On the distance spectra of some graphs

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Abstract. *The D-eigenvalues of a connected graph G are the eigenvalues of its distance matrix D, and form the D-spectrum of G. The D-energy $E_D(G)$ of the graph G is the sum of the absolute values of its D-eigenvalues. Two (connected) graphs are said to be D-equienergetic if they have equal D-energies. The D-spectra of some graphs and their D-energies are calculated. A pair of D-equienergetic bipartite graphs on $24t$, $t \geq 3$, vertices is constructed.*

Key words: *distance eigenvalue (of a graph), distance spectrum (of a graph), distance energy (of a graph), distance–equienergetic graphs*

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1. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix $D = D(G)$ of G is defined so that its (i, j)-entry is equal to $d_G(v_i, v_j)$, the distance (= length of the shortest path [2]) between the vertices v_i and v_j of G. The eigenvalues of the $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by $spec_D(G)$.

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [4]. In what follows we denote the ordinary eigenvalues of the graph G by λ_i , $i = 1, 2, \ldots, p$, and the respective spectrum by $spec(G)$.

Since the distance matrix is symmetric, all its eigenvalues μ_i , $i = 1, 2, \ldots, p$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. If $\mu_{i_1} > \mu_{i_2} > \cdots > \mu_{i_g}$ are the distinct D-eigenvalues, then the D-spectrum can be written as

$$spec_D(G) = \begin{pmatrix} \mu_{i_1} & \mu_{i_2} & \ldots & \mu_{i_g} \\ m_1 & m_2 & \ldots & m_g \end{pmatrix}$$

where m_j indicates the algebraic multiplicity of the eigenvalue μ_{i_j}. Of course, $m_1 + m_2 + \cdots + m_g = p$.

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Two graphs G and H for which $spec_D(G) = spec_D(H)$ are said to be D-cospectral. Otherwise, they are non-D-cospectral.

The D-energy, $E_D(G)$, of G is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i| \ . \tag{1}$$

Two graphs with equal D-energy are said to be D-equienergetic. D-cospectral graphs are evidently D-equienergetic. Therefore, in what follows we focus our attention to D-equienergetic non-D-cospectral graphs.

The concept of D-energy, Eq. (1), was recently introduced [11]. This definition was motivated by the much older [7] and nowadays extensively studied [8, 9, 10, 13, 14, 15, 16] graph energy, defined in a manner fully analogous to Eq. (1), but in terms of the ordinary graph eigenvalues (eigenvalues of the adjacency matrix, see [4]).

In this paper we first derive a Hoffman–type relation for the distance matrix of distance regular graphs. By means of it, the distance spectra of some graphs and their energies are obtained. Also pairs of D-equienergetic bipartite graphs on $24\,t$, $t \geq 3$, vertices are constructed. All graphs considered in this paper are simple and we follow [4] for spectral graph theoretic terminology.

The considerations in the subsequent sections are based on the applications of the following lemmas:

Lemma 1 [see [4]]. *Let G be a graph with adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then* $\det A = \prod_{i=1}^{p} \lambda_i$. *In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence* $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2 [see [5]]. *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 block symmetric matrix. Then the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Lemma 3 [see [4]]. *Let M, N, P, and Q be matrices, and let M be invertible. Let*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix} .$$

Then $\det S = \det M \det(Q - PM^{-1}N)$. *Besides, if M and P commute, then* $\det S = \det(MQ - PN)$.

Lemma 4 [see [4]]. *Let G be a connected r-regular graph, $r \geq 3$, with ordinary spectrum $spec(G)\{r, \lambda_2, \ldots, \lambda_p\}$. Then*

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & p(r-2)/2 \end{pmatrix} .$$

Lemma 5 [see [4]]. *For every $t \geq 3$, there exists a pair of non-cospectral cubic graphs on $2t$ vertices.*

Lemma 6 [see [6]]. *The distance spectrum of the cycle C_n is given by*

n	*greatest eigenvalue*	*j even*	*j odd*
even	$\frac{n^2}{4}$	0	$-cosec^2\left(\frac{\pi j}{n}\right)$
odd	$\frac{n^2-1}{4}$	$-\frac{1}{4}sec^2\left(\frac{\pi j}{2n}\right)$	$-\frac{1}{4}cosec^2\left(\frac{\pi j}{2n}\right)$

Definition 1 [see [12]]. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$. Take another copy of G with the vertices labelled by $\{u_1, u_2, \dots, u_p\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The resulting graph, denoted by D_2G, is called the double graph of G.*

Definition 2 [see [4]]. *Let G be a graph. Attach a pendant vertex to each vertex of G. The resulting graph, denoted by $G \circ K_1$, is called the corona of G with K_1.*

We first prove the following auxiliary theorem.

Theorem 1. *Let M be a real symmetric irreducible square matrix of order p in which each row sum is equal to a constant k. Then there exists a polynomial $Q(x)$ such that $Q(M) = J$, where J is the all one square matrix whose order is same as that of M.*

Proof. Since M is a real symmetric irreducible matrix in which each row sums to k, by the Frobenius theorem [4], k is a simple and greatest eigenvalue of M. The matrix M is diagonalizable because it is real and symmetric. Therefore there exists an orthonormal basis of characteristic vectors of M, associated with the eigenvalues of M.

Let $\lambda_1 = k, \lambda_2, \dots, \lambda_g$ be the distinct eigenvalues of M. Let $\Im(\lambda_i)$ be the eigenspace spanned by the orthonormal set of characteristic vectors $\{x_1^i, x_2^i, \dots, x_{p_i}^i\}$ associated with λ_i, $i = 1, 2, \dots, g$. Then M has a spectral decomposition

$$M = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_g T_g$$

where T_i is the projection of M onto $\Im(\lambda_i)$, treating M as a linear operator. Then $T_i^2 = T_i$, $T_i T_j = 0$, $i \neq j$ and

$$T_i = x_1^i \left(x_1^i\right)^T + x_2^i \left(x_2^i\right)^T + \cdots + x_{p_i}^i \left(x_{p_i}^i\right)^T .$$

Now, corresponding to the greatest eigenvalue k of M, there exists a unique (one-dimensional) orthonormal basis

$$x_1 = \begin{bmatrix} 1/\sqrt{p} \\ 1/\sqrt{p} \\ \vdots \\ 1/\sqrt{p} \end{bmatrix}$$

for $\Im(\lambda_1) = \Im(k)$, such that $M = k\,T_1 + \lambda_2\,T_2 + \cdots + \lambda_g\,T_g$ where

$$\begin{aligned} T_1 &= \begin{bmatrix} 1/\sqrt{p} \\ 1/\sqrt{p} \\ \vdots \\ 1/\sqrt{p} \end{bmatrix} \begin{bmatrix} 1/\sqrt{p}, & 1/\sqrt{p}, & \cdots, & 1/\sqrt{p} \end{bmatrix} \\ &= \begin{bmatrix} 1/p & 1/p & \cdots & 1/p \\ 1/p & 1/p & \cdots & 1/p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1/p & 1/p & \cdots & 1/p \end{bmatrix} = \frac{1}{p}\,J\,. \end{aligned}$$

Because the T_i's are projections, we have $f(M) = f(k)\,T_1 + f(\lambda_2)\,T_2 + \cdots + f(\lambda_g)\,T_g$ for any polynomial $f(x)$. As M is diagonalizable, the minimal polynomial of M is $(x-k)(x-\lambda_2)\cdots(x-\lambda_g)$.

Let $S(x) = (x-\lambda_2)\cdots(x-\lambda_g)$. Then $S(\lambda_i) = 0$, $\lambda_i \neq k$. Thus $S(M) = S(k)\,T_1 S(k)\,(1/p)\,J$. Choose $Q(x) = p\,S(x)/S(k)$. This $Q(x)$ satisfies the requirement of the theorem. □

Theorem 2. *Let D be the distance matrix of a connected distance regular graph G. Then D is irreducible and there exists a polynomial $P(x)$ such that $P(D) = J$. In this case*

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_g)}{(k-\lambda_2)(k-\lambda_3)\ldots(k-\lambda_g)}$$

where k is the unique sum of each row which is also the greatest simple eigenvalue of D, whereas $\lambda_2, \lambda_3, \ldots, \lambda_g$ are the other distinct eigenvalues of D.

Proof. The theorem follows from Theorem 1 due to the observation that the distance matrix of a connected distance regular graph is irreducible, symmetric and each row sums to a constant. □

The rest of this paper is organized as follows. In the next section we obtain the distance spectra of $D_2(G)$, $G \times K_2$, $G[K_2]$, the lexicographic product of G with K_2, and $G \circ K_1$. Using this, the distance energies of $D_2(C_{2n})$, $C_n \times K_2$, $C_{2n}[K_2]$, and $C_n \circ K_1$ are calculated. In the third section the D-spectrum of the extended double cover graphs of regular graphs of diameter 2 is discussed and a pair of D-equienergetic bipartite graphs on $24t$, $t \geq 3$ vertices is constructed.

For operations on graphs that are not defined in this paper see [4].

2. Distance spectra of some graphs

In this section we obtain the distance spectra of the double graph of C_n, the Cartesian product of C_n with K_2 and the corona of C_n with K_1.

2.1. The double graph of G

Theorem 3. *Let G be a graph with distance spectrum $spec_D(G) = \{\mu_1, \mu_2, \ldots, \mu_p\}$. Then*

$$spec_D(D_2G) = \begin{pmatrix} 2(\mu_i + 1) & -2 \\ 1 & p \end{pmatrix}, \; i = 1, 2, \ldots, p.$$

Proof. By definition of $D_2(G)$ we have:

$$\begin{aligned} d_{D_2G}(v_i, v_j) &= d_G(v_i, v_j) \\ d_{D_2G}(v_i, u_i) &= 2 \\ d_{D_2G}(v_i, u_j) &= d_G(v_i, v_j) \\ d_{D_2G}(v_j, u_i) &= d_G(v_j, v_i). \end{aligned}$$

Hence a suitable ordering of vertices yields the distance matrix of D_2G of the form

$$\begin{bmatrix} D & D + 2I \\ D + 2I & D \end{bmatrix}$$

and the theorem follows from Lemma 2. □

Theorem 4. $E_D(D_2C_{2n}) = 4n(n + 1)$.

Proof. By Lemma 6 and Theorem 3 we have

$$spec_D(D_2C_{2n}) = \begin{pmatrix} 2\left(n^2 + 1\right) & 2 & -2\cot^2(\pi j/2n) & -2 \\ 1 & n - 1 & 1 & 2n \end{pmatrix}, \; j = 1, 3, 5, \ldots, 2n{-}1.$$

Thus $E_D(D_2C_{2n}) = 2 \times [2(n^2 + 1) + 2(n - 1)]4n(n + 1)$. □

2.2. The Cartesian product $G \times K_2$

Theorem 5. *Let G be a distance regular graph with distance regularity k, distance matrix D, and D-spectrum $\{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then*

$$spec_D(G \times K_2) = \begin{pmatrix} 2k + p & -p & 2\mu_i & 0 \\ 1 & 1 & 1 & p - 1 \end{pmatrix}, \; i = 2, 3, \ldots, p.$$

Proof. The theorem follows from the fact that the distance matrix of $G \times K_2$ has the form

$$\begin{bmatrix} D & D + J \\ D + J & D \end{bmatrix}$$

and from Theorem 1 and Lemma 2. □

Corollary 1. $E_D(G \times K_2) = 2(E_D(G) + p)$.

2.3. The corona of G and K_1

Theorem 6. *Let G be a connected distance regular graph with distance regularity k , distance matrix D , and $spec_D(G) = \{\mu_1 = k, \mu_2, \dots, \mu_p\}$. Then $spec_D(G \circ K_1)$ consists of the numbers*

$$p+k-1+\sqrt{(p+k)^2+(p-1)^2} \quad , \quad p+k-1-\sqrt{(p+k)^2+(p-1)^2}$$

$$\mu_i - 1 + \sqrt{\mu_i^2+1} \ , \ \ \mu_i - 1 - \sqrt{\mu_i^2+1} \quad , \quad i = 2,3,\dots,p \ .$$

Proof. From the definition of $G \circ K_1$, it follows that the distance matrix H of $G \circ K_1$ is of the form

$$\begin{bmatrix} D & D+J \\ D+J & D+2(J-I) \end{bmatrix} .$$

Now the characteristic equation of H is

$$\begin{aligned} |\lambda I - H| &= 0 \Rightarrow \begin{vmatrix} \lambda I - D & -(D+J) \\ -(D+J) & \lambda I - D - 2(J-I) \end{vmatrix} = 0 \\ &\Rightarrow \left| (\lambda I - D)(\lambda I - D - 2(J-I)) - (D+J)^2 \right| = 0 \text{ by Lemma 3} \end{aligned}$$

Now D being the distance matrix of a distance regular graph, it satisfies the requirement in Theorem 2. Then the $D-$ spectrum of $G \circ K_1$ follows from Theorem 2 and Lemma 1. □

Corollary 2.

$$\begin{aligned} E_D(C_{2n} \circ K_1) &= 2\left[(n-1)^2 + \sqrt{(n-1)^4 + 6\,n^2}\right] \\ E_D(C_{2n+1} \circ K_1) &= 2\left[n^2 + 3n + \sqrt{(n^2+3n)^2 + 6\,n^2 + 6n + 1}\right] . \end{aligned}$$

2.4. The lexicographic product of G with K_2

Theorem 7. *Let G be a connected graph with distance spectrum $spec_D(G)\{\mu_1 = k, \mu_2, \dots, \mu_p\}$. Then*

$$spec_D(G[K_2]) = \begin{pmatrix} 2\mu_i + 1 & -1 \\ 1 & p \end{pmatrix} \ , \ i = 1,2,\dots,p \ .$$

Proof. From the definition of the lexicographic product of G with K_2 , its distance matrix can be written as

$$\begin{bmatrix} D & D+I \\ D+I & D \end{bmatrix}$$

and the theorem follows from Lemma 2. □

Corollary 3. $E_D(C_{2n}[K_2]) = 2n(2n+1)$.

Proof. From Lemma 6 and Theorem 7 we have

$$spec_D(C_{2n}[K_2]) = \begin{pmatrix} 2\,n^2+1 & 1 & -1 & 1-2\,cosec^2(\pi j/2n) \\ 1 & n-1 & 2n & 1 \end{pmatrix} \, , \; j = 1,3,5,\dots .$$

Since $1-2\,cosec^2\theta = -(cot^2\theta + coesc^2\theta)$, the only positive eigenvalues are $2\,n^2+1$ and 1 with multiplicities 1 and $n-1$, respectively. Thus $E_D(C_{2n}[K_2]) = 2n(2n+1)$. □

3. The extended double cover graph of regular graphs of diameter 2

In [1] N. Alon introduced the concept of extended double cover graph of a graph as follows.

Let G be a graph on the vertex set $\{v_1, v_2, \dots , v_p\}$. Define a bipartite graph H with $V(H) = \{v_1, v_2, \dots , v_p, u_1, u_2, \dots , u_p\}$ in which v_i is adjacent to u_i for each $i = 1, 2, \dots , p$ and v_i is adjacent to u_j if v_i is adjacent to v_j in G . The graph H is known as the extended double cover graph (EDC-graph) of G . The ordinary spectrum of H has been determined in [3].

In this section we obtain the distance spectrum of the $EDC-$graph of a regular graph of diameter 2 and use it to construct regular D-equienergetic bipartite graphs on $24\,t$ vertices, for $t \geq 3$.

Theorem 8. *Let G be an r-regular graph of diameter 2 on p vertices with (ordinary) spectrum $\{r, \lambda_2, \dots , \lambda_p\}$. Then the D-spectrum of the EDC-graph of G consists of the numbers $5p-2r-4$, $2r-p$, $-2\,(\lambda_i+2)$, $i = 2, 3, \dots , p$, and $2\,\lambda_i$, $i = 2, 3, \dots , p$.*

Proof. Let A and $\overline{A}$ be, respectively, the adjacency matrices of G and $\overline{G}$. Then by the definition of the EDC-graph, its distance matrix can be written as

$$\begin{bmatrix} 2\,(J-I) & A+3\overline{A}+I \\ A+3\overline{A}+I & 2\,(J-I) \end{bmatrix}$$

and the theorem follows from Lemmas 1 and 3 and also from the observation that $\overline{A} = J-I-A$. □

Corollary 4.

$$E_D\left(EDC\left(C_p \nabla C_p\right)\right) = \begin{cases} 40 \ , p=3 \\ 4\left[E\left(C_p\right) + 5p - 10\right] \ , p \geqslant 4 \end{cases}$$

where $C_p \nabla C_p$ is the join [4] of C_p with itself.

Proof. The join of C_p with itself is a regular graph diameter 2 with the ordinary spectrum

$$\begin{pmatrix} p+2 & 2-p & \lambda_i \\ 1 & 1 & 2 \end{pmatrix} \, , \; i = 2, 3, \dots , p$$

where $\{2, \lambda_2, \dots, \lambda_p\}$ is the ordinary spectrum of C_p. Then by the above theorem, the distance spectrum of $EDC(C_p \nabla C_p)$ is

$$\begin{pmatrix} 8p-8 & 4 & -2(\lambda_i+2) & 2p-8 & 4-2p & 2\lambda_i \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \; i = 2, 3, \dots, p$$

and hence the corollary follows as $E(C_3) = 4$. □

3.1. On a pair of D-equienergetic bipartite graphs

Theorem 9. *There exists a pair of regular non-D-cospectral D-equienergetic bipartite graphs on $24t$ vertices, for each $t \geq 3$.*

Proof. Let G be a cubic graph on $2t$ vertices, $t \geq 3$. Consider $L^2(G)$, its second iterated line graph. Then by Lemma 4 and Theorem 8, we calculate that for $F = L^2(G) \nabla L^2(G)$, the D-spectrum of $EDC(F)$ is

$$\begin{pmatrix} 16(3t-1) & 12 & 0 & 2(\lambda_i+3) & 12t-16 & -4 & -12(t-1) & -2(\lambda_i+5) \\ 1 & 1 & 8t & 2 & 1 & 8t & 1 & 2 \end{pmatrix},$$

$i = 2, 3, \dots, 2t$. Thus

$$\begin{aligned} E_D(EDC(F)) &= 2 \times \left[12(t-1) + 32t + 4 \sum_{i=2}^{2t} (\lambda_i + 5) \right] \\ &= 2 \times [12t - 12 + 32t + 4(-3 + 5(2t-1))] \\ &= 8(21t - 11) . \end{aligned}$$

Now let G_1 and G_2 be the two non-cospectral cubic graphs on $2t$ vertices as given by Lemma 5. Further, let H_1 and H_2 be the EDC-graphs of $L^2(G_1) \nabla L^2(G_1)$ and $L^2(G_2) \nabla L^2(G_2)$, respectively. Then H_1 and H_2 are bipartite and $E_D(H_1) = E_D(H_2) = 8(21t - 11)$, proving the theorem. □

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