Common fixed point theorems through generalized altering distance functions

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Abstract. *In this paper we obtain a unique common fixed point theorem, for four self maps using a generalized altering distance function in four variables, which generalizes and improves the main theorem of Choudhury [2]. We also obtain a Gregus type common fixed point theorem for four maps as a corollary.*

Key words: *fixed points, semi compatible mappings, altering distance functions*

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1. Introduction

M. S. Khan et al. [12] introduced the altering distances and used it for solving fixed points problems in metric spaces. Recently, for example authors of [1, 4, 5, 8, 9, 10] used the altering distance function and obtained some fixed point theorems. In 2005 Choudhury [2] introduced a generalized distance function in three variables and obtained a common fixed point theorem for a pair of self maps in a complete metric space. The main aim of this paper is to prove the existence and uniqueness of common fixed points of two pairs of semi compatible or compatible pair of type(β) mappings by using a generalized altering distance function of four variables. Our results extend and generalize the main theorem of [2]. We also give an example to show that our theorem is not valid when the generalized altering distance function is in five variables or the second generalized altering distance function is dropped.

In [14], Cho et al. introduced the non symmetrical concept of semi compatibility of maps in d-complete topological spaces. They defined a pair of self maps(S,T) to be semi compatible if (i) $Sy = Ty$ implies $STy = TSy$ and (ii) $\lim\limits_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$ implies $\lim_{n\to\infty} STx_n = Tx$ hold. Bijendra

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Singh and Jain [3] observed that (ii) implies (i). Hence they defined the semi compatibility by the condition (ii) only.

In [6], Pathak et. al defined a pair of self maps (S, T) to be compatible of type (P) or (β) if $\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$. It is clear that in this case $STy = TSy$ whenever $Sy = Ty$ for any $y \in X$.

Remark 1.1. *The concepts semi compatible pair and compatible pair of type (β) maps are independent in view of the following examples.*

Example 1.2. *Let $X = [0, 2]$ with usual metric. Let*

$$f(x) = \begin{cases} 2-x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ 2 & \text{if } x \ge 1. \end{cases}$$

Let $x_n = 1 - 1/n$,$n > 1$. Then $fx_n = 1 + 1/n \to 1$, $Sx_n = x_n = 1 - 1/n \to 1$. $fSx_n = 1 + 1/n \to 1 \neq S(1)$ and $d(f^2x_n, S^2x_n) = d(1, 1 - 1/n) \to 0$. Thus (f, S) is not semi compatible and (f, S) is compatible pair of type (β).

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Let $x_n = 1 - 1/n$,$n > 1$. Then $fx_n = 1 + 1/n \to 1$, $Sx_n = 1 - 1/n \to 1$. $fSx_n = 1 + 1/n \to 1 = S(1)$, $d(f^2x_n, S^2x_n) = d(2, 1 - 1/n) = 1 + 1/n \to 1 \neq 0$. Thus (f, S) is semi compatible and (f, S) is not compatible pair of type (β).

Let Ψ_n denote the set of all functions $\psi : [0, \infty)^n \to [0, \infty)$ such that

(i) ψ is continuous,

(ii) ψ is monotone increasing in all the variables,

(iii) $\psi(t_1, t_2, ..., t_n) = 0$ if and only if $t_1 = t_2 = ... = t_n = 0$.

The functions in Ψ_n are called generalized altering distance functions.

Theorem 1.4. (Theorem 1, [2]) *Let (X, d) be a complete metric space and S and T be self maps on X such that*

$$\phi_1(d(Sx, Ty)) \le \psi_1(d(x, y), d(x, Sx), d(y, Ty)) - \psi_2(d(x, y), d(x, Sx), d(y, Ty))$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_3$ and $\phi_1(x) = \psi_1(x, x, x) \forall x \in [0, \infty)$. Then S and T have a unique common fixed point.

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space and $f, g, S, T : X \to X$ be such that*

(i) $$\phi_1(d(fx, gy)) \le \psi_1 \begin{pmatrix} d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ \frac{1}{2}[d(Sx, gy) + d(Ty, fx)] \end{pmatrix} - \psi_2 \begin{pmatrix} d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ \frac{1}{2}[d(Sx, gy) + d(Ty, fx)] \end{pmatrix}$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_4$ and $\phi_1(x) = \psi_1(x, x, x, x) \forall x \in [0, \infty)$, *(ii) one of four mappings f, g, S and T is continuous,*

(iii) (f, S) and (g, T) are semi compatible pairs ,

(iv) $f(X) \subseteq T(X), g(X) \subseteq S(X)$.

Then f, g, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. From (iv), construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $fx_{2n} = Tx_{2n+1} = y_{2n}, gx_{2n+1} = Sx_{2n+2} = y_{2n+1}$, $n = 0, 1, 2, \ldots$. Let $a_n = d(y_n, y_{n+1})$. Putting $x = x_{2n}, y = x_{2n+1}$ in (i) we get

$$\begin{aligned}\phi_1(a_{2n}) \leq \psi_1\Big(a_{2n-1}, a_{2n-1}, a_{2n}, \frac{1}{2}[a_{2n-1} + a_{2n}]\Big) \\ -\psi_2\Big(a_{2n-1}, a_{2n-1}, a_{2n}, \frac{1}{2}d(y_{2n-1}, y_{2n+1})\Big). \qquad (1)\end{aligned}$$

If $a_{2n-1} < a_{2n}$ then

$$\begin{aligned}\phi_1(a_{2n}) &\leq \psi_1(a_{2n}, a_{2n}, a_{2n}, a_{2n}) - \psi_2(a_{2n-1}, a_{2n-1}, a_{2n}, \frac{1}{2}d(y_{2n-1}, y_{2n+1})) \\ &< \phi_1(a_{2n})\end{aligned}$$

which is a contradiction. Hence $a_{2n} \leq a_{2n-1}$, $n = 1, 2, \ldots$. Similarly by putting $x = x_{2n+2}, y = x_{2n+1}$ in (i) we can show that $a_{2n+1} \leq a_{2n}$, $n = 0, 1, 2, \ldots$. Thus $a_{n+1} \leq a_n, n = 0, 1, 2, ...$ so that $\{a_n\}$ is a decreasing sequence of non negative real numbers and hence convergent to some $a \in R$.

Let $b = \lim\limits_{n\to\infty} \frac{1}{2}d(y_n, y_{n+2})$. Letting $n \to \infty$ in (1) we get

$$\phi_1(a) \leq \psi_1(a, a, a, a) - \psi_2(a, a, a, b) = \phi_1(a) - \psi_2(a, a, a, b).$$

Thus $\psi_2(a, a, a, b) = 0$ so that $a = b = 0$. Hence

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \qquad (2)$$

To show that$\{y_n\}$ is a Cauchy sequence, it is sufficient to show that the sub sequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in view of (2). If $\{y_{2n}\}$ is not Cauchy, there exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k) > m(k)$, $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$ and

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon \qquad (3)$$

From (3),

$$\begin{aligned}\epsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \epsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})\end{aligned}$$

Letting $k \to \infty$, using (2), we have

$$\lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \qquad (4)$$

Letting $k \to \infty$, using (4), and (2) in

$$|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1})$$

we get

$$\lim_{n\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) = \epsilon. \tag{5}$$

Letting $k \to \infty$ and using (4), (2) in

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1})$$

we get

$$\lim_{n\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \epsilon. \tag{6}$$

Letting $k \to \infty$ and using (6) and (2) in

$$|d(y_{2m(k)-1}, y_{2n(k)+1}) - d(y_{2m(k)-1}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1})$$

we get

$$\lim_{n\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) = \epsilon. \tag{7}$$

Putting $x = x_{2m(k)}, y = x_{2n(k)-1}$ in (i) we have

$$\begin{aligned}\phi_1(d(y_{2m(k)}, y_{2n(k)+1})) \leq \psi_1 &\begin{pmatrix} d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), \\ d(y_{2n(k)}, y_{2n(k)+1}), \\ \frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}) + d(y_{2n(k)}, y_{2m(k)})] \end{pmatrix} \\ &- \psi_2 \begin{pmatrix} d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), \\ d(y_{2n(k)}, y_{2n(k)+1}), \\ \frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}) + d(y_{2n(k)}, y_{2m(k)})] \end{pmatrix}.\end{aligned}$$

Letting $k \to \infty$ and using (2), (4), (5), (6) and (7) we get

$$\phi_1(\epsilon) \leq \psi_1(\epsilon, 0, 0, \epsilon) - \psi_2(\epsilon, 0, 0, \epsilon) < \psi_1(\epsilon, \epsilon, \epsilon, \epsilon) = \phi_1(\epsilon).$$

It is a contradiction. Therefore $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence from(2). Since X is complete, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Case: Suppose S is continuous. Then $Sfx_{2n} \to Sz, S^2x_{2n} \to Sz$. Since (f, S) is semi compatible , we have $fSx_{2n} \to Sz$.

Step I: Putting $x = Sx_{2n}, y = x_{2n+1}$ in (i) we have

$$\begin{aligned}\phi_1(d(fSx_{2n}, gx_{2n+1})) \leq \psi_1 &\begin{pmatrix} d(S^2x_{2n}, Tx_{2n+1}), d(S^2x_{2n}, fSx_{2n}), \\ d(Tx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}[d(S^2x_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fSx_{2n})] \end{pmatrix} \\ &- \psi_2 \begin{pmatrix} d(S^2x_{2n}, Tx_{2n+1}), d(S^2x_{2n}, fSx_{2n}), \\ d(Tx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}[d(S^2x_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fSx_{2n})] \end{pmatrix}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\phi_1(d(Sz,z)) &\leq \psi_1(d(Sz,z),0,0,d(Sz,z)) - \psi_2(d(Sz,z),0,0,d(Sz,z)) \\ &< \psi_1(d(Sz,z),d(Sz,z),d(Sz,z),d(Sz,z)) \\ &= \phi_1(d(Sz,z).\end{aligned}$$

It is a contradiction if $Sz \neq z$. Hence $Sz = z$.
Step II : Putting $x = z, y = x_{2n+1}$ in (i) and letting $n \to \infty$ we get $fz = z$.
Step III: Since $z = fz \in f(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$. Putting $x = x_{2n}, y = u$ in (i) and letting $n \to \infty$, we get $z = gu$ and hence $gu = Tu$. Since (g, T) is semi compatible, we have $gTu = Tgu$ so that $gz = Tz$.
Step IV: Putting $x = z, y = z$ in (i), we get $gz = z$ so that $gz = z = Tz$. Thus z is a common fixed point of f, g, S and T.
Case: Suppose f is continuous. Then $f^2x_{2n} \to fz, fSx_{2n} \to fz$. Since (f, S) is semi compatible, $fSx_{2n} \to Sz$. Hence $fz = Sz$. The rest of the proof follows from Steps II,III, IV. Similarly we can show that z is a common fixed point of f, g, S and T when g or T is continuous as in previous two cases. Uniqueness of common fixed point follows easily from (i). □

Remark 2.2. Thoerem 2.1 *is not true if the condition (i) is replaced by*
(A) $\phi_1(d(fx,gy)) \leq \psi_1(d(Sx,Ty),d(Sx,fx),d(Ty,gy),d(Sx,gy),d(Ty,fx))$
$\qquad -\psi_2(d(Sx,Ty),d(Sx,fx),d(Ty,gy),d(Sx,gy),d(Ty,fx))$
for all $x, y \in X$, *where* $\psi_1, \psi_2 \in \Psi_5$ *and* $\phi_1(x) = \psi_1(x,x,x,x,x)\forall x \in [0,\infty)$ *in view of the following Example even when* $S = T = I$ *(Identity map).*

Example 2.3.(Ex.6, [7]) *Let* $X = \{1,2,3,4\}, d(n,n) = 0\forall n \in X, d(1,2) = d(3,4) = 2, d(1,3) = d(2,4) = 1, d(1,4) = d(2,3) = \frac{3}{2}$. $f, g : X \to X$ *as* $f1 = f4 = 2, f2 = f3 = 1$ *and* $g1 = g3 = 4, g2 = g4 = 3$.
Clearly,

$$d(fx,gy) \leq \frac{3}{4} max\{d(x,y),d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}$$

$\forall x, y \in X$. *Let* $\psi_1(t_1,t_2,t_3,t_4,t_5) = max\{t_1,t_2,t_3,t_4,t_5\}, \psi_2 = \frac{1}{4}\psi_1$. *Then* $\phi_1(t) = t\forall t \in [0,\infty)$. *All conditions of* Theorem 2.1 *are satisfied except (i). Here the condition (A) is satisfied with* $S = T = I$ *(identity map). But* f *and* g *have no common fixed point in* X.

Remark 2.4. Example 2.3 *also shows the importance of second generalized altering distance function* ψ_2. Theorem 2.1 *is not true if the condition (i) is replaced by*
(B) $\phi_1(d(fx,gy)) \leq \psi_1(d(Sx,Ty),d(Sx,fx),d(Ty,gy),\frac{1}{2}[d(Sx,gy)+d(Ty,fx)])$
for all $x, y \in X$, *where* $\psi_1 \in \Psi_4$ *and* $\phi_1(x) = \psi_1(x,x,x,x)\forall x \in [0,\infty)$.

In Example 2.3, *we have* $d(fx,gy) \leq max\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}[d(x,gy)+d(y,fx)]\}$ *for all* $x, y \in X$. *Let* $\psi_1(t_1,t_2,t_3,t_4) = max\{t_1,t_2,t_3,t_4\}$ *and* $S = T = I$ *(Identity map). Then* $\phi_1(t) = t\forall t \in [0,\infty)$. *All conditions of* Theorem 2.1 *are satisfied except (i). Here the condition (B) is satisfied with* $S = T = I$ *(identity map). But* f *and* g *have no common fixed point in* X.

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. Here, we give some corollaries to our theorem.

Corollary 2.5. Theorem 2.1 *with the inequality (i) is replaced by*

$$d(fx, gy) \leq kmax\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{1}{2}[d(Sx, gy) + d(Ty, fx)]\}$$

$\forall x, y \in X$, *where* $0 < k < 1$.

Proof. Let $\psi_1(t_1, t_2, t_3, t_4) = max\{t_1, t_2, t_3, t_4\}$ and $\psi_2(t_1, t_2, t_3, t_4) = (1-k)$ $\psi_1(t_1, t_2, t_3, t_4)$.Then $\phi_1(t) = t \forall t \in [0, \infty)$. The corollary then follows from *Theorem 2.1.* □

Now we give a Gregus type [11] common fixed point theorem for four maps.

Corollary 2.6. Theorem 2.1 *with the inequality (i) is replaced by*

$$d^s(fx, gy) \leq \phi(ad^s(Sx, Ty) + (1-a)max\{d^s(Sx, fx), d^s(Ty, gy)\})$$

$\forall x, y \in X$, *where s is any positive integer and $\phi : [0, \infty) \to [0, \infty)$ is continuous, decreasing and $\phi(t) > 0 \forall t > 0$.*

Proof. Let $\psi_1(t_1, t_2, t_3, t_4) = at_1^s + (1-a)max\{t_2^s, t_3^s\}, \psi_2(t_1, t_2, t_3, t_4) = \psi_1(t_1, t_2, t_3, t_4) - \phi(\psi_1(t_1, t_2, t_3, t_4))$. Then $\phi_1(t) = \psi_1(t, t, t, t) = t^s$. Now the corollary follows from *Theorem 2.1.* □

Note: *Corollary 2.6* with $S = T = I$ is a different version of theorem of Sessa and Fisher [13].

Finally we give the following theorem for compatible of type(β) pairs of mappings.

Theorem 2.7. Theorem 2.1 *with condition (iii) is replaced by*
(iii)^1 *(f, S) and (g, T) are compatible of type (β) pairs.*

Proof. As in *Theorem 2.1*, the sequence $\{y_n\}$ converges to $z \in X$.

Case: Suppose S is continuous. Then $Sfx_{2n} \to Sz, S^2x_{2n} \to Sz$. Since (f, S) is compatible of type (β), we have $f^2x_{2n} \to Sz$. Putting $x = fx_{2n}, y = x_{2n+1}$ in (i) and letting $n \to \infty$, we get $Sz = z$. Putting $x = z, y = x_{2n+1}$ in (i) and letting $n \to \infty$, we get $fz = z$. As in Steps III, IV in *Theorem 2.1*, we have $gz = z = Tz$.

Case: Suppose f is continuous. Then $fSx_{2n} \to fz, f^2x_{2n} \to fz$. Since (f, S) is compatible pair of type(β), we have $S^2x_{2n} \to fz$.

Putting $x = Sx_{2n}, y = x_{2n+1}$ in (i) and letting $n \to \infty$, we get$fz = z$. Since $z = fz \in f(X) \subseteq T(X)$, there exists $w \in X$such that $z = Tw$.

Putting $x = x_{2n}, y = w$ in (i) and letting $n \to \infty$, we get $z = gw$ so that $gw = Tw$. Since (g, T) is compatible pair of type (β), we have $gz = Tz$. Putting $x = x_{2n}, y = z$ in (i) and letting $n \to \infty$, we get $gz = z$ so that $Tz = z$. Since $z = gz \in g(X) \subseteq S(X)$, there exists $v \in X$ such that $z = Sv$. Putting $x = v, y = x_{2n+1}$ in (i) and letting $n \to \infty$, we get $fv = z$ so that $z = fv = Sv$. Since (f, S) is compatible pair of type (β), we have $fz = Sz$ so that $Sz = fz = z$. Similarly we can show that z is a common fixed point of f, g, S and T when T or g is continuous. Uniqueness of z follows easily from (i). □

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