_{2n} = \frac{1}{\left(a+\frac{1}{2}\right)_n 2^{4n}\, n!}\,; \tag{14}$$

similarly, in (13) changing n to $2n+1$ by the result (8) again we conclude

$$a_{2n+1} = -\frac{1-2a/d}{(2a+1)\left(a+\frac{3}{2}\right)_n 2^{4n+1}\, n!}\,. \tag{15}$$

Substituting the values of a_{2n} and a_{2n+1} in (12) and summing the series, we arrive at the right–hand side of (11). □

Remark 1. *In* (11)*, if we take* $d = 2a$*, we get Kummer's second theorem* (4)*. Thus,* (11) *may be regarded as the generalization of Kummer's second theorem* (4)*.*

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