Angle bisectors of a triangle in I_2

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Abstract. *The concept of an angle bisector of the triangle will be introduced in an isotropic plane. Some statements about relationships between the introduced concepts and some other previously studied geometric concepts about triangles will be investigated in an isotropic plane. A number of these statements seems to be new, and some of them are known in Euclidean geometry.*

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Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, into the so-called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a+b+c = 0$. With the labels $p = abc$, $q = bc+ca+ab$ the equalities $q = bc - a^2 = ca - b^2 = ab - c^2$, $(c - a)(a - b) = 2q - 3bc$, $(a - b)(b - c) = 2q - 3ca$, $(b - c)(c - a) = 2q - 3ab$ and $q = -(b^2 + bc + c^2)$ are valid.

Theorem 1. *The bisectors of angles A, B, C of the triangle ABC with vertices*

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2) \tag{1}$$

have the following equations

$$y = \frac{a}{2}x + \frac{a^2}{2}, \quad y = \frac{b}{2}x + \frac{b^2}{2}, \quad y = \frac{c}{2}x + \frac{c^2}{2}. \tag{2}$$

They meet the lines BC, CA, AB at the points D, E, F, respectively, with the same ordinate $-\frac{q}{3}$ and with the abscissas

$$d = \frac{q - 3bc}{3a}, \quad e = \frac{q - 3ca}{3b}, \quad f = \frac{q - 3ab}{3c}. \tag{3}$$

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Proof. According to [5] the lines AB and AC have the equations

$$y = -cx - ab, \quad y = -bx - ca,$$

whose addition, because of $b + c = -a$, gives equation $2y = ax + a^2$, and it is the first equation of (2). From that equation and the equation $y = -ax - bc$ of the line BC we get equation $\frac{3}{2}ax = -\frac{a^2}{2} - bc$, wherefrom multiplied by 2, because of $a^2 = bc - q$, follows $3ax = q - 3bc$ with the solution $x = d$. Besides that solution, from the equation of the line BC for ordinate is got

$$-ad - bc = \frac{1}{3}(3bc - q) - bc = -\frac{q}{3}.$$

□

The orthic axis of the triangle ABC has according to [5] the equation $y = -\frac{q}{3}$, wherefrom follows:

Corollary 1. *The angle bisectors intersect the opposite sides of the triangle at three points which lie on the orthic axis of the triangle.*

If the bisectors of the angles of the triangle ABC are understood as their "outer" bisectors, where their "inner" bisectors would be the isotropic lines through the points A, B, C, then the role of the intersection of "inner" bisector has the absolute point of the plane, and the line through the points D, E, F should be called, by the analogy with Euclidean case, antiorthic axis of the triangle ABC. Then Corollary 1 means that orthic and antiorthic axis in an isotropic plane coincide. In Euclidean plane it is not generally the truth, and it holds if and only if the triangle is isosceles.

Theorem 2. *Any angle bisector of the triangle intersects its opposite side in the ratio of lengths of its adjacent sides.*

Proof. Owing to the first equlity (3) we get

$$BD = d - b = \frac{1}{3a}(q - 3bc - 3ab) = \frac{1}{3a}(3ca - 2q) = -\frac{1}{3a}(a - b)(b - c)$$

and similarly

$$CD = \frac{1}{3a}(3ab - 2q) = -\frac{1}{3a}(b - c)(c - a),$$

and therefore

$$\frac{BD}{CD} = \frac{a - b}{c - a} = \frac{AB}{CA}.$$

□

Theorem 3. *If $\mathcal{A}$ is the bisector of the angle A of the triangle ABC, then the angle $\angle(BC, \mathcal{A})$ is the arithmetic mean of the angles $\angle(BC, CA)$ and $\angle(BC, AB)$.*

Proof. The lines BC, CA, AB and $\mathcal{A}$ have the slopes $-a, -b, -c$ and $\frac{a}{2}$, wherefrom

$$\begin{aligned}\angle(BC, \mathcal{A}) &= \frac{a}{2} + a = \frac{3}{2}a,\\ \angle(BC, CA) &= -b + a,\\ \angle(BC, AB) &= -c + a\end{aligned}$$

and then

$$\angle(BC, CA) + \angle(BC, AB) = 2a - b - c = 3a = 2 \cdot \angle(BC, \mathcal{A}).$$

□

Theorem 4. *The angle bisectors of the triangle ABC determine the triangle $A_sB_sC_s$, whose vertices are parallel to the points A, B, C and they are the midpoints of the altitudes AA_h, BB_h, CC_h of the triangle ABC.*

Proof. The orthic triangle $A_hB_hC_h$ of the triangle ABC with the vertices (1) has according to [5] the vertices

$$A_h = (a, q - 2bc), \quad B_h = (b, q - 2ca), \quad C_h = (c, q - 2ab).$$

The points A and A_h have for the midpoint the first one of the three analogous points

$$A_s = (a, -\frac{bc}{2}), \quad B_s = (b, -\frac{ca}{2}), \quad C_s = (c, -\frac{ab}{2}), \tag{4}$$

because of $a^2 + q - 2bc = -bc$. The point A_s from (4) lies on the second and third line from (2) because, for example, for the second line (2) we get

$$\frac{b}{2} \cdot a + \frac{b^2}{2} = -\frac{bc}{2}.$$

□

The triangle $A_sB_sC_s$ from Theorem 4 will be called *symmetral triangle* of the triangle ABC.

Corollary 2. *The points A_s, B_s, C_s from Theorem 4 lie on the mid-lines of the triangle ABC which are parallel to the lines BC, CA, AB, respectively.*

Corollary 3. *The points A, B, C are the feet of the altitudes of the triangle $A_sB_sC_s$ i.e. the triangle ABC is the orthic triangle of its symmetral triangle.*

Corollary 4. *The symmetral triangle $A_sB_sC_s$ of the standard triangle ABC has the vertices given with (4) and the sides with the equations (2).*

Theorem 5. *The symmetral triangle $A_sB_sC_s$ from Theorem 4 is inscribed in the polar circle of the triangle ABC.*

Proof. The polar circle of the triangle ABC has according to [1] the equation $y = -\frac{1}{2}x^2 - \frac{q}{2}$, and the tangent line at the points A_s, B_s, C_s on that circle are the mid-lines of that triangle ABC i.e. the polar circle of triangle is inscribed to its complementary triangle, and the mid-line B_mC_m has, according to [5] the equation $y = -ax - q + \frac{bc}{2}$. From these two equations the equation for the abscissas of the common points follows

$$\frac{x^2}{2} - ax + \frac{bc - q}{2} = 0.$$

Because of $bc - q = a^2$ this equation is of the form $\frac{1}{2}(x - a)^2 = 0$, and it has double solution $x = a$ which corresponds to the point A_s of the line B_mC_m from Corollary 4. □

Because of ([6], Corollary 6) and the fact (from Theorem 4) that points A_s, B_s, C_s lie on polar circle and they are the midpoints of the pairs of points A, A_h; B, B_h; C, C_h it follows

Corollary 5. *The points A_h, B_h, C_h from Theorem 4 are inverse images to the points A, B, C with respect to the polar circle of the triangle ABC.*

Theorem 6. *If any isotropic line meets the lines BC, CA, AB at the points U,V,W, then the centroid of points U,V,W lies on the orthic axis of the triangle ABC.*

Proof. By the addition of the equations $y=-ax-bc$, $y=-bx-ca$, $y=-cx-ab$ of the lines BC, CA, AB we get equation $3y=-q$ of the orthic axis of the triangle ABC. $\square$

According to [1] the contact triangle $A_iB_iC_i$ of the standard triangle ABC, where A_i,B_i,C_i are the touching points of the lines BC,CA,AB with the the inscribed circle of the triangle ABC, has for example the vertex $A_i=(-2a,bc-2q)$, and according to [5] the triangle ABC has centroid $G=(0,-\frac{2}{3}\,q)$. Because of the first equality (4) the equality $A_i+2A_s=3G$ is valid and according to [5] it means that the point A_s is complementary to the point A_i in the triangle ABC, i.e. we get the following corollary.

Corollary 6. *The symmetral triangle $A_sB_sC_s$ of the allowable triangle ABC is complementary triangle to its contact triangle $A_iB_iC_i$ with respect to that triangle.*

From (4) immediately follows

Corollary 7. *The symmetral triangle $A_sB_sC_s$ of the standard triangle ABC has the centroid $G_s=(0,-\frac{q}{6})$.*

Theorem 7. *If the bisector of the angle A meets the line BC at the point D, then the point L' isogonal to midpoint L of the segment AD with respect to the triangle ABC lies on the line which joins midpoint of the side BC and the midpoint of the corresponding altitude AA_h of that triangle.* (The Euclidean version of that theorem is due to MINEUR, [7]).

Proof. The point

$$D=\left(\frac{q-3bc}{3a},-\frac{q}{3}\right) \tag{5}$$

from the proof of Theorem 1 and the point $A=(a,a^2)$ have, according to

$$\frac{q-3bc+3a^2}{6a}=\frac{-2q}{6a}=-\frac{q}{3a},$$

the midpoint

$$L=\left(-\frac{q}{3a},\frac{3a^2-q}{6}\right). \tag{6}$$

For the coordinates x,y of the point L we get successively

$$y-x^2=\frac{3a^2-q}{6}-\frac{q^2}{9a^2}=\frac{1}{18a^2}(9a^4-3a^2q-2q^2)=\frac{1}{18a^2}(3a^2+q)(3a^2-2q),$$

$$\begin{aligned}xy+qx-p&=-\frac{1}{18a}(3a^2q-q^2)-\frac{q^2}{3a}-p=-\frac{1}{18a}(5q^2+3a^2q+18a^2bc)\\&=-\frac{1}{18a}[5q^2+3a^2q+18a^2(a^2+q)]=-\frac{1}{18a}(18a^4+21a^2q+5q^2)\\&=-\frac{1}{18a}(3a^2+q)(6a^2+5q),\end{aligned}$$

$$\begin{aligned} px - qy - y^2 &= -\frac{pq}{3a} + \frac{q^2 - 3a^2 q}{6} - \frac{1}{36}(q^2 - 6a^2 q + 9a^4) \\ &= -\frac{1}{36}[12q(a^2 + q) - 6(q^2 - 3a^2 q) + q^2 - 6a^2 q + 9a^4] \\ &= -\frac{1}{36}(9a^4 + 24a^2 q + 7q^2) = -\frac{1}{36}(3a^2 + q)(3a^2 + 7q), \end{aligned}$$

and then, owing to [6], the coordinates of the point L' are

$$\begin{aligned} x' &= \frac{xy + qx - p}{y - x^2} = -a \cdot \frac{6a^2 + 5q}{3a^2 - 2q}, \\ y' &= \frac{px - qy - y^2}{y - x^2} = -\frac{a^2}{2} \cdot \frac{3a^2 + 7q}{3a^2 - 2q}. \end{aligned}$$

The line which joins the midpoints of the side BC and altitude AA_h has according to [4] the equation

$$y = \frac{q}{3a}x - \frac{q}{3} - \frac{bc}{2}.$$

The point L' lies on that line because we get

$$\begin{aligned} y' - \frac{q}{3a}x' &= -\frac{a^2}{2} \cdot \frac{3a^2 + 7q}{3a^2 - 2q} + \frac{q}{3} \cdot \frac{6a^2 + 5q}{3a^2 - 2q} = -\frac{9a^4 + 9a^2 q - 10q^2}{6(3a^2 - 2q)} \\ &= -\frac{1}{6}(3a^2 + 5q) = -\frac{1}{6}(3bc + 2q) = -\frac{q}{3} - \frac{bc}{2}. \end{aligned}$$

□

Theorem 8. *If the bisectors of the angles A, B, C of the triangle ABC meet respectively the lines BC, CA, AB at the points D, E, F, then the midpoints L, M, N of segments AD, BE, CF lie on one line, which passes through the centroid G of the triangle ABC.*

Proof. The line with the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}\,q \tag{7}$$

passes through the points L, M, N because for example for the point L from (6) we get

$$y + \frac{3p}{2q}x = \frac{3a^2 - q}{6} - \frac{3bc}{6} = -\frac{4q}{6} = -\frac{2}{3}\,q.$$

The line (7) passes obviously through the centroid $G = (0, -\frac{2}{3}\,q)$. □

The line (7) is indeed the harmonic polar line of the Gergonne's point Γ of the triangle ABC.

Theorem 9. *If the bisector of the angle A of the triangle ABC meets the line BC at the point D, then the equality*

$$AD^2 - BD \cdot CD = -CA \cdot AB \tag{8}$$

follows.

Proof. With d from (3) we get

$$\begin{aligned}AD = d - a &= \frac{1}{3a}(q - 3bc - 3a^2) = \frac{1}{3a}(4q - 6bc) = \frac{2}{3a}(2q - 3bc)\\ &= \frac{2}{3a}(c-a)(a-b),\\ BD = d - b &= \frac{1}{3a}(q - 3bc - 3ab) = \frac{1}{3a}(3ca - 2q) = -\frac{1}{3a}(a-b)(b-c)\end{aligned}$$

and similarly $CD = -\frac{1}{3a}(b-c)(c-a)$, wherefrom

$$\begin{aligned}AD^2 - BD \cdot CD &= \frac{(c-a)(a-b)}{9a^2}[4(c-a)(a-b) - (b-c)^2]\\ &= \frac{(c-a)(a-b)}{9a^2}[4ca + 4ab - 4a^2 - (b+c)^2]\\ &= \frac{(c-a)(a-b)}{9a^2}(-4a^2 - 4a^2 - a^2)\\ &= -(c-a)(a-b) = -CA \cdot AB.\end{aligned}$$

□

Theorem 9 suggests the question under which condition the point D on the line BC will satisfy the equality (8). The answer is the same as at [2] in Euclidean case because of

Theorem 10. *If for the point D on the line BC the equality (8) holds, then either AD is the bisector of the angle A of the triangle ABC or the triangle ABC is isosceles with $CA = AB$.*

Proof. Let d be abscissa of the point D. Then

$$\begin{aligned}AD^2 - BD \cdot CD + CA \cdot AB &= (d-a)^2 - (d-b)(d-c) + (a-c)(b-a)\\ &= (-2a + b + c)d + ca + ab - 2bc = -3ad + q - 3bc\\ &= -3a\left(d - \frac{q - 3bc}{3a}\right)\end{aligned}$$

and the condition (8) is satisfied if d is given by the formula (3) or if $a = 0$. The condition $CA = AB$ is of the form $a - c = b - a$, i.e. $2a = b + c$ or $3a = 0$, i.e. $a = 0$. □

Theorem 11. *The angle bisectors of an allowable triangle meet the bisectors of its opposite sides at three points which lie on its circumscribed circle.*

Proof. For points on the bisector of the side BC is $x = -\frac{a}{2}$, and with this value, from the equation $y = x^2$ of the circumscribed circle and $y = \frac{a}{2}x + \frac{a^2}{2}$ of the bisector of the angle A we get the same value $y = \frac{a^2}{4}$, and the intersection which is looking for is the point $U = (-\frac{a}{2}, \frac{a^2}{4})$. □

Theorem 12. *A circle $\mathcal{K}$ through the point A meets circumscribed circle of the triangle ABC and the lines CA and AB once again at the points D, E, F. The segments BC and EF have the same bisector if and only if AD is the bisector of the angle A.*

Proof. The circle with the equation $y = ux^2+vx+w$ passes through the point A under condition $a^2 = a^2u+av+w$, wherefrom $w = a^2(1-u)-av$ follows, so the circle $\mathcal{K}$ has the equation of the form

$$y = ux^2 + vx + a^2(1-u) - av. \tag{9}$$

From the equation (9) and the equation $y = x^2$ of the circumscribed circle we get the equation $(u-1)x^2 + vx + a^2(1-u) - av = 0$, which can be written in the form $(x-a)[(u-1)(x+a)+v] = 0$ with the solutions $x = a$ and $x = d$, where

$$d = \frac{a-au-v}{u-1}$$

is the abscissa of the point D. The line AD is bisector of the angle A if and only if the point D is coincident with the point U from proof of Theorem 12, i.e. under condition

$$\frac{a-au-v}{u-1} = -\frac{a}{2}. \tag{10}$$

From the equation (9) and the equation $y = -bx-ca$ of the line CA we get equation $ux^2+(v+b)x-a^2u-av-ab = 0$ because of $a^2+ca = -ab$. That equation can be written in the form $(x-a)[u(x+a)+v+b] = 0$ with solutions $x = a$ and $x = e$, where e is given by the first equality out of the two analogous equalities

$$e = -\frac{au+v+b}{u}, \qquad f = -\frac{au+v+c}{u},$$

while e and f are the abscissas of the points E and F. The segments BC and EF have the same bisector under condition $e+f = b+c$, i.e. $e+f = -a$ or

$$\frac{2au+2v-a}{u} = a.$$

It is easily seen that this condition is coincident with (10). $\square$

Theorem 13. *The angle bisector at the vertex A_m of the complementary triangle $A_mB_mC_m$ of the triangle ABC meet the lines CA and AB at the points E_a and F_a such that the equalities $E_aA_m = AB$ and $F_aA_m = AC$ are valid.*

Proof. According to [5] we get

$$A_m = \left(-\frac{a}{2}, -\frac{1}{2}(q+bc)\right).$$

The line with the equation

$$y = \frac{a}{2}x - \frac{3}{4}q - \frac{1}{4}bc \tag{11}$$

is parallel to the bisector of the angle A from (2) and passes through the point A_m because of

$$-\frac{a^2}{4} - \frac{3}{4}q - \frac{1}{4}bc = -\frac{1}{4}(bc-q) - \frac{3}{4}q - \frac{1}{4}bc = -\frac{1}{2}(q+bc).$$

From the equation (11) and the equation $y = -bx - ca$ of the line CA we get for the abscissa of the point E_a the equation

$$\left(\frac{a}{2} + b\right) x = \frac{3}{4}q + \frac{1}{4}bc - ca.$$

As the right side here is equal to

$$\frac{3}{4}(ca - b^2) + \frac{1}{4}bc - ca = \frac{1}{4}c(b-a) - \frac{3}{4}b^2 = \frac{1}{4}(a^2-b^2) - \frac{3}{4}b^2 = \frac{1}{4}a^2 - b^2 = \left(\frac{a}{2} + b\right)\left(\frac{a}{2} - b\right),$$

it follows $e = \frac{a}{2} - b$. And then

$$E_a A_m = -\frac{a}{2} - \left(\frac{a}{2} - b\right) = b - a = AB.$$

□

Theorem 14. *If E and F are variable points of the lines CA and AB such that $CE = BF$, then the midpoint of the segment EF scribes bisector at the vertex A_m of the complementary triangle of the triangle ABC.* (In Euclidean geometry this statement can be found in [3], p. 447–448.)

Proof. If $CE = BF = t$, then the points E and F have abscissas $c + t$ and $b + t$. From the equation $y = -bx - ca$ of the line CA for ordinate of the point E we get

$$y = -b(c+t) - ca = c^2 - bt,$$

wherefrom $E = (c+t,\ c^2 - bt)$, and similarly $F = (b+t, b^2 - ct)$. The points E and F have midpoint

$$\left(t - \frac{a}{2},\ \frac{a}{2}t - q - \frac{1}{2}a^2\right).$$

It lies on the line (11) because of

$$\frac{a}{2}t - q - \frac{1}{2}a^2 - \frac{a}{2}\left(t - \frac{a}{2}\right) + \frac{3}{4}q + \frac{1}{4}bc = \frac{1}{4}(bc - q - a^2) = 0.$$

□

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