On sensitivity analysis of general variational inequalities

MUHAMMAD ASLAM NOOR* AND KHALIDA INAYAT NOOR†

Abstract. *It is well known that the Wiener-Hopf equations are equivalent to the general variational inequalities. We use this alternative equivalent formulation to study the sensitivity of the general variational inequalities without assuming the differentiability of the given data. Since the general variational inequalities include classical variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. In fact, our results can be considered as a significant extension of previously known results.*

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1. Introduction

Variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, optimization , operations research and engineering sciences, see, for example [1-25] and the references therein. The behavior of such equilibrium solutions as a result of changes in the problem data is always of concern. In this paper, we study the sensitivity analysis of a class of variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed. We remark that sensitivity analysis is important for several reasons. First, since estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing systems. Third, sensitivity analysis provides useful information for designing or planning various equilibrium

*Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan, e-mail: `noormaslam@hotmail.com`

†Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan, e-mail: `khalidanoor@hotmail.com`

systems. Furthermore, from mathematical and engineering points of view, sensitivity analysis can provide new insight regarding problems being studied and can stimulate new ideas for problem solving. Over the last decade, there has been increasing interest in studying the sensitivity analysis of variational inequalities and variational inclusions. Sensitivity analysis for variational inclusions and inequalities has been studied by many authors including Tobin [23], Kyparisis [6,7], Dafermos [3], Qiu and Magnanti [20], Yen [24], Noor [11-14], Moudafi and Noor [9], Noor and Noor [18] and Liu [8] using quite different techniques. The techniques suggested so far vary with the problem being studied. Dafermos [3] used the fixed-point formulation to consider the sensitivity analysis of the classical variational inequalities, whereas Noor [13] used the Wiener-Hopf equations technique. These techniques have been modified and extended by many authors for studying the sensitivity analysis of other classes of variational inequalities and variational inclusions, see [1,9,11,12, 18-20, 23-25] and the references therein.

Variational inequalities have been extended and generalized in various directions using the novel and innovative techniques, which proved to be productive and useful. Noor [15] has introduced a new class of variational inequalities involving three operators and is known as the general variational inequality. It has been shown in [15] that the minimum of a class of differentiable nonconvex functions on the nonconvex set can be characterized via the general variational inequalities. We would like to mention this new class of general variational inequalities is quite different than the general variational inequalities,which were introduced and studied by Noor [10] in 1988. For the recent developments in general variational inequalities, see Noor [14] and the references therein. We would like to point out this new class of variational inequalities is quite general and includes the variational inequalities introduced by Stampacchia [22] in 1964 and several other optimization problems as special cases. This clearly shows that the general variational inequalities are unifying one and has significant applications in different fields of pure and applied sciences.

In this paper, we study the sensitivity analysis of the general variational inequalities. Noor [15] has established the equivalence between the general variational inequalities and the Wiener-Hopf equations by using the projection operator method. This fixed-point formulation is obtained by a suitable and appropriate rearrangement of the Wiener-Hopf equations. We would like to point out that the Wiener-Hopf equations technique is quite general, unified, flexible and provides us with a new approach to study the sensitivity analysis of variational inclusions and related optimization problems. We use this equivalence to develop sensitivity analysis for the general variational inequalities without assuming the differentiability of the given data. Our results can be considered as significant extensions of the results of Dafermos [3], Moudafi and Noor [9], Noor [13] and others in this area. The techniques and idea of this paper can be used to study the sensitivity analysis for the multivalued general variational inequalities.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g : H \to H$, consider the problem of finding $u \in K$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.1}$$

where $\rho > 0$ is a constant. Inequality of type (2.1) was introduced and studied by Noor [15]. It is clear that, if $g(u) = u$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.2}$$

which is known as the general variational inequality. We would like to point out that the general variational inequality (2.2) is quite different than the general variational inequality, which was introduced by Noor [10] in 1988. See also [13,14, 19]. It has been shown [15] that the minimum of a class of differentiable nonconvex function on a nonconvex set can be characterized by the general variational inequalities (2.2). For $g = I$, the identity operator, the general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \forall v \in K,$$

which is known as the classical variational inequality and was introduced in 1964 by Stampacchia [22]. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see [1-25] and the references therein.

From the above discussion, it is clear that the general variational inequalities (2.1) is most general and includes several previously known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

We also need the following concepts and results.

Lemma 2.1. *Let K be a closed convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K z$, *where P_K is the projection of H onto the closed convex set K in H.*

It is well known that the projection operator P_K is a nonexpansive operator.

Related to the general variational inequalities (2.1), we consider the problem of solving the Wiener-Hopf equations. To be precise, let $Q_K = I - gP_K$, where I is the identity operator. For given nonlinear operators T, g, we consider the problem of finding $z \in H$ such that

$$TP_K z + \rho^{-1} Q_K z = 0, \tag{2.3}$$

which is called the general Wiener-Hopf equation. We note that, if $g = I$, then one can obtain the original Wiener-Hopf equations, which are mainly due to Shi [21]. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimization problems, see[13-15, 17-19,21] and the references therein.

We now consider the parametric versions of the problem (2.1) and (2.3). To formulate the problem, let M be an open subset of H in which the parameter λ takes values. Let $T(u, \lambda)$ be given operator defined on $H \times H \times M$ and take value in $H \times H$.

From now onward, we denote $T_\lambda(.) \equiv T(., \lambda)$ unless otherwise specified.

The parametric general variational inequality problem is to find $(u, \lambda) \in H \times M$ such that

$$\langle \rho T_\lambda u + u - g(u), g(v) - u \rangle \geq 0, \forall v \in H : g(v) \in K. \tag{2.4}$$

We also assume that for some $\overline{\lambda} \in M$, problem (2.4) has a unique solution $\overline{u}$.

Related to the parametric variational inclusion (2.4), we consider the parametric Wiener-Hopf equations. We consider the problem of finding $(z, \lambda) \in H \times M$, such that

$$T_\lambda P_K z + \rho^{-1} Q_K z = 0, \tag{2.5}$$

where $\rho > 0$ is a constant and $Q_K z$ is defined on the set of (z, λ) with $\lambda \in M$ and takes values in H. The equations of the type (2.5) are called the parametric Wiener-Hopf equations.

One can establish the equivalence between the problems (2.4) and (2.5) by using the projection operator technique, see Noor [13,14,15].

Lemma 2.2. *The parametric general variational inequality (2.4) has a solution $(u, \lambda) \in H \times M$ if and only if the parametric Wiener-Hopf equations (2.5) have a solution $(z, \lambda) \in H \times M$, where*

$$u = P_K z \tag{2.6}$$

$$z = g(u) - \rho T_\lambda(u). \tag{2.7}$$

From *Lemma 2.2*, we see that the parametric general variational inequalities (2.4) and the parametric Wiener-Hopf equations (2.5) are equivalent. We use this equivalence to study the sensitivity analysis of the general variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (2.5) has a solution $\overline{z}$ and X is a closure of a ball in H centered at $\overline{z}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (2.9) has a unique solution $z(\lambda)$ near $\overline{z}$ and the function $z(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 2.1. *Let $T_\lambda(.)$ be an operator on $X \times M$. Then, the operator $T_\lambda(.)$ is said to :*
(a) Locally strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall \lambda \in M, u, v \in X.$$

(b) Locally Lipschitz continuous if there exists a constant $\beta > 0$ *such that*

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta\|u - v\|, \quad \forall\lambda \in M, u, v \in X.$$

3. Main results

We consider the case, when the solutions of the parametric Wiener-Hopf equations (2.5) lie in the interior of X. Following the ideas of Dafermos [3] and Noor [13,14], we consider the map

$$\begin{aligned} F_\lambda(z) &= P_K z - \rho T_\lambda(u), \quad \forall \quad (z, \lambda) \in X \times M \\ &= g(u) - \rho T_\lambda(u), \end{aligned} \tag{3.1}$$

where

$$u = P_K z. \tag{3.2}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of the parametric Wiener-Hopf equations (2.5). First of all, we prove that the map $F_\lambda(z)$, defined by (3.1), is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 3.1. *Let* $T_\lambda(.)$ *be a locally strongly monotone with constant* $\alpha > 0$ *and locally Lipschitz continuous with constant* $\beta > 0$. *If that the operators* g *is strongly monotone with constants* $\sigma > 0$ *and Lipschitz continuous with constants* $\delta > 0$ *respectively, then, for all* $z_1, z_2 \in X$ *and* $\lambda \in M$, *we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta\|z_1 - z_2\|,$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\} \tag{3.3}$$

for

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.4}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}. \tag{3.5}$$

Proof. For all $z_1, z_2 \in X$, $\lambda \in M$, we have, from (3.1),

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &= \|g(u_1) - g(u_2) - \rho(T_\lambda(u_1) - T_\lambda(u_2))\| \\ &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|. \end{aligned} \tag{3.6}$$

Using the strong monotonicity and Lipschitz continuity of the operator g, we have

$$\begin{aligned}\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 &\leq \|u_1 - u_2\|^2 - 2\langle u_1 - u_2, g(u_1) - g(u_2)\rangle \\ &\quad + \|g(u_1) - g(u_2)\|^2 \\ &\leq (1 - 2\delta + \sigma^2)\|u_1 - u_2\|^2. \end{aligned} \tag{3.7}$$

In a similar way, we have

$$\|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|^2 \leq (1 - 2\rho\alpha + \beta^2\rho^2)\|u_1 - u_2\|^2, \tag{3.8}$$

where $\alpha > 0$ is the strongly monotonicity constant and $\beta > 0$ is the Lipschitz continuity constant of the operator T_λ respectively.

From (3.6), (3.7) and (3.8), we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \left\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\}\|u_1 - u_2\|. \tag{3.9}$$

From (3.2) and using the nonexpnsivity of the operator P_K, we have

$$\|u_1 - u_2\| \leq \|P_K z_1 - P_K z_2\| \leq \|z_1 - z_2\|. \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\begin{aligned}\|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \left\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right\}\|z_1 - z_2\| \\ &= \theta\|z_1 - z_2\|, \quad \text{using (3.3)}.\end{aligned}$$

From (3.3), and (3.4), it follows that $\theta < 1$ and consequently the map $F_\lambda(z)$ defined by (3.1) is a contraction map and has a fixed point $z(\lambda)$, which is the solution of the Wiener-Hopf equation (2.5). □

Remark 3.1. *From* Lemma 3.1*, we see that the map $F_\lambda(z)$ defined by (3.1) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also, by assumption, the function $\overline{z}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric Wiener-Hopf equations (2.5). Again using* Lemma 3.1*, we see that $\overline{z}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\overline{\lambda}}(z)$. Consequently, we conclude that*

$$z(\overline{\lambda}) = \overline{z} = F_{\overline{\lambda}}(z(\overline{\lambda})).$$

Using Lemma 3.1*, we can prove the continuity of the solution $z(\lambda)$ of the parametric Wiener-Hopf equations (2.9) using the technique of Noor [13,14]. However, for the sake of completeness and to convey an idea of the techniques involved, we give its proof.*

Lemma 3.2. *Assume that the operator $T_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operator $T_\lambda(.)$ is Locally Lipschitz continuous and the map $\lambda \to P_{K_\lambda} z$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (2.5) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. For all $\lambda \in M$, invoking *Lemma 3.1* and the triangle inequality, we have

$$\begin{aligned}\|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}(z(\bar{\lambda})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\bar{\lambda})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|.\end{aligned} \tag{3.11}$$

From (3.1) and the fact that the operator T_λ is a Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned}\|F_\lambda(z(\bar\lambda)) - F_{\bar\lambda}(z(\bar\lambda))\| &= \|u(\bar\lambda) - u(\bar\lambda) + \rho(T_\lambda(u(\bar\lambda), u(\bar\lambda)) - T_{\bar\lambda}(u(\bar\lambda), u(\bar\lambda)))\| \\ &\leq \rho\mu\|\lambda - \bar\lambda\|. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\|z(\lambda) - z(\bar\lambda)\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar\lambda\|, \quad \text{for all } \lambda, \bar\lambda \in M,$$

from which the required result follows. □

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 3.1. *Let $\overline{u}$ be the solution of the parametric general variational inequality (2.9) and $\overline{z}$ be the solution of the parametric Wiener-Hopf equations (2.5) for $\lambda = \overline{\lambda}$. Let $T_\lambda(u)$ be the locally strongly monotone Lipschitz continuous operator for all $u, v \in X$. If the the map $\lambda \to P_K$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric Wiener-Hopf equations (2.5) have a unique solution $z(\lambda)$ in the interior of $X, z(\overline{\lambda}) = \overline{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from *Lemmas 3.1*, *3.2* and *Remark 3.1*. □

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