Common fixed points and best approximants in nonconvex domain

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Abstract. *The aim of the paper is to show the validity of results of Imdad [7] in a domain which is not necessarily starshaped and mappings are not necessarily linear. Our results also improve, extend and generalize various existing known results in the literature.*

Key words: *best approximant, fixed point, compatible pair, contractive jointly continuous family, starshaped subset*

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1. Introduction and preliminaries

Let $\mathcal{X}$ be a normed space and let $\mathcal{C}$ be a nonempty subset of $\mathcal{X}$. Let $x \in \mathcal{X}$. An element $y \in \mathcal{C}$ is called a best $\mathcal{C}$-approximant to $x \in \mathcal{X}$ if

$$\|x - y\| = dist(x, \mathcal{C}) = \inf\{\|x - z\| : z \in \mathcal{C}\}.$$

The set of best $\mathcal{C}$-approximants to x is denoted by $\mathcal{D}$ and is defined as $\mathcal{D} = \{y \in \mathcal{C} : \|x - y\| = dist(x, \mathcal{C})\}$. Let $\mathcal{I}, \mathcal{J} : \mathcal{C} \to \mathcal{C}$ be two mappings. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called an $(\mathcal{I}, \mathcal{J})$-contraction if there exists $0 \leq k < 1$ such that $d(\mathcal{T}x, \mathcal{T}y) \leq kd(\mathcal{I}x, \mathcal{J}y)$ for any $x, y \in \mathcal{C}$. If $k = 1$, then $\mathcal{T}$ is called $(\mathcal{I}, \mathcal{J})$-nonexpansive. Also if $\mathcal{I} = \mathcal{J}$, we say that $\mathcal{T}$ is called $\mathcal{I}$-nonexpansive. The set of fixed points of $\mathcal{T}$(resp. $\mathcal{I}$) is denoted by $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$). A point $x \in \mathcal{C}$ is a common fixed point of $\mathcal{I}$ and $\mathcal{T}$ if $x = \mathcal{I}x = \mathcal{T}x$. The pair $(\mathcal{I}, \mathcal{T})$ is called: (1) commuting if $\mathcal{I}\mathcal{T}x = \mathcal{T}\mathcal{I}x$ for all $x \in \mathcal{C}$; (2) compatible if $\lim_{n\to\infty} \|\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n\| = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{C}$ such that

$$\lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} \mathcal{I}x_n = t \in \mathcal{C}.$$

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Every commuting pair of mappings is compatible but the converse is not true in general [9].

Let $\mathcal{X}$ be a normed space. A set $\mathcal{C}$ in $\mathcal{X}$ is said to be convex, if $\lambda x+(1-\lambda)y \in \mathcal{C}$, whenever $x, y \in \mathcal{C}$ and $0 \leq \lambda \leq 1$.

The set $\mathcal{C}$ is $p-$starshaped if the line segment $[p, x] = \{(1-\lambda)p + \lambda x\}$ joining p to x, is contained in $\mathcal{C}$ for all $x \in \mathcal{C}$ and $0 < \lambda < 1$. In this case p is called the starcenter of $\mathcal{C}$.

Each convex set is starshaped with respect to each of its points, but not conversely.

We give the definition providing the notion of contractive jointly continuous family introduced by Dotson [3].

Let $\Im = \{f_\alpha\}_{\alpha\in\mathcal{C}}$ a family of functions from $[0,1]$ into $\mathcal{C}$ such that $f_\alpha(1) = \alpha$ for each $\alpha \in \mathcal{C}$. The family $\Im$ is said to be contractive, if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in \mathcal{C}$ and all $t \in (0,1)$, we have

$$\|f_\alpha(t) - f_\beta(t)\| \leq \phi(t)\|\alpha - \beta\|.$$

The family $\Im$ is said to be jointly continuous if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in $\mathcal{X}$, then $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in $\mathcal{X}$.

If $\mathcal{X}$ is a normed linear space and $\Im$ is a family as above, then $\Im$ is said to be *jointly weakly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in $\mathcal{C}$ imply that $f_\alpha(t) \to^w f_{\alpha_0}(t_0))$ in $\mathcal{X}$.

Hence, *property* (Γ) on contractive jointly continuous family $\Im$ can now be defined as:

A self mapping $\mathcal{T}$ of $\mathcal{C}$ is said to satisfy the *property* (Γ), if for any $t \in [0,1]$, for all $\alpha \in \mathcal{C}$ and for all $f_\alpha \in \Im$, we have $\mathcal{T}(f_\alpha(t)) = f_{\mathcal{T}\alpha}(t)$.

For clarification of a metric space that satisfies the notion of a contractive and jointly continuous family of functions, a lemma is presented below, it gives the concept of contractive and jointly continuous family of functions. It also implies that in Euclidean n-space such a set must be connected.

Lemma 1 [see [14]]. *Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{C}$ a nonempty subset which (as a subspace) is not connected. Suppose that $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, $\mathcal{C}_0 \cap \mathcal{C}_1 = \phi$ where $\mathcal{C}_0$ and $\mathcal{C}_1$ are both open and closed, and suppose that there exist $x \in \mathcal{C}_0$ and $y \in \mathcal{C}_1$ such that $d(x, y) = d(\mathcal{C}_0, \mathcal{C}_1)$. Then $\mathcal{C}$ does not admit a jointly continuous contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{C}}$; i.e. $\mathcal{C}$ does not have the property of contractiveness and joint continuity.*

A consequence of this lemma is that, in a finite-dimensional Banach space, every bounded subset (considered as a metric space) that has the property of contractiveness and joint continuity must be connected. For closed bounded sets are compact, and the conditions of the lemma are satisfied in this case.

Existence of fixed point have been used at many places in the field of approximation theory. One of them is to prove existence of best approximation with help of fixed point. In 1963, Meinardus [12] employed the Schauder fixed-point theorem to establish the existence of an invariant approximation. Further, Brosowski [1] obtained a celebrated result and generalized the Meinardus's result. Afterwards, a number of results has been proved in the direction of Brosowski [1] (see in [6, 15, 16, 18]).

Recently, Imdad [7] proved result on common fixed point via best approximation for noncommuting and linear mappings in a domain which is starshaped.

Here it is important to remark that Dotson [2] proved the existence of fixed point for nonexpansive mapping in the setup of starshaped. He further extended his result without starshapedness under non-convex condition [3]. This idea was utilized by Mukherjee and Som [13] to prove existence of fixed point as best approximant. In this way, they extended the result of Singh [16] without starshapedness condition.

The aim of the paper is to show the validity of results of Imdad [7] in a domain which is not necessarily starshapedness and mappings are not necessarily linear. Incidently, results of Dotson [2], Habiniak [4], Sahab, Khan and Sessa [15], Singh [16, 17] are extended, improved and generalized.

The following common fixed point result is needed in the sequel.

Theorem 1 [see [8]]. . *Let $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ be self-maps of a complete metric space $(\mathcal{X}, d)$ with $\mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$ such that for each $x, y \in \mathcal{X}$ and $0 \leq h < 1$*

$$d(\mathcal{T}x, \mathcal{T}y) \leq h \max\{d(\mathcal{I}x, \mathcal{J}y), \tfrac{1}{2}[d(\mathcal{I}x, \mathcal{T}x) + d(\mathcal{J}y, \mathcal{T}y)], \tfrac{1}{2}[d(\mathcal{I}x, \mathcal{T}y) + d(\mathcal{J}y, \mathcal{T}x)]\}$$

then $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ has a unique common fixed point z in $\mathcal{X}$ provided any one of the following holds:

(a) $(\mathcal{T}, \mathcal{I})$ is compatible, $\mathcal{I}$ or $\mathcal{T}$ is continuous and $(\mathcal{T}, \mathcal{J})$ are coincidently commuting.

(a') $(\mathcal{T}, \mathcal{J})$ is compatible, $\mathcal{J}$ or $\mathcal{T}$ is continuous and $(\mathcal{T}, \mathcal{I})$ are coincidently commuting.

Moreover, z remains the common fixed point of the pairs $(\mathcal{T}, \mathcal{I})$ and $(\mathcal{T}, \mathcal{J})$ separately.

2. Main result

First, we prove best approximation result for three mappings:

Theorem 2. *Let $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ be a self mappings of a normed space $\mathcal{X}$ and $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$. Let $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. Assume $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ satisfy the condition*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \mathcal{M}(x, y)$$

where

$$\mathcal{M}(x, y) = \max\{\|\mathcal{I}x - \mathcal{J}y\|, \tfrac{1}{2}[\|\mathcal{I}x - \mathcal{T}x\| + \|\mathcal{J}y - \mathcal{T}y\|], \tfrac{1}{2}[\|\mathcal{I}x - \mathcal{T}y\| + \|\mathcal{J}y - \mathcal{T}x\|]\} \tag{1}$$

for all $x, y \in \mathcal{D} \cup \{\overline{x}\}$*.*
Further, suppose that the pair $(\mathcal{T}, \mathcal{I})$ *and* $(\mathcal{T}, \mathcal{J})$ *are compatible and any one of* $\mathcal{T}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *is continuous. If* $\mathcal{D}$ *is nonempty compact and has a contractive jointly continuous family* $\Im = \{f_x\}_{x\in\mathcal{D}}$ *such that* $\mathcal{I}$ *and* $\mathcal{J}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0, 1]$ *and* $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$*, then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \phi.$$

Proof. First, we show that $\mathcal{T}$ is a self map on $\mathcal{D}$, i.e., $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. Let $y \in \mathcal{D}$, then $\mathcal{I}y \in \mathcal{D}$, since $\mathcal{I}(\mathcal{D}) = \mathcal{D}$. Also, if $y \in \partial\mathcal{C}$, then $\mathcal{T}y \in \mathcal{C}$ since $\mathcal{T}(\partial\mathcal{C}) \subset \mathcal{C}$. Now

$$\begin{aligned}\|\mathcal{T}y - \overline{x}\| &= \|\mathcal{T}y - \mathcal{T}\overline{x}\| \\ &\le \max\{\|\mathcal{I}y - \mathcal{J}\overline{x}\|, \tfrac{1}{2}[\|\mathcal{I}y - \mathcal{T}y\| + \|\mathcal{J}\overline{x} - \mathcal{T}\overline{x}\|], \\ &\qquad \tfrac{1}{2}[\|\mathcal{I}y - \mathcal{T}\overline{x}\| + \|\mathcal{J}\overline{x} - \mathcal{T}y\|]\}\end{aligned}$$

yielding thereby $\mathcal{T}y \in \mathcal{D}$. Thus $\mathcal{T}$ is a self mapping of $\mathcal{D}$.

Choose $k_n \in [0, 1)$ such that $\{k_n\} \to 1$. Then define sequence $\{\mathcal{T}_n\}$ as

$$\mathcal{T}_n(x) = f_{\mathcal{T}x}(k_n)$$

for all $x \in \mathcal{D}$ and for each n. The map $\{\mathcal{T}_n\}$ is a well-defined map from $\mathcal{D}$ into $\mathcal{D}$ for each n. Since $\mathcal{I}$ satisfy property (Γ), we have

$$\mathcal{T}_n\mathcal{I}x_n = f_{\mathcal{T}\mathcal{I}x_n}(k_n)$$

$$\mathcal{I}\mathcal{T}_n x_n = \mathcal{I}f_{\mathcal{T}x_n}(k_n) = f_{\mathcal{I}\mathcal{T}x_n}(k_n)$$

Since $(\mathcal{T}, \mathcal{I})$ are compatible, therefore

$$0 \le \lim_{n\to\infty} \|\mathcal{T}_n\mathcal{I}x_n - \mathcal{I}\mathcal{T}_n x_n\| = \lim_{n\to\infty} \|f_{\mathcal{T}\mathcal{I}x_n}(k_n) - f_{\mathcal{I}\mathcal{T}x_n}(k_n)\|$$

$$\le \lim_{n\to\infty} \phi(k_n)\|\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n\| = 0,$$

whenever $\lim_{n\to\infty} \mathcal{I}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t \in \mathcal{D}$ for all n.
Hence $\{\mathcal{T}_n\}$ and $\mathcal{I}$ are compatible on $\mathcal{D}$ for each n and $\mathcal{T}_n(\mathcal{D}) \subset \mathcal{D} = \mathcal{I}(\mathcal{D})$. Similarly it can be shown that $\{\mathcal{T}_n\}$ and $\mathcal{J}$ are compatible on $\mathcal{D}$ for each n and $\mathcal{T}_n(\mathcal{D}) \subset \mathcal{D} = \mathcal{J}(\mathcal{D})$. It follows from (1) and contractiveness of $\Im$ that

$$\begin{aligned}\|\mathcal{T}_n x - \mathcal{T}_n y\| = \|f_{\mathcal{T}x}(k_n) - f_{\mathcal{T}y}(k_n)\| &\le \phi(k_n)\|\mathcal{T}x - \mathcal{T}y\| \\ &\le \phi(k_n)\mathcal{M}(x, y) < \mathcal{M}(x, y)\end{aligned}$$

for all $x, y \in \mathcal{D}$. As $\mathcal{D}$ is compact, therefore by Theorem 1,

$$\mathcal{F}(\mathcal{T}_n) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) = \{x_n\}$$

for each n. Also, since $\mathcal{D}$ is compact, there exists a subsequence of $\{x_n\}$ in $\mathcal{D}$, denoted by $\{x_m\}$, converging to a point, say, $y \in \mathcal{D}$ and hence $\mathcal{T}x_m \to \mathcal{T}y$. The jointly continuity of $\Im$ gives

$$x_m = \mathcal{T}_m x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}y}(1) = \mathcal{T}y$$

and thus the uniqueness of the limit implies $\mathcal{T}y = y$ giving thereby $y \in \mathcal{D} \cap \mathcal{F}(\mathcal{T})$, provided $\mathcal{T}$ is taken to be continuous. Now since $\mathcal{T}(\mathcal{D}) \subset \mathcal{I}(\mathcal{D})$ there exists a point z in $\mathcal{X}$ such that

$$\mathcal{T}y = y = \mathcal{I}z.$$

For $x_0 \in \mathcal{D}$ arbitrary, let $x_1 \in \mathcal{D}$ be such that $\mathcal{T}x_0 = \mathcal{I}x_1$ and for this point x_1, there exists a point x_2 in $\mathcal{D}$ such that $\mathcal{T}x_1 = \mathcal{J}x_2$ and so on. Inductively, one can choose x_n such that $\mathcal{T}x_{2n} = \mathcal{I}x_{2n+1}$ and $\mathcal{T}x_{2n+1} = \mathcal{J}x_{2n+2}$ (cf. [5]). This is possible because $\mathcal{T}(\mathcal{D}) \subset \mathcal{I}(\mathcal{D})$ and $\mathcal{T}(\mathcal{D}) \subset \mathcal{J}(\mathcal{D})$. Therefore

$$\begin{aligned}\|\mathcal{T}z - \mathcal{T}x_{2n+2}\| \le \max\{&\|\mathcal{I}z - \mathcal{J}x_{2n+2}\|, \tfrac{1}{2}[\|\mathcal{I}z - \mathcal{T}z\| + \|\mathcal{J}x_{2n+2} - \mathcal{T}x_{2n+2}\|], \\ &\tfrac{1}{2}[\|\mathcal{I}z - \mathcal{T}x_{2n+2}\| + \|\mathcal{J}x_{2n+2} - \mathcal{T}z\|]\}\end{aligned}$$

which on letting $n \to \infty$, one gets

$$\|\mathcal{T}z - y\| \le \frac{1}{2}\|\mathcal{T}z - y\|$$

yielding thereby $\mathcal{T}z = y$. Since $(\mathcal{T}, \mathcal{I})$ are compatible hence coincidently commuting, therefore

$$d(\mathcal{T}y, \mathcal{I}y) = d(\mathcal{T}\mathcal{I}z, \mathcal{I}\mathcal{T}z) \le d(\mathcal{I}z, \mathcal{T}z) = d(z, z) = 0$$

yielding thereby $\mathcal{T}y = \mathcal{I}y$. Hence $\mathcal{I}y = \mathcal{T}y = y$. Also since $\mathcal{T}(\mathcal{D}) \subset \mathcal{I}(\mathcal{D})$, there exists a point v in $\mathcal{X}$ and $\mathcal{T}y = y = \mathcal{I}v$. Now,

$$\begin{aligned}\|\mathcal{T}x_{2m+1} - \mathcal{T}v\| \le \max\{&\|\mathcal{I}x_{2m+1} - \mathcal{J}v\|, \tfrac{1}{2}[\|\mathcal{I}x_{2m+1} - \mathcal{T}x_{2m+1}\| + \|\mathcal{J}v - \mathcal{T}v\|], \\ &\tfrac{1}{2}[\|\mathcal{I}x_{2m+1} - \mathcal{T}v\| + \|\mathcal{J}v - \mathcal{T}x_{2m+1}\|]\}\end{aligned}$$

which on letting $m \to \infty$, reduced to

$$\|y - \mathcal{T}v\| \le \frac{1}{2}\|y - \mathcal{T}v\|$$

yielding thereby $y = \mathcal{T}v = \mathcal{J}v$. Since $(\mathcal{T}, \mathcal{J})$ are compatible hence coincidently commuting, therefore

$$\mathcal{J}y = \mathcal{J}(\mathcal{T}v) = \mathcal{T}(\mathcal{J}v) = y$$

which show that y is a common fixed point of $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$. This completes the proof. $\square$

Corollary 1. *Let $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ be a self mappings of a normed space $\mathcal{X}$ and $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$. Let $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. Assume $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ satisfy the condition*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \mathcal{M}(x,y)$$

where

$$\mathcal{M}(x,y) = \max\{\|\mathcal{I}x - \mathcal{J}y\|, \tfrac{1}{2}\|\mathcal{I}x - \mathcal{T}x\|, \tfrac{1}{2}\|\mathcal{J}y - \mathcal{T}y\|, \tfrac{1}{2}\|\mathcal{I}x - \mathcal{T}y\|, \tfrac{1}{2}\|\mathcal{J}y - \mathcal{T}x\|\} \tag{2}$$

for all $x, y \in \mathcal{D} \cup \{\overline{x}\}$.
Further, suppose that the pair $(\mathcal{T},\mathcal{I})$ and $(\mathcal{T},\mathcal{J})$ are compatible and any one of $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ is continuous. If $\mathcal{D}$ is nonempty compact and has a contractive jointly continuous family $\Im = \{f_x\}_{x\in\mathcal{D}}$ such that $\mathcal{I}$ and $\mathcal{J}$ satisfy property (Γ) for all $x \in \mathcal{D}$ and $t \in [0,1]$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$, then

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \phi.$$

The following result is needed in the sequel:

Theorem 3 [see [7, Theorem 2.2]]. *Let $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ be a self mappings of a complete metric space $(\mathcal{X}, d)$ with $\mathcal{T}(\mathcal{X}) \subset \mathcal{IA}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X}) \subset \mathcal{JB}(\mathcal{X})$ such that for all $x, y \in \mathcal{X}$ and $0 \leq h < 1$*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq h\mathcal{M}(x,y)$$

where

$$\mathcal{M}(x,y) = \max\{\|\mathcal{IA}x - \mathcal{JB}y\|, \tfrac{1}{2}[\|\mathcal{IA}x - \mathcal{T}x\| + \|\mathcal{JB}y - \mathcal{T}y\|], \tfrac{1}{2}[\|\mathcal{IA}x - \mathcal{T}y\| + \|\mathcal{JB}y - \mathcal{T}x\|]\}$$

with $\mathcal{M}(x,y) > 0$, then $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point z in $\mathcal{X}$ provided any one of the following holds:

(a) $(\mathcal{T},\mathcal{IA})$ is compatible, $\mathcal{IA}$ or $\mathcal{T}$ is continuous and $(\mathcal{T},\mathcal{JB})$ are coincidently commuting.

(a') $(\mathcal{T},\mathcal{JB})$ is compatible, $\mathcal{JB}$ or $\mathcal{T}$ is continuous and $(\mathcal{T},\mathcal{IA})$ are coincidently commuting.

Moreover, if the pairs $(\mathcal{I},\mathcal{A})$, $(\mathcal{I},\mathcal{A}^2)$, $(\mathcal{J},\mathcal{B})$, $(\mathcal{J},\mathcal{B}^2)$, $(\mathcal{T},\mathcal{I})$, $(\mathcal{T},\mathcal{A})$, $(\mathcal{T},\mathcal{J})$ and $(\mathcal{T},\mathcal{B})$ commute at the common fixed point z, then z remains the unique common fixed point of the pairs $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$.

Theorem 4. *Let $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ be a self mappings of a normed space $\mathcal{X}$ and $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$. Let $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})$. Assume $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \mathcal{M}(x,y)$$

where

$$\mathcal{M}(x,y) = \max\{\|\mathcal{IA}x - \mathcal{JB}y\|, \tfrac{1}{2}[\|\mathcal{IA}x - \mathcal{T}x\| + \|\mathcal{JB}y - \mathcal{T}x\|], \tfrac{1}{2}[\|\mathcal{IA}x - \mathcal{T}y\| + \|\mathcal{JB}y - \mathcal{T}y\|]\}, \tag{3}$$

for all $x, y \in \mathcal{D} \cup \{\overline{x}\}$.
Further, suppose that the pair $(\mathcal{T}, \mathcal{IA})$ *and* $(\mathcal{T}, \mathcal{JB})$ *are compatible, and one of* $\mathcal{T}$, $\mathcal{IA}$ *and* $\mathcal{JB}$ *is continuous. If* $\mathcal{D}$ *is nonempty compact and has a contractive jointly continuous family* $\Im = \{f_x\}_{x \in \mathcal{D}}$ *such that* $\mathcal{IA}$ *and* $\mathcal{JB}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0,1]$ *and* $\mathcal{IA}(\mathcal{D}) = \mathcal{D} = \mathcal{JB}(\mathcal{D})$, *then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{IA}) \cap \mathcal{F}(\mathcal{JB}) \neq \phi.$$

Moreover, if common fixed point $(\mathcal{IA}, \mathcal{T})$ *and* $(\mathcal{JB}, \mathcal{T})$ *is unique and the pairs* $(\mathcal{I}, \mathcal{A})$, $(\mathcal{I}, \mathcal{A}^2)$, $(\mathcal{J}, \mathcal{B}^2)$, $(\mathcal{J}, \mathcal{B})$, $(\mathcal{T}, \mathcal{I})$, $(\mathcal{T}, \mathcal{A})$, $(\mathcal{T}, \mathcal{J})$ *and* $(\mathcal{T}, \mathcal{B})$ *commute at the common fixed point of* $\mathcal{IA}$, $\mathcal{JB}$ *and* $\mathcal{T}$, *then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \neq \phi.$$

Proof. First, we show that $\mathcal{T}$ is a self map on $\mathcal{D}$, i.e., $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. Let $y \in \mathcal{D}$, then $\mathcal{IA}y \in \mathcal{D}$, since $\mathcal{IA}(\mathcal{D}) = \mathcal{D}$. Also, if $y \in \partial\mathcal{C}$, then $\mathcal{T}y \in \mathcal{C}$, since $\mathcal{T}(\partial\mathcal{C}) \subset \mathcal{C}$. Now

$$\begin{aligned}\|\mathcal{T}y - \overline{x}\| &= \|\mathcal{T}y - \mathcal{T}\overline{x}\| \\ &\leq \max\{\|\mathcal{IA}y - \mathcal{JB}\overline{x}\|, \tfrac{1}{2}[\|\mathcal{IA}y - \mathcal{T}y\| + \|\mathcal{JB}\overline{x} - \mathcal{T}\overline{x}\|], \\ &\qquad \tfrac{1}{2}[\|\mathcal{IA}y - \mathcal{T}\overline{x}\| + \|\mathcal{JB}\overline{x} - \mathcal{T}y\|]\}\end{aligned}$$

yielding thereby $\mathcal{T}y \in \mathcal{D}$. Thus $\mathcal{T}$ is a self mapping of $\mathcal{D}$.

Choose $k_n \in [0,1)$ such that$\{k_n\} \to 1$. Then define sequence $\{\mathcal{T}_n\}$ as

$$\mathcal{T}_n(x) = f_{\mathcal{T}x}(k_n)$$

for all $x \in \mathcal{D}$ and for each n. Then, each $\{\mathcal{T}_n\}$ is a well-defined map from $\mathcal{D}$ into $\mathcal{D}$ for each n. Since $\mathcal{IA}$ satisfy property (Γ), we have

$$\mathcal{T}_n\mathcal{I}x_n = f_{\mathcal{T}\mathcal{I}x_n}(k_n)$$

$$\mathcal{I}\mathcal{T}_n x_n = \mathcal{I}f_{\mathcal{T}x_n}(k_n) = f_{\mathcal{I}\mathcal{T}x_n}(k_n)$$

Since $(\mathcal{T}, \mathcal{IA})$ are compatible, therefore

$$0 \leq \lim_{n\to\infty} \|\mathcal{T}_n(\mathcal{IA}x_n) - (\mathcal{IA})\mathcal{T}_n x_n\| \leq \lim_{n\to\infty} \|f_{\mathcal{T}\mathcal{IA}x_n}(k_n) - (f_{\mathcal{IA})\mathcal{T}x_n}(k_n)\|$$

$$\leq \lim_{n\to\infty} k_n\|\mathcal{T}\mathcal{I}x_n - \mathcal{IA}\mathcal{T}x_n\| = 0,$$

whenever $\lim_{n\to\infty} \mathcal{IA}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t \in \mathcal{D}$ for all n.
Hence $\{\mathcal{T}_n\}$ and $\mathcal{IA}$ are compatible on $\mathcal{D}$ for each n and $\mathcal{T}_n(\mathcal{D}) \subseteq \mathcal{D} = \mathcal{IA}(\mathcal{D})$. Similarly it can be shown that $\{\mathcal{T}_n\}$ and $\mathcal{JB}$ are compatible on $\mathcal{D}$ for each n and $\mathcal{T}_n(\mathcal{D}) \subseteq \mathcal{D} = \mathcal{JB}(\mathcal{D})$. It follows from (1) and contractiveness of $\Im$ that

$$\begin{aligned}\|\mathcal{T}_n x - \mathcal{T}_n y\| &= \|f_{\mathcal{T}x}(k_n) - f_{\mathcal{T}y}(k_n)\| \leq \phi(k_n)\|\mathcal{T}x - \mathcal{T}y\| \\ &\leq \phi(k_n)\mathcal{M}(x,y) < \mathcal{M}(x,y)\end{aligned}$$

for all $x, y \in \mathcal{D}$. As $\mathcal{D}$ is compact, therefore by Theorem 3,

$$\mathcal{F}(\mathcal{T}_n) \cap \mathcal{F}(\mathcal{IA}) \cap \mathcal{F}(\mathcal{JB}) = \{x_n\}$$

for each n. Also, since $\mathcal{D}$ is compact, there exists a subsequence of $\{x_n\}$ in $\mathcal{D}$, denoted by $\{x_m\}$, converging to a point, say, $y \in \mathcal{D}$ and hence $\mathcal{T}x_m \to \mathcal{T}y$. The jointly continuity of $\Im$ gives

$$x_m = \mathcal{T}_m x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}y}(1) = \mathcal{T}y$$

and thus the uniqueness of the limit implies $\mathcal{T}y = y$ giving thereby $y \in \mathcal{D} \cap \mathcal{F}(\mathcal{T})$, provided $\mathcal{T}$ is taken to be continuous. Now since $\mathcal{T}(\mathcal{D}) \subset \mathcal{IA}(\mathcal{D})$ there exists a point z in $\mathcal{X}$ such that

$$\mathcal{T}y = y = \mathcal{IA}z$$

For $x_0 \in \mathcal{D}$ arbitrary, let $x_1 \in \mathcal{D}$ be such that $\mathcal{T}x_0 = \mathcal{IA}x_1$ and for this point x_1, there exists a point x_2 in $\mathcal{D}$ such that $\mathcal{T}x_1 = \mathcal{JB}x_2$ and so on. Inductively, one can choose x_n such that $\mathcal{T}x_{2n} = \mathcal{IA}x_{2n+1}$ and $\mathcal{T}x_{2n+1} = \mathcal{JB}x_{2n+2}$ (cf. [5]). This is possible because $\mathcal{T}(\mathcal{D}) \subset \mathcal{IA}(\mathcal{D})$ and $\mathcal{T}(\mathcal{D}) \subset \mathcal{JB}(\mathcal{D})$. Therefore

$$\begin{aligned}\|\mathcal{T}z - \mathcal{T}x_{2n+2}\| \le \max\{\|\mathcal{IA}z - \mathcal{JB}x_{2n+2}\|, \tfrac{1}{2}[\|\mathcal{IA}z - \mathcal{T}z\| \\ +\|\mathcal{JB}x_{2n+2} - \mathcal{T}x_{2n+2}\|], \\ \tfrac{1}{2}[\|\mathcal{IA}z - \mathcal{T}x_{2n+2}\| + \|\mathcal{JB}x_{2n+2} - \mathcal{T}z\|]\}\end{aligned}$$

which on letting $n \to \infty$, one gets

$$\|\mathcal{T}z - y\| \le \frac{1}{2}\|\mathcal{T}z - y\|$$

yielding thereby $\mathcal{T}z = y$. Since $(\mathcal{T}, \mathcal{IA})$ are compatible hence coincidently commuting, therefore

$$d(\mathcal{T}y, \mathcal{I}y) = d(\mathcal{T}(\mathcal{IA})z, (\mathcal{IA})\mathcal{T}z) \le d(\mathcal{IA}z, \mathcal{T}z) = d(z, z) = 0$$

yielding thereby $\mathcal{T}y = \mathcal{IA}y$. Hence $\mathcal{IA}y = \mathcal{T}y = y$. Also since $\mathcal{T}(\mathcal{D}) \subset \mathcal{IA}(\mathcal{D})$, there exists a point v in $\mathcal{X}$ and $\mathcal{T}y = y = \mathcal{IA}v$. Now,

$$\begin{aligned}\|\mathcal{T}x_{2m+1} - \mathcal{T}v\| \le \max\{\|\mathcal{IA}x_{2m+1} - \mathcal{JB}v\|, \tfrac{1}{2}[\|\mathcal{IA}x_{2m+1} - \mathcal{T}x_{2m+1}\| \\ +\|\mathcal{JB}v - \mathcal{T}v\|], \\ \tfrac{1}{2}[\|\mathcal{IA}x_{2m+1} - \mathcal{T}v\| + \|\mathcal{JB}v - \mathcal{T}x_{2m+1}\|]\},\end{aligned}$$

which on letting $m \to \infty$, reduced to

$$\|y - \mathcal{T}v\| \le \frac{1}{2}\|y - \mathcal{T}v\|$$

yielding thereby $y = \mathcal{T}v = \mathcal{JB}v$. Since $(\mathcal{T}, \mathcal{JB})$ are compatible hence coincidently commuting, therefore

$$\mathcal{JB}y = \mathcal{JB}(\mathcal{T}v) = \mathcal{T}(\mathcal{JB}v) = y$$

which show that y is a common fixed point of $\mathcal{T}$, $\mathcal{IA}$ and $\mathcal{JB}$.

Moreover, if y is the unique common fixed point of the pairs $(\mathcal{T}, \mathcal{IA})$ and $(\mathcal{T}, \mathcal{JB})$, then in the line of the proof of Theorem 2.2 of Imdad [7], it can be shown that y is the unique common fixed point of $\mathcal{I}$, $\mathcal{A}$, $\mathcal{J}$, $\mathcal{B}$ and $\mathcal{T}$. Hence

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \neq \phi.$$

This completes the proof. □

Corollary 2. *Let $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ be a self mappings of a normed space $\mathcal{X}$ and $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$. Let $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})$. Assume $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \mathcal{M}(x, y)$$

where

$$\mathcal{M}(x,y) = \max\{\|\mathcal{IA}x - \mathcal{JB}y\|, \tfrac{1}{2}\|\mathcal{IA}x - \mathcal{T}x\|, \tfrac{1}{2}\|\mathcal{JB}y - \mathcal{T}y\|, \tfrac{1}{2}\|\mathcal{IA}x - \mathcal{T}y\|, \tfrac{1}{2}\|\mathcal{JB}y - \mathcal{T}x\|\}, \tag{4}$$

for all $x, y \in \mathcal{D} \cup \{\overline{x}\}$.
Further, suppose that the pair $(\mathcal{T}, \mathcal{IA})$ and $(\mathcal{T}, \mathcal{JB})$ are compatible, and one of $\mathcal{T}$, $\mathcal{IA}$ and $\mathcal{JB}$ is continuous. If $\mathcal{D}$ is nonempty compact and has a contractive jointly continuous family $\Im = \{f_x\}_{x \in \mathcal{D}}$ such that $\mathcal{IA}$ and $\mathcal{JB}$ satisfy property (Γ) for all $x \in \mathcal{D}$ and $t \in [0,1]$ and $\mathcal{IA}(\mathcal{D}) = \mathcal{D} = \mathcal{JB}(\mathcal{D})$, then

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{IA}) \cap \mathcal{F}(\mathcal{JB}) \neq \phi.$$

Moreover, if common fixed point $(\mathcal{IA}, \mathcal{T})$ and $(\mathcal{JB}, \mathcal{T})$ is unique and the pairs $(\mathcal{I}, \mathcal{A})$, $(\mathcal{I}, \mathcal{A}^2)$, $(\mathcal{J}, \mathcal{B})$, $(\mathcal{J}, \mathcal{B}^2)$, $(\mathcal{T}, \mathcal{I})$, $(\mathcal{T}, \mathcal{A})$, $(\mathcal{T}, \mathcal{J})$ and $(\mathcal{T}, \mathcal{B})$ commute at the common fixed point of $\mathcal{IA}$, $\mathcal{JB}$ and $\mathcal{T}$, then

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \neq \phi.$$

Suppose that $\mathcal{H} = \{f_\alpha\}_{\alpha \in \mathcal{C}}$ is a family of functions from $[0,1]$ into $\mathcal{C}$ having the property that for each sequence (λ_n) in $(0,1]$ with $\lambda_n \to 1$ as $n \to \infty$, we have

$$f_\alpha(\lambda_n) = \lambda_n \alpha. \tag{$*$}$$

It is observed that $\mathcal{H} \subseteq \Im$ and it has additional property that it is contractive, jointly continuous and weakly jointly continuous [11].

Example 1 [see [11]]. *Any subspace, a convex set with* 0*, a star-shaped subset with center* 0 *and a cone of a normed space have the family of functions associated with them which satisfy condition* $(*)$.

Theorem 5. *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{C}$ *be subset of* $\mathcal{X}$*. Let* $\mathcal{I}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$ *and* $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$*. Assume* $\mathcal{T}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *satisfy the condition (1) for all* $x, y \in \mathcal{D} \cup \{\overline{x}\}$*. Further, suppose that the pair* $(\mathcal{T}, \mathcal{I})$ *and* $(\mathcal{T}, \mathcal{J})$ *are compatible and any one of* $\mathcal{T}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *is continuous. If* $\mathcal{D}$ *is nonempty compact and has a family* $\mathcal{H}$ *satisfying condition* $(*)$ *and* $\mathcal{I}$ *and* $\mathcal{J}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0,1]$ *and* $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$*, then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \phi.$$

Proof. In the line of proof of Theorem 2. □

Theorem 6. *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{C}$ *be subset of* $\mathcal{X}$*. Let* $\mathcal{T}, \mathcal{I}, \mathcal{J}, \mathcal{A}, \mathcal{B} : \mathcal{X} \to \mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$*. Let* $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})$*. Assume* $\mathcal{T}$*,* $\mathcal{I}$*,* $\mathcal{J}$*,* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy the condition (3) for all* $x, y \in \mathcal{D} \cup \{\overline{x}\}$*. Further, suppose that the pair* $(\mathcal{T}, \mathcal{I}\mathcal{A})$ *and* $(\mathcal{T}, \mathcal{J}\mathcal{B})$ *are compatible, and one of* $\mathcal{T}$*,* $\mathcal{I}\mathcal{A}$ *and* $\mathcal{J}\mathcal{B}$ *is continuous. If* $\mathcal{D}$ *is nonempty compact and has a family* $\mathcal{H}$ *satisfying condition* $(*)$ *such that* $\mathcal{I}\mathcal{A}$ *and* $\mathcal{J}\mathcal{B}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0,1]$ *and* $\mathcal{I}\mathcal{A}(\mathcal{D}) = \mathcal{D} = \mathcal{J}\mathcal{B}(\mathcal{D})$*, then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}\mathcal{A}) \cap \mathcal{F}(\mathcal{J}\mathcal{B}) \neq \phi.$$

Moreover, if common fixed point $(\mathcal{I}\mathcal{A}, \mathcal{T})$ *and* $(\mathcal{J}\mathcal{B}, \mathcal{T})$ *is unique and the pairs* $(\mathcal{I}, \mathcal{A})$*,* $(\mathcal{I}, \mathcal{A}^2)$*,* $(\mathcal{J}, \mathcal{B}^2)$*,* $(\mathcal{J}, \mathcal{B})$*,* $(\mathcal{T}, \mathcal{I})$*,* $(\mathcal{T}, \mathcal{A})$*,* $(\mathcal{T}, \mathcal{J})$ *and* $(\mathcal{T}, \mathcal{B})$ *commute at the common fixed point of* $\mathcal{I}\mathcal{A}$*,* $\mathcal{J}\mathcal{B}$ *and* $\mathcal{T}$*, then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \neq \phi.$$

Proof. In the line of proof of Theorem 4. □

Following is a application of fixed point theorem to best approximation on weakly compact subset.

Theorem 7. *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{C}$ *be subset of* $\mathcal{X}$*. Let* $\mathcal{I}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$ *and* $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$*. Assume* $\mathcal{T}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *satisfy the condition (1) for all* $x, y \in \mathcal{D} \cup \{\overline{x}\}$*. Further, suppose that the pair* $(\mathcal{T}, \mathcal{I})$ *and* $(\mathcal{T}, \mathcal{J})$ *are compatible and any one of* $\mathcal{T}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *is weakly continuous. If* $\mathcal{D}$ *is nonempty weakly compact and has a family* $\mathcal{H}$ *satisfying condition* $(*)$ *and* $\mathcal{I}$ *and* $\mathcal{J}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0,1]$ *and* $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$*, then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \neq \phi.$$

Proof. In the line of proof of Theorem 2. □

Theorem 8. *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{C}$ *be subset of* $\mathcal{X}$*. Let* $\mathcal{T}, \mathcal{I}, \mathcal{J}, \mathcal{A}, \mathcal{B} : \mathcal{X} \to \mathcal{X}$ *such that* $\mathcal{T}(\partial\mathcal{C}) \subseteq \mathcal{C}$*. Let* $\overline{x} \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})$*. Assume* $\mathcal{T}$*,* $\mathcal{I}$*,* $\mathcal{J}$*,* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy the condition (3) for all* $x, y \in \mathcal{D} \cup \{\overline{x}\}$*. Further, suppose* *that the pair* $(\mathcal{T},\mathcal{IA})$ *and* $(\mathcal{T},\mathcal{JB})$ *are compatible, and one of* $\mathcal{T}$, $\mathcal{IA}$ *and* $\mathcal{JB}$ *is weakly continuous. If* $\mathcal{D}$ *is nonempty weakly compact and has a family* $\mathcal{H}$ *satisfying condition* $(*)$ *such that* $\mathcal{IA}$ *and* $\mathcal{JB}$ *satisfy property* (Γ) *for all* $x \in \mathcal{D}$ *and* $t \in [0,1]$ *and* $\mathcal{IA}(\mathcal{D}) = \mathcal{D} = \mathcal{JB}(\mathcal{D})$, *then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{IA}) \cap \mathcal{F}(\mathcal{JB}) \neq \phi.$$

Moreover, if common fixed point $(\mathcal{IA},\mathcal{T})$ *and* $(\mathcal{JB},\mathcal{T})$ *is unique and the pairs* $(\mathcal{I},\mathcal{A})$, $(\mathcal{I},\mathcal{A}^2)$, $(\mathcal{J},\mathcal{B}^2)$, $(\mathcal{J},\mathcal{B})$, $(\mathcal{T},\mathcal{I})$, $(\mathcal{T},\mathcal{A})$, $(\mathcal{T},\mathcal{J})$ *and* $(\mathcal{T},\mathcal{B})$ *commute at the common fixed point of* $\mathcal{IA}$, $\mathcal{JB}$ *and* $\mathcal{T}$, *then*

$$\mathcal{D} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J}) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \neq \phi.$$

Remark 1. *In the light of the comment given by Dotson [3] and Khan, Latif, Bano and Hussain [10] that if* $\mathcal{C} \subseteq \mathcal{X}$ *is* $p-$*starshaped and* $f_\alpha(t) = (1-t)p + t\alpha$, $(\alpha \in \mathcal{C}, t \in [0,1])$, *then* $\{f_\alpha\}_{\alpha\in\mathcal{C}}$ *is a contractive jointly continuous family with* $\phi(t) = t$. *Thus the class of subsets of* $\mathcal{X}$ *with the property of contractiveness and jointly continuity contains the class of starshaped sets which in turns contains the class of convex sets. If for a subset* $\mathcal{C}$ *of* $\mathcal{X}$, *there exists a contractive jointly continuous family* $\Im = \{f_\alpha\}_{\alpha\in\mathcal{C}}$, *then we say that* $\mathcal{C}$ *has the property of contractiveness and joint continuity.*

Remark 2. *With Remark 1, Theorem 4 and Corollary 2 generalize the results of Imdad [7] in a domain which is not necessarily starshaped and mappings are not necessarily linear.*

Remark 3. *Theorem 2, Corollary 1, Theorem 4, and Corollary 2 generalize and improve the result of Mukherjee and Som [13] by increasing the number of mappings and generalized form of nonexpansive mapping.*

Remark 4. *With Remark 1, Theorem 2, Corollary 1, Theorem 4, and Corollary 2 also generalize the results of Sahab, Khan and Sessa [15] by increasing the number of mappings and by employing the compatible mappings instead of commuting mappings in a domain which is not necessarily starshaped and mappings are not necessarily linear. Further, the conditions (1), (2), (3) and (4) are much general than the condition of Sahab, Khan and Sessa [15].*

Remark 5. *With Remark 1, Theorem 2, Corollary 1, Theorem 3, Theorem 4, and Corollary 2 also generalize the results of Brosowski [1], Hicks and Humpheries [6] and Singh [16] by increasing the number of mappings and by considering generalized form of mapping.*

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