# Partitions of positive integers into sets without infinite progressions

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**Abstract**. *We prove a result which implies that, for any real numbers $a$ and $b$ satisfying $0 \le a \le b \le 1$, there exists an infinite sequence of positive integers $A$ with lower density $a$ and upper density $b$ such that the sets $A$ and $\mathbb{N} \setminus A$ contain no infinite arithmetic and geometric progressions. Furthermore, for any $m \ge 2$ and any positive numbers $a_1, \dots, a_m$ satisfying $a_1 + \dots + a_m = 1$, we give an explicit partition of $\mathbb{N}$ into $m$ disjoint sets $\cup_{j=1}^{m} A_j$ such that $d_P(A_j) = a_j$ for each $j = 1, \dots, m$ and each infinite arithmetic and geometric progression $P$, where $d_P(A_j)$ denotes the proportion between the elements of $P$ that belong to $A_j$ and all elements of $P$, if a corresponding limit exists. In particular, for $a = 1/2$ and $m = 2$, this gives an explicit partition of $\mathbb{N}$ into two disjoint sets such that half of elements in each infinite arithmetic and geometric progression will be in one set and half in another.*

**Key words:** *infinite sequence, partition of integers, density, arithmetic and geometric progression*

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## 1. Introduction

Let $A = \{a_1 < a_2 < a_3 < \dots\}$ be an infinite sequence of positive integers. Throughout the paper we shall write $A(n)$ for the intersection $A \cap \{1, 2, \dots, n\}$. Recall that the *lower density* of $A$ is defined by the formulae

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A(n)|}{n} = \liminf_{n \to \infty} \frac{n}{a_n}$$

and the *upper density* of $A$ by the formulae

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A(n)|}{n} = \limsup_{n \to \infty} \frac{n}{a_n}.$$

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If $\underline{d}(A) = \overline{d}(A)$, then their common value $d(A) = \lim_{n\to\infty} n/a_n$ is called the *density* of $A$. Similarly, taking into account only elements of $A$ that belong to a subset $P$ of $\mathbb{N}$, one can define

$$\underline{d}_P(A) = \liminf_{n\to\infty} \frac{|(P\cap A)(n)|}{|P(n)|}, \qquad \overline{d}_P(A) = \limsup_{n\to\infty} \frac{|(P\cap A)(n)|}{|P(n)|},$$

and call their common value $d_P(A)$ the *density of $A$ with respect to $P$* if $\underline{d}_P(A) = \overline{d}_P(A)$.

A famous theorem of Szemerédi [5] states that, for any $k \in \mathbb{N}$, every set of natural numbers of upper density $> 0$ contains an arithmetic progression of length $k$. Two important alternative proofs of Szemerédi's theorem were later found by Furstenberg [2] and Gowers [1]. All these results are related to van der Waerden's theorem claiming that, for each pair of positive integers $k$ and $r$, there exists a positive integer $M$ such that in any coloring of integers $1, 2, \dots, M$ into $r$ colors there is a monochromatic (i.e., colored in one color) arithmetic progression of length $k$.

There is no 'infinite version' of Szemerédi's theorem. Even a sequence of integers of density 1 not necessarily contains an infinite arithmetic progression. Moreover, Wagstaff [6] proved that, for any two real numbers $a, b$ satisfying $0 \le a \le b \le 1$, there is an infinite sequence $A \subset \mathbb{N}$ with $\underline{d}(A) = a$ and $\overline{d}(A) = b$ which contains no infinite arithmetic progression. The set $A$ which was considered in [6] satisfies $a_{n+1} - a_n > \sqrt{n}$ for infinitely many $n \in \mathbb{N}$. Of course, such a set cannot contain an infinite arithmetic progression. How far one can extend this gap on two consecutive elements of $A$?

**Theorem 1.** *Suppose $0 \le a \le b \le 1$ and suppose $\delta(n)$, $n = 1, 2, \dots,$ is an arbitrary sequence of positive numbers satisfying $\lim_{n\to\infty} \delta(n) = 0$. Then there is a sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ with lower density $a$ and upper density $b$ such that $a_{n+1} - a_n > n\delta(n)$ for infinitely many $n \in \mathbb{N}$.*

The condition $\lim_{n\to\infty} \delta(n) = 0$ is necessary for Theorem 1 to hold. Indeed, for any set $A \subset \mathbb{N}$ with density $a > 0$, we have $\lim_{n\to\infty} n/a_n = a$, so $\lim_{n\to\infty} a_{n+1}/n = \lim_{n\to\infty} a_n/n$. It follows that $\lim_{n\to\infty}(a_{n+1} - a_n)/n = 0$. Thus one cannot replace $\delta(n)$ by a positive constant.

Is it possible to construct infinite sequences with prescribed densities which do not contain not only infinite arithmetic but also infinite geometric progressions or some other sparse sets? Moreover, is it possible to partition $\mathbb{N}$ into two disjoint sets $A$ and $B$ with prescribed lower and upper densities in such a way that not only $A$ but both $A$ and $B$ do not contain infinite arithmetic and geometric progressions? (Of course, since $\mathbb{N} = A \cup B$ and $A \cap B = \emptyset$, an obvious restriction is $\underline{d}(A) + \overline{d}(B) = 1$ and $\overline{d}(A) + \underline{d}(B) = 1$.)

The next theorem implies a positive answer to both these questions:

**Theorem 2.** *For any real numbers $a$ and $b$ satisfying $0 \le a \le b \le 1$ and any countable collection of increasing infinite sequences of positive integers $S_1, S_2, S_3, \dots,$ there are two disjoint infinite sets of positive integers $A$ and $B$ such that $A \cup B = \mathbb{N}$, $\underline{d}(A) = a$, $\overline{d}(A) = b$, $\underline{d}(B) = 1 - b$, $\overline{d}(B) = 1 - a$ and each of the sets $A$ and $B$ contains infinitely many elements of $S_j$ for every $j \ge 1$.*

Here, the sequences $S_j$, $j = 1, 2, 3, \dots,$ are not necessarily disjoint or even distinct. Clearly, the set of all infinite arithmetic and geometric progressions lying in $\mathbb{N}$ is countable, so Theorem 2 implies that there exists an infinite sequence of positive integers $A$ with lower density $a$ and upper density $b$ such that the sets $A$ and $\mathbb{N} \setminus A$ contain no infinite arithmetic and geometric progressions. Similarly, we can take $S_j = \{p_j(n) \mid n \geq n(p_j)\}$, where $p_j$, $j = 1, 2, \dots$, are polynomials with integer coefficients and positive leading coefficients and $n(p_j)$ is the least positive integer such that $p_j(x) > 0$ for $x \geq n(p_j)$.

Finally, let $a_1, \dots, a_m$ be positive numbers satisfying $a_1 + \dots + a_m = 1$. Is there a partition of $\mathbb{N}$ into $m$ disjoint sets $A_j$, $j = 1, \dots, m$, such that $d_P(A_j) = a_j$ for any infinite arithmetic and geometric progression $P$? This means that, for each infinite progression $P \subset \mathbb{N}$, not just that the set $P \cap A_j$ is infinite, but also that $a_j$th part of $P$ belongs to $A_j$. The answer is 'yes' and the proof is constructive.

**Theorem 3.** *For any positive real numbers $a_1, \dots, a_m$ satisfying $a_1 + \dots + a_m = 1$, there is an explicit partition of $\mathbb{N}$ into $m$ disjoint subsets $\mathbb{N} = \cup_{j=1}^m A_j$ such that $d_P(A_j) = a_j$ for $j = 1, \dots, m$ and every infinite arithmetic and geometric progression $P \subset \mathbb{N}$.*

Note that $\mathbb{N}$ itself is an arithmetic progression, so $d_{\mathbb{N}}(A_j) = d(A_j) = a_j$. In case $m = 2$ and $a_1 = a_2 = 1/2$, the theorem shows how $\mathbb{N}$ can be colored into two colors so that 'half' of all elements in every infinite progression (arithmetic or geometric) are colored into one color and 'half' into another.

## 2. Sequences with large gaps: proof of Theorem 1

It is clear that every infinite sequence $A \subset \mathbb{N}$ can be given by strings of consecutive elements that belong and do not belong to $A$. So each $A = \{a_1 < a_2 < a_3 < \dots\}$ for which $\mathbb{N} \setminus A$ is infinite can be represented by the sequence

$$e_1, f_1, e_2, f_2, e_3, f_3, \dots,$$

where $e_1 \geq 0, e_2, e_3, \dots \geq 1$ stand for $e_1, e_2, e_3, \dots$ consecutive 'empty' places which are free of elements of $A$, and $f_1, f_2, f_3, \dots \geq 1$ stand for $f_1, f_2, f_3, \dots$ consecutive occupied ('full') places. Put $E_n = e_1 + \dots + e_n$, $F_n = f_1 + \dots + f_n$ and $E_0 = F_0 = 0$.

For any positive integer $k$, there is a unique nonnegative integer $n$ such that $E_n + F_n < k \leq E_{n+1} + F_{n+1}$. Then

$$A(k) = \begin{cases} F_n, & \text{if } E_n + F_n < k \leq E_{n+1} + F_n, \\ k - E_{n+1}, & \text{if } E_{n+1} + F_n < k \leq E_{n+1} + F_{n+1}. \end{cases}$$

Using this, one can easily check that

$$\frac{F_n}{F_n + E_{n+1}} \leq \frac{A(k)}{k} \leq \frac{F_{n+1}}{F_{n+1} + E_{n+1}}.$$

In particular, this yields

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{F_n}{F_n + E_{n+1}}, \quad \overline{d}(A) = \limsup_{n \to \infty} \frac{F_n}{F_n + E_n}. \tag{1}$$

Moreover, we have $A(E_n + F_n) = A(E_{n+1} + F_n) = F_n$. Thus $a_{F_n} = E_n + F_n$ and $a_{F_n+1} = E_{n+1} + F_n + 1$, giving

$$a_{F_n+1} - a_{F_n} = E_{n+1} - E_n + 1. \tag{2}$$

We shall consider the following six cases:
($i$) $a = 0$, $0 \le b < 1$,
($ii$) $a = 0$, $b = 1$,
($iii$) $0 < a < b < 1$,
($iv$) $0 < a < b = 1$,
($v$) $0 < a = b < 1$,
($vi$) $a = b = 1$.

For $a = b = 0$, we can take any sufficiently fast increasing sequence $A$. For $a = 0$, $0 < b < 1$, fix any $c$ satisfying $c \min\{b, 1-b\} > 1$. Put $F_n = [cb2^{2^n}]$ and $E_n = [c(1-b)2^{2^n}]$. Then (1) implies that $\underline{d}(A) = 0$ and $\overline{d}(A) = b$. Also, (2) implies that $(a_{F_{n+1}} - a_{F_n})/F_n \to \infty$ as $n \to \infty$, because $E_{n+1}/F_n \to \infty$ as $n \to \infty$. This completes the proof in the case ($i$). Of course, in this case, the gap on $a_{n+1} - a_n$ can be greater than any given function $g(n)$ for infinitely many $n \in \mathbb{N}$.

In case ($ii$), we can take $F_n = 2^{2^n}$ and $E_n = [2^{2^n}/n]$. Then (1) shows that $A$ has lower density 0 and upper density 1, whereas (2) implies that $(a_{F_{n+1}} - a_{F_n})/F_n \to \infty$ as $n \to \infty$.

In case ($iii$), put $w = b(1-a)/(a(1-b))$, $F_n = [cbw^n]$, $E_n = [c(1-b)w^n]$, where $c$ is so large that the sequences $F_n$, $n = 1, 2, \dots$, and $E_n$, $n = 1, 2, \dots$, are increasing. Then (1) implies that $A$ has lower density $a$ and upper density $b$. Also, (2) implies that $\lim_{n\to\infty}(a_{F_{n+1}} - a_{F_n})/F_n = (b-a)/(ab)$.

In case ($iv$), we take $c$ so large that $c \min\{a, 1-a\} > 1$ and put $F_n = [ca2^{2^n}]$, $E_n = [c(1-a)2^{2^{n-1}}]$. Then (1) and (2) imply that $\underline{d}(A) = a$, $\overline{d}(A) = 1$ and $\lim_{n\to\infty}(a_{F_{n+1}} - a_{F_n})/F_n = 1/a - 1$.

In cases ($v$) and ($vi$), we can assume without loss of generality that the sequence $\delta(n)$, $n = 1, 2, \dots$, is decreasing and the sequence $n\delta(n)$, $n = 1, 2, \dots$, is increasing and unbounded. Consider the sequence $x_1 = 1$,

$$x_{n+1} = x_n\bigl(1 + \sqrt{\delta(x_n)}\bigr) \tag{3}$$

for $n \ge 1$. Clearly, $x_n \to \infty$ as $n \to \infty$.

In case ($v$), put $F_n = [cax_n]$ and $E_n = [c(1-a)x_n]$, where $c$ is so large that the sequences $F_n$, $n = 1, 2, \dots$, and $E_n$, $n = 1, 2, \dots$, are increasing. By (3), $\lim_{n\to\infty} x_{n+1}/x_n = 1$. Thus (1) implies that $d(A) = a$. Also, (2) and (3) imply that $a_{F_{n+1}} - a_{F_n}$ is at least

$$E_{n+1} - E_n + 1 > c(1-a)(x_{n+1} - x_n) = c(1-a)x_n\sqrt{\delta(x_n)}.$$

This is greater than $F_n\delta(F_n)$ for each sufficiently large $n \in \mathbb{N}$.

In case ($vi$), let $F_n = [cx_n]$ for each $n \ge 1$, whereas $E_n = [cx_{n-1}(\delta(x_{n-1}))^{1/3}]$ for $n$ odd and $E_n = [cx_{n-1}(\delta(x_{n-2}))^{1/3}]$ for $n$ even. Here, $c$ is so large that the sequences $F_n$, $n = 1, 2, \dots$, and $E_n$, $n = 1, 2, \dots$, are increasing. Now, (1) implies that $d(A) = 1$. Also, (2) and (3) imply that $a_{F_{2n+2}} - a_{F_{2n+1}}$ is at least

$$E_{2n+2} - E_{2n+1} > c(x_{2n+1} - x_{2n})\delta(x_{2n})^{1/3} = cx_{2n}\delta(x_{2n})^{5/6}.$$

This is greater than $F_{2n+1}\delta(F_{2n+1})$ for each sufficiently large $n \in \mathbb{N}$.

## 3. Proof of Theorem 2

Let us consider the collection of sequences

$$S_1, S_1, S_2, S_1, S_2, S_3, S_1, S_2, S_3, S_4, \dots.$$

It contains each $S_j$ infinitely many times. We shall denote this collection of sequences by $T_1, T_2, T_3, \dots$.

Given $a$ and $b$ satisfying $0 \le a \le b \le 1$, by Theorem 1, there is set $A$ with lower density $a$ and upper density $b$ such that both $A$ and $B = \mathbb{N} \setminus A$ are infinite. We shall construct two sequences of positive integers $u_1 < u_2 < u_3 < \dots$ and $v_1 < v_2 < v_3 < \dots$ as follows. Take any $u_1, v_1 \in T_1$ satisfying $u_1 < v_1$. Suppose $u_1 < v_1 < \dots < u_n < v_n$, where $u_j, v_j \in T_j$, are given. Take $u_{n+1} > \max\{2u_n, v_n\}$ in $T_{n+1}$. Then take $v_{n+1} > \max\{2v_n, u_{n+1}\}$ in $T_{n+1}$, and so on.

Put $A' = A \cup \{v_1, v_2, v_3, \dots\} \setminus \{u_1, u_2, u_3, \dots\}$ and $B' = \mathbb{N} \setminus A'$. By the above construction, $u_{j+1} > 2u_j$ and $v_{j+1} > 2v_j$, so $v_j > u_j \ge 2^{j-1}$. Hence the sets $\{v_1, v_2, \dots\}$ and $\{u_1, u_2, \dots\}$ are both of density zero. This implies that $A'$ is still of lower density $a$ and of upper density $b$. Since $v_j \in A' \cap T_j$ and $u_j \in B' \cap T_j$, each of the sets $A'$ and $B'$ contains at least one element of $T_j$, where $j \ge 1$. By the construction of $T_j$, for every fixed $i \ge 1$, we have $T_j = S_i$ for infinitely many $j$'s. So both $A'$ and $B'$ contain infinitely many elements of $S_i$.

## 4. Proof of Theorem 3

Let $c$ be a positive integer satisfying $c \min_{1 \le j \le m} a_j > 1$. At the first step, for each $j = 1, \dots, m$, we subsequently color $[ca_j e]$ elements into $j$th color. Then, for every $n \ge 2$, at the $n$th step, for each $j = 1, \dots, m$, we subsequently color $[ca_j e^{n^{2/3}}] - [ca_j e^{(n-1)^{2/3}}]$ elements into $j$th color. Suppose that $A_j$ is the set of all elements colored into $j$th color. Put $F_{j,n} = [ca_j e^{n^{2/3}}]$. Then $A_j(F_{1,n} + \dots + F_{j,n} + F_{j+1,n-1} + \dots + F_{m,n-1}) = F_{j,n}$ and $A_j(F_{1,n+1} + \dots + F_{j-1,n+1} + F_{j,n} + \dots + F_{m,n}) = F_{j,n}$. Since the quotients

$$F_{j,n}/(F_{1,n} + \dots + F_{j,n} + F_{j+1,n-1} + \dots + F_{m,n-1}) \tag{4}$$

and

$$F_{j,n}/(F_{1,n+1} + \dots + F_{j-1,n+1} + F_{j,n} + \dots + F_{m,n}) \tag{5}$$

both tend to $a_j$ as $n \to \infty$, we obtain that $d(A_j) = a_j$.

Consider the arithmetic progression $P$ of the form $um+v$, $m = 0, 1, 2, \dots$, where $u$ and $v$ are some positive integers. Note that $um + v$ is an element of the $j$th color in the $n$th string if

$$\begin{aligned} &F_{1,n} + \dots + F_{j-1,n} + F_{j,n-1} + \dots + F_{m,n-1} \\ &\quad < um + v \\ &\quad \le F_{1,n} + \dots + F_{j,n} + F_{j+1,n-1} + \dots + F_{m,n-1}. \end{aligned}$$

There are

$$\begin{aligned}&[(F_{1,n}+\dots+F_{j,n}+F_{j+1,n-1}+\dots+F_{m,n-1}-v)/u]\\&\quad-[(F_{1,n}+\dots+F_{j-1,n}+F_{j,n-1}+\dots+F_{m,n-1}-v)/u]\\&=(F_{j,n}-F_{j,n-1})/u+\theta_n\end{aligned}$$

of such integers $m$, where $|\theta_n|<1$. Since $\lim_{n\to\infty}(F_{j,n}-n)=\infty$, we have

$$\sum_{k=1}^{n}((F_{j,k}-F_{j,k-1})/u+\theta_k)\sim F_{j,n}/u$$

as $n\to\infty$. Also, there are $\sim(F_{1,n}+\dots+F_{j,n}+F_{j+1,n-1}+\dots+F_{m,n-1})/u$ elements of $P$ which belong to $A_j$ and do not exceed $F_{1,n}+\dots+F_{j,n}+F_{j+1,n-1}+\dots+F_{m,n-1}$ and $\sim(F_{1,n+1}+\dots+F_{j-1,n+1}+F_{j,n}+\dots+F_{m,n})/u$ elements which do not exceed $F_{1,n+1}+\dots+F_{j-1,n+1}+F_{j,n}+\dots+F_{m,n}$. The same factor $1/u$ occurs in all expressions, so, using the fact that (4) and (5) tend to $a_j$ as $n\to\infty$, we obtain that $d_P(A_j)=a_j$.

Suppose next that $P$ is the geometric progression $uv^m$, $m=0,1,2,\dots$, where $u\ge 1$ and $v\ge 2$ are some positive integers. Fix $\varepsilon>0$. We claim that $\underline{d}_P(A_j)\ge a_j-3\varepsilon$ for every $j=1,\dots,m$. Since $\varepsilon>0$ is arbitrary and $a_1+\dots+a_m=1$, this would imply that $d_P(A_j)=a_j$. Indeed, using $\underline{d}_P(A_i)\ge a_i-3\varepsilon$ for each $i$, we obtain that

$$\begin{aligned}1-\overline{d}_P(A_j)&=\underline{d}_P(\mathbb{N}\setminus A_j)=\underline{d}_P(\cup_{i\ne j}A_i)\\&\ge\sum_{i\ne j}\underline{d}_P(A_i)\ge\sum_{i\ne j}(a_i-3\varepsilon)=1-a_j-3(m-1)\varepsilon.\end{aligned}$$

Hence $\overline{d}_P(A_j)\le a_j+3(m-1)\varepsilon$. Combined with $\underline{d}_P(A_j)\ge a_j-3\varepsilon$ this implies that $d_P(A_j)=a_j$.

It remains to prove the inequality $\underline{d}_P(A_j)\ge a_j-3\varepsilon$. For every nonnegative integer $m$, there is a unique $n\in\mathbb{N}$ such that $\sum_{j=1}^m F_{j,n-1}<uv^m\le\sum_{j=1}^m F_{j,n}$. Using

$$uv^m\le\sum_{j=1}^{m}F_{j,n}\le c(a_1+\dots+a_m)e^{n^{2/3}}=ce^{n^{2/3}},$$

we deduce that

$$n\ge(m\log v+\log(u/c))^{3/2}.$$

The term $uv^m$ belongs to $A_j$ if and only if

$$\begin{aligned}&F_{1,n}+\dots+F_{j-1,n}+F_{j,n-1}+\dots+F_{m,n-1}\\&\quad<uv^m\le F_{1,n}+\dots+F_{j,n}+F_{j+1,n-1}+\dots+F_{m,n-1}\end{aligned}$$

for some $n$. We shall prove that, for $m$ large enough, this happens if the fractional part $\ell(m)=\{(m\log v+\log(u/c))^{3/2}\}$ belongs to the interval $(a_1+\dots+a_{j-1}+\varepsilon,a_1+\dots+a_j-\varepsilon)$.

Put $n(m) = [(m \log v + \log(u/c))^{3/2}]$. Since $(m \log v + \log(u/c))^{3/2} \notin \mathbb{Z}$, we have $n \geq [(m \log v + \log(u/c))^{3/2}] + 1 = n(m) + 1$, giving

$$F_{1,n} + \cdots + F_{j,n} + F_{j+1,n-1} + \cdots + F_{m,n-1} > c(a_1 + \cdots + a_j)e^{(n(m)+1)^{2/3}} \\ + c(a_{j+1} + \cdots + a_m)e^{n(m)^{2/3}} - m$$

Similarly, for $n = n(m) + 1$, we have

$$F_{1,n} + \cdots + F_{j-1,n} + F_{j,n-1} + \cdots + F_{m,n-1} \leq c(a_1 + \cdots + a_{j-1})e^{(n(m)+1)^{2/3}} \\ + c(a_j + \cdots + a_m)e^{n(m)^{2/3}}.$$

Note that $uv^m = ce^{(n(m)+\ell(m))^{2/3}}$. So it suffices to show that

$$(a_1 + \cdots + a_{j-1})e^{(n(m)+1)^{2/3}} + (a_j + \cdots + a_m)e^{n(m)^{2/3}} < e^{(n(m)+\ell(m))^{2/3}} \\ < (a_1 + \cdots + a_j)e^{(n(m)+1)^{2/3}} + (a_{j+1} + \cdots + a_m)e^{n(m)^{2/3}} - m/c.$$

Since $a_1 + \cdots + a_m = 1$, subtracting $e^{n(m)^{2/3}}$, we obtain

$$(a_1 + \cdots + a_{j-1})(e^{(n(m)+1)^{2/3}} - e^{n(m)^{2/3}}) < e^{(n(m)+\ell(m))^{2/3}} - e^{n(m)^{2/3}} \\ < (a_1 + \cdots + a_j)(e^{(n(m)+1)^{2/3}} - e^{n(m)^{2/3}}) - m/c.$$

Now, multiplying by $(3/2)n(m)^{1/3}e^{-n(m)^{2/3}}$, and letting $n(m) \to \infty$, we deduce that $a_1 + \cdots + a_{j-1} < \ell(m) < a_1 + \cdots + a_j$. Hence, if $\varepsilon > 0$ is fixed and $m$ is large enough, then the required inequalities for $uv^m$ hold (i.e., $\sum_{j=1}^{m} F_{j,n-1} < uv^m \leq \sum_{j=1}^{m} F_{j,n}$) provided that $a_1 + \cdots + a_{j-1} + \varepsilon < \ell(m) < a_1 + \cdots + a_j - \varepsilon$.

The length of the interval $(a_1 + \cdots + a_{j-1} + \varepsilon, a_1 + \cdots + a_j - \varepsilon)$ is $a_j - 2\varepsilon$. By a version of Fejér's difference theorem (see, for instance, Theorem 3.4 in [3] or, more precisely, 2.6.1 in [4]), the sequence $g(m)$, $m = 0, 1, 2, \ldots$, where $g(x) = (x \log v + \log(au/c))^{3/2}$, is uniformly distributed modulo 1, because $g''(x)$ tends monotonically to zero as $x \to \infty$ and $x|g''(x)| \to \infty$ as $x \to \infty$. It follows that the sequence $\ell(m)$, $m = 1, 2, \ldots$, is uniformly distributed in $[0, 1)$. In particular, the part of $m \in \mathbb{N}$ for which $\ell(m) \in (a_1 + \cdots + a_{j-1} + \varepsilon, a_1 + \cdots + a_j - \varepsilon)$ is at least $a_j - 3\varepsilon$, namely, $\underline{d}_P(A_j) \geq a_j - 3\varepsilon$. This completes the proof of the theorem.

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