On the existence of mild solutions of a nonconvex evolution inclusion

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Abstract. *We prove a Filippov type existence theorem for mild solutions of a nonconvex evolution inclusion by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions.*

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1. Introduction

In this paper we study differential inclusions of the form

$$x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map Lipschitzian with respect to the second variable, X is a separable Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t,\tau), t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X, $D = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : D \to \mathbf{R}$ is continuous and $x_0 \in X$.

Existence of mild solutions of problem (1.1) have been obtained in [2-7,16] etc. via fixed point techniques. The aim of our paper is to provide a Filippov type result concerning the existence of solutions to problem (1.1). We recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem ([13]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution.

Our approach is different from the ones in [2-7,16] and consists in applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion.

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The idea of applying the set-valued contraction principle due to Covitz and Nadler ([12]) in the space of derivatives of the solutions belongs to Tallos ([14,17]) and it was already used for obtaining similar results for other classes of differential inclusions ([9-11]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. Preliminaries

Let denote by I the interval $[0,T]$, $T>0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$.

In the sequel $\{A(t); t\in I\}$ is the infinitesimal generator of the strongly continuous evolution system $G(t,s)$, $0\leq s\leq t\leq T$.

Recall that a family of bounded linear operators $G(t,s)$ on X, $0\leq s\leq t\leq T$ depending on two parameters is said to be a strongly continuous evolution system if there are fulfilled the following conditions: $G(s,s)=I$, $G(t,r)G(r,s)=G(t,s)$ for $0\leq s\leq r\leq t\leq T$ and $(t,s)\to G(t,s)$ is strongly continuous for $0\leq s\leq t\leq T$, i.e, $\lim_{t\to s, t>s} G(t,s)x=x$ for all $x\in X$.

In what follows we are concerned with the evolution inclusion

$$x'(t)\in A(t)x(t)+\int_0^t K(t,s)F(s,x(s))ds, \quad x(0)=x_0, \tag{2.1}$$

where $F: I\times X\to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t,\tau), t\geq 0, \tau\geq 0\}$ of bounded linear operators of X into X, $D=\{(t,s)\in I\times I; t\geq s\}$, $K(.,.): D\to \mathbf{R}$ is continuous and $x_0\in X$.

A continuous mapping $x(.)\in C(I,X)$ is called a *mild solution* of problem (2.1) if there exists a (Bochner) integrable function $f(.)\in L^1(I,X)$ such that

$$f(t)\in F(t,x(t)) \quad a.e.\,(I) \tag{2.2}$$

$$x(t)=G(t,0)x_0+\int_0^t G(t,\tau)\int_0^\tau K(\tau,s)f(s)dsd\tau \quad \forall t\in I. \tag{2.3}$$

In this case we shall call $(x(.), f(.))$ a *trajectory-selection pair* of (2.1).

We note that relation (2.3) can be rewritten as

$$x(t)=G(t,0)x_0+\int_0^t U(t,s)f(s)ds \quad \forall t\in I, \tag{2.4}$$

where $U(t,s)=\int_s^t G(t,\tau)K(\tau,s)d\tau$.

In what follows the following conditions are satisfied.

Hypothesis 2.1.

(i) *$\{A(t); t\in I\}$ is the infinitesimal generator of the strongly continuous evolution system $G(t,s)$, $0\leq s\leq t\leq T$.*

(ii) $F(.,.) : I \times X \to \mathcal{P}(X)$ *has nonempty closed values and for every* $x \in X$, $F(.,x)$ *is measurable.*

(iii) *There exists* $L(.) \in L^1(I, \mathbf{R}_+)$ *such that for almost all* $t \in I, F(t,.)$ *is* $L(t)$-*Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall\ x, y \in X,$$

where $d_H(A,B)$ *is the Hausdorff distance between* $A, B \subset X$

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

(iv) $d(0, F(t,0)) \leq L(t)$ *a.e.* (I)

Let $m(t) = \int_0^t L(u)du$, denote $M := \sup_{t,s \in I} |G(t,s)|$ and $M_0 := \sup_{(t,s) \in D} |K(t,s)|$ and remark that $|U(t,s)| \leq MM_0(t-s) \leq MM_0T$.

Given $\alpha \in \mathbf{R}$ we consider on $L^1(I,X)$ the following norm

$$|f|_1 = \int_0^T e^{-\alpha m(t)}|f(t)|dt, \quad f \in L^1(I,X),$$

which is equivalent with the usual norm on $L^1(I,X)$.

Consider the following norm on $C(I,X) \times L^1(I,X)$

$$|(x,f)|_{C \times L} = |x|_C + |f|_1 \quad \forall\ (x,f) \in C(I,X) \times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t \in I} |x(t)|\ \forall\ x \in C(I,X)$.

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x,y) \quad \forall x, y \in Z.$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ ([12]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Proposition 2.2.([15]) *Let* Z *be a complete metric space and suppose that* T_1, T_2 *are* λ-*contractions with closed values in* Z. *Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in Z} d_H(T_1(z), T_2(z)).$$

3. The main result

We are ready now to present an existence theorem concerning mild solutions for the Cauchy problem (2.1).

Theorem 3.1. *Let Hypothesis 2.1 be satisfied and let $\alpha > M_0MT$ and let $y(.)$ be a mild solution of the problem*

$$y'(t) = A(t)y(t) + \int_0^t K(t,s)g(s)ds, \quad y(0) = y_0,$$

where $g(.) \in L^1(I,X)$ and there exists $p(.) \in L^1(I,\mathbf{R})$ such that

$$d(g(t), F(t, y(t))) \leq p(t), \quad a.e.\ (I).$$

Then for every $\varepsilon > 0$ there exists $x(.)$ a mild solution of (2.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq (M + \frac{M^2M_0T}{\alpha - MM_0T}e^{\alpha m(t)})|x_0 - y_0| + \frac{\alpha MM_0Te^{\alpha m(t)}}{\alpha - MM_0T}\int_0^T e^{-\alpha m(s)}p(s)ds + \varepsilon. \tag{3.1}$$

Proof. Let us consider $x_0 \in X, f(.) \in L^1(I,X)$ and define the following set valued maps

$$M_{x_0,f}(t) = F(t, G(t,0)x_0 + \int_0^t U(t,s)f(s)ds), \quad t \geq 0, \tag{3.2}$$

$$T_{x_0}(f) = \{\phi(.) \in L^1(I,X); \quad \phi(t) \in M_{x_0,f}(t) \quad a.e.\ (I)\}. \tag{3.3}$$

We shall prove first that $T_{x_0}(f)$ is nonempty and closed for every $f \in L^1(I,X)$. The fact that the set valued map $M_{x_0,f}(.)$ is measurable is well known. For example, the map $t \to G(t,0)x_0 + \int_0^t U(t,s)f(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [8]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [8]) we infer that $M_{x_0,f}(.)$ admits a measurable selection ϕ. According to Hypothesis 2.1 one has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t,x(t))) \leq L(t)(1 + |x(t)|)]] \leq L(t)(1 + M|x_0| + \int_0^t MM_0(t-s)|f(s)|ds).$$

Thus integrating by parts we obtain

$$\int_0^T e^{-\alpha m(t)}|\phi(t)|dt \leq \int_0^T e^{-\alpha m(t)}L(t)(1 + M|x_0| + \int_0^t MM_0(t-s)|f(s)|ds)dt \leq \frac{1 + M|x_0|}{\alpha} + \frac{MM_0T|f|_1}{\alpha}.$$

Hence, if $\phi(.)$ is a measurable selection of $M_{x_0,f}(.)$, then $\phi(.) \in L^1(I,X)$ and thus $T_{x_0}(f) \neq \emptyset$.

The set $T_{x_0}(f)$ is closed. Indeed, if $\phi_n \in T_{x_0}(f)$ and $|\phi_n - \phi|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T_{x_0}(f)$.

The next step of the proof will show that $T_{x_0}(.)$ is a contraction on $L^1(I,X)$.

Let $f, g \in L^1(I,X)$ be given, $\phi \in T_{x_0}(f)$ and let $\delta > 0$. Consider the following set valued map

$$G(t) = M_{x_0,g}(t) \cap \{x \in X;\ |\phi(t) - x| \leq L(t)|\int_0^t U(t,s)(f(s)-g(s))ds| + \delta\}.$$

Since

$$d(\phi(t), M_{x_0,g}(t)) \leq d(F(t, G(t,0)x_0 + \int_0^t U(t,s)f(s)ds),$$

$$F(t, G(t,0)x_0 + \int_0^t U(t,s)g(s)ds)) \leq L(t)|\int_0^t U(t,s)(f(s)-g(s))ds|$$

we deduce that $G(.)$ has nonempty closed values. Moreover, according to Proposition III.4 in [8], $G(.)$ is measurable. Let $\psi(.)$ be a measurable selection of $G(.)$. It follows that $\psi \in T_{x_0}(g)$ and

$$\begin{aligned} |\phi - \psi|_1 &= \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \\ &\leq \int_0^T e^{-\alpha m(t)} L(t)(\int_0^t MM_0(t-s)|f(s)-g(s)|ds)dt + \int_0^T \delta e^{-\alpha m(t)}dt \\ &\leq \frac{MM_0T}{\alpha}|f-g|_1 + \delta \int_0^T e^{-\alpha m(t)}dt. \end{aligned}$$

Since δ was arbitrary, we deduce that

$$d(\phi, T_{x_0}(g)) \leq \frac{MM_0T}{\alpha}|f-g|_1.$$

Replacing f by g we obtain

$$d(T_{x_0}(f), T_{x_0}(g)) \leq \frac{MM_0T}{\alpha}|f-g|_1,$$

hence $T_{x_0}(.)$ is a contraction on $L^1(I,X)$.

We consider next the following set-valued maps

$$\tilde{F}(t,x) = F(t,x) + p(t)B, \quad (t,x) \in I \times X,$$

$$\tilde{M}_{y_0,f}(t) = \tilde{F}(t, G(t,0)y_0 + \int_0^t U(t,s)f(s)ds),\ t \in I, y_0 \in X,$$

$$\tilde{T}_{y_0}(f) = \{\phi(.) \in L^1(I,X); \quad \phi(t) \in \tilde{M}_{y_0,f}(t) \quad a.e.\ (I)\},\ f \in L^1(I,X),$$

where B denotes the closed unit ball in X. Obviously, $\tilde{F}(.,.)$ satisfies Hypothesis 2.1.

Repeating the previous step of the proof we obtain that $\tilde{T}_{y_0}(.)$ is also a $\frac{MM_0T}{\alpha}$-contraction on $L^1(I,X)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_{x_0}(f),\tilde{T}_{y_0}(f)) \leq \frac{M}{\alpha}|x_0-y_0|+\int_0^T e^{-\alpha m(t)}p(t)dt, \qquad \forall\, f(.)\in L^1(I,X). \quad (3.4)$$

Let $\phi\in T_{x_0}(f)$, $\delta>0$ and, for $t\in I$, define

$$G_1(t)=\tilde{M}_{y_0,f}(t)\ \cap\ \{z\in X;\ |\phi(t)-z|\leq L(t)|G(t,0)|\cdot|x_0-y_0|+p(t)+\delta\}$$

With the same arguments used for the set-valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.)\in\tilde{T}_{y_0}(f)$ and one has

$$\begin{aligned}|\phi-\psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t)-\psi(t)|dt \leq \int_0^T e^{-\alpha m(t)}[L(t)|G(t,0)|\cdot|x_0-y_0|\\ +p(t)+\delta]dt\leq\frac{M}{\alpha}|x_0-y_0|+\int_0^T e^{-\alpha m(t)}p(t)dt+\delta\int_0^T e^{-\alpha m(t)}p(t)dt.\end{aligned}$$

Since $\delta>0$ was arbitrary, as above, we obtain (3.4). Applying Proposition 2.2 we get

$$d_H(Fix(T_{x_0}),Fix(\tilde{T}_{y_0}))\leq\frac{M}{\alpha-MM_0T}|x_0-y_0|+\frac{\alpha}{\alpha-M_0MT}\int_0^T e^{-\alpha m(t)}p(t)dt.$$

Since $g(.)\in Fix(\tilde{T}_{y_0})$ it follows that there exists $f(.)\in Fix(T_{x_0})$ such that for any $\varepsilon>0$

$$\begin{aligned}|g-f|_1\leq\frac{M}{\alpha-MM_0T}|x_0-y_0|+\frac{\alpha}{\alpha-MM_0T}\int_0^T e^{-\alpha m(t)}p(t)dt\\ +\frac{\varepsilon}{MM_0Te^{\alpha m(T)}}.\end{aligned} \quad (3.5)$$

We define $x(t)=G(t,0)x_0+\int_0^t U(t,s)f(s)ds$, $t\in I$ and we have

$$|x(t)-y(t)|\leq M|x_0-y_0|+MM_0Te^{\alpha m(t)}|f-g|_1.$$

Combining the last inequality with (3.5) we obtain (3.1).

Remark 3.2. *If $A(t)\equiv A$ and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t);t\geq 0\}$ from X to X then problem (1.1) reduces to the problem*

$$x'(t)\in Ax(t)+\int_0^t K(t,s)F(s,x(s))ds,\quad x(0)=x_0, \quad (3.6)$$

well known ([1-7,16] etc.) as an integrodifferential inclusion.

Obviously, a similar result to the one in Theorem 3.1 may be obtained for problem (3.6).

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