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# Λ − φ generalized synchronization: application to fractional hyperchaotic systems with arbitrary dimensions and orders

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**ABSTRACT**

This paper investigates the $\Lambda - \phi$ generalized synchronization between non-identical fractional-order systems characterized by different dimensions and different orders. The $\Lambda - \phi$ generalized synchronization combines the inverse matrix projective synchronization with the generalized synchronization. In particular, the proposed approach enables $\Lambda - \phi$ generalized synchronization to be achieved between $n$-dimensional master system and $m$-dimensional slave system in different dimensions. The technique, which exploits nonlinear controllers, stability property of integer-order linear systems and Lyapunov stability theory, proves to be effective in achieving the $\Lambda - \phi$ generalized synchronization. Finally, the approach is applied between 4-D and 5-D fractional hyperchaotic systems with the aim to illustrate the capabilities of the novel scheme proposed herein.

## 1. Introduction

Over the last years, great efforts have been devoted to the study of synchronization in chaotic systems [1,2]. Chaotic systems are, nonlinear dynamical systems, characterized by trajectories that separate exponentially in the course of the time, even when they start from two nearby initial states (i.e, sensitivity to initial conditions). Given two systems in the master-slave configuration, the objective in synchronization is to make the slave system variables synchronized in time with the corresponding master system variables. Year after year, different types of synchronization have been proposed in the literature, for continuous-time systems as well as discrete-time systems [3–6]. Recently, the topic of synchronization between different dimensional chaotic and hyperchaotic systems attract more and more attentions. Until now, many effective control schemes have been introduced to achieve chaos synchronization between dynamical systems with different dimensions such as full state hybrid projective synchronization [7], inverse matrix projective synchronization [8], generalized synchronization [9], inverse generalized synchronization [10], hybrid synchronization [11], $\Phi - \Theta$ synchronization [12], Q-S synchronization [13], reduced order synchronization [14] and increased order generalized synchronization [15]. Recently, a new type of synchronization, called $\Lambda - \phi$ generalized synchronization, has been proposed to synchronize chaotic and hyperchaotic systems with different dimensions.

This new type is constructed by combining the inverse matrix projective synchronization (based on a matrix) with the generalized synchronization (based on a functional relationship). The $\Lambda - \phi$ generalized synchronization was applied with successfully in discrete-time and continuous-time systems [16,17].

Referring to fractional-order systems, researches have shown that fractional-order differential systems, as generalizations of well-known integer-order differential systems, are characterized by chaotic (hyperchaotic) dynamics [18]. Specifically, researches have shown that chaos is achievable when the system order is less than 3, whereas hyperchaos can be obtained when the system order is less than 4 [19]. Recently, research about fractional-order hyperchaotic systems gains a lot of interest from both theoretical and applied point of view include colour image encryption algorithm and applications of different types of synchronization [20,21]. However, few types of generalized synchronization have been proposed for fractional chaotic and hyperchaotic systems compared to integer-order ones. Moreover, most of the approaches are related to the generalized chaos synchronization of fractional-order systems with identical dimensions [22,23]. Very few methods for synchronizing different dimensional fractional chaotic and hyperchaotic systems have been illustrated [24–29].

In this work, a further contribution to the topic of chaos synchronization of fractional-order systems with

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different dimensions is provided. Namely, the paper investigates the $\Lambda - \phi$ generalized synchronization, with index $d$, of non-identical fractional-order systems characterized by different dimensions and different orders. Note that the synchronization index $d$ corresponds to the dimension of the synchronization error. Specifically, by exploiting fractional Laplace transform and classical Lyapunov stability theory, the $\Lambda - \phi$ generalized synchronization between two fractional-order systems for the case $d = m$ is proved, showing that the zero solution of the error system is globally asymptotically stable. Additionally, by using the stability theory of linear integer-order systems, the $\Lambda - \phi$ generalized synchronization for the case $d < m$ is demonstrated. The approach presents the remarkable feature of being both rigorous and applicable to a wide class of commensurate and incommensurate fractional-order systems with different dimension and different orders. The conceived scheme is general and the only restriction on the scaling functions is that they must be differentiable functions.

The paper is organized as follows. In Section 2, some basic notions on fractional calculus are given. In Section 3, the $\Lambda - \phi$ generalized synchronization with index $d$ is defined. In Section 4, two different theorems are provided, which cover different synchronization cases, with indices $d = m$, and $d < m$, respectively. Finally, in order to show the capabilities of the conceived synchronization schemes, Section 5 illustrated the $\Lambda - \phi$ generalized synchronization between two new 4-D and 5-D fractional hyperchaotic systems, when the synchronization indices are $d = 5$, $d = 4$, $d = 3$ and $d = 2$. Concluding remarks are given in Section 6.

## 2. Preliminaries

**Definition 2.1 ([30]):** The Riemann–Liouville fractional integral operator of order $p > 0$ of the function $f(t)$ is defined as,

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} f(\tau) \, d\tau, \quad t > 0. \quad (1)$$

**Definition 2.2 ([31]):** The Caputo fractional derivative of $f(t)$ is defined as,

$$D_t^p f(t) = J^{m-p} \left( \frac{d^m}{dt^m} f(t) \right)$$

$$= \frac{1}{\Gamma(m-p)} \int_0^t \frac{f^{(m)}(\tau)}{(t - \tau)^{p-m+1}} \, d\tau, \quad (2)$$

for $m - 1 < p \leq m$, $m \in N$, $t > 0$.

**Lemma 2.1 ([32]):** *The Laplace transform of the Caputo fractional derivative rule reads*

$$\mathbf{L}\left\{ D_t^p f(t) \right\} = s^p \mathbf{F}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

$$(p > 0, n - 1 < p \leq n). \quad (3)$$

*Particularly, when $p \in (0, 1]$, we have $\mathbf{L}\{D_t^p f(t)\} = s^p \mathbf{F}(s) - s^{p-1} f(0)$.*

**Lemma 2.2 ([33]):** *The Laplace transform of the Riemann–Liouville fractional integral rule satisfies*

$$\mathbf{L}\left\{ J^q f(t) \right\} = s^{-q} \mathbf{F}(s), \quad (q > 0). \quad (4)$$

**Lemma 2.3 ([34]):** *Suppose $f(t)$ has a continuous $k$th derivative on $[0, t]$ ($k \in N$, $t > 0$), and let $p$, $q > 0$ be such that there exists some $\ell \in N$ with $\ell \leq k$ and $p$, $p + q \in [\ell - 1, \ell]$. Then*

$$D_t^p D_t^q f(t) = D_t^{p+q} f(t) \quad (5)$$

**Remark 2.1:** Note that the condition requiring the existence of the number $\ell$ with the above restrictions in the property is essential. In this study, we consider the case that $p, q \in ]0, 1]$ and $p + q \in ]0, 1]$. Apparently, under such conditions this property holds.

## 3. Problem statement

The master and the slave systems are in the following forms

$$D_t^p X(t) = F(X(t)), \quad (6)$$

$$D_t^q Y(t) = G(Y(t)) + U, \quad (7)$$

where $X(t) \in \mathbf{R}^n$, $Y(t) \in \mathbf{R}^m$ are states of the master system (6) and the slave system (7), respectively, $0 < p$, $q < 1$, $D_t^p$, $D_t^q$ are the Caputo fractional derivatives of orders $p$ and $q$, respectively, $F : \mathbf{R}^n \to \mathbf{R}^n$, $G : \mathbf{R}^m \to \mathbf{R}^m$ and $U = (u_i)_{1 \leq i \leq m}$ is a vector controller.

**Definition 3.1:** The master system (6) and the slave system (7) are said to be $\Lambda - \phi$ generalized synchronized in dimension $d$, if there exists a controller $U = (u_i)_{1 \leq i \leq m}$, a function matrix $\Lambda(t) = (\Lambda_{ij}(t))_{d \times m}$ and differentiable function $\phi : \mathbf{R}^n \to \mathbf{R}^d$ such that the synchronization error

$$e(t) = \Lambda(t) Y(t) - \phi(X(t)), \quad (8)$$

converge to zero asymptotically, i.e, $\lim_{t \to +\infty} \|e(t)\| = 0$.

**Remark 3.1:** When $(\Lambda, \phi(.)) = (I, X(t))$, $(\Lambda, \phi(.)) = (I, -X(t))$, $(\Lambda, \phi(.)) = (\Lambda(t), X(t))$ and $(\Lambda, \phi(.)) = (I, \phi(X(t)))$ complete synchronization, anti- synchronization, inverse matrix projective synchronization and generalized synchronization will appear, respectively.

## 4. Different schemes of synchronization

In this section, we discuss two schemes of $\Lambda - \phi$ generalized synchronization.

### 4.1. Master and salve systems description

Consider the following master system

$$D_t^p X(t) = f(X(t)), \qquad (9)$$

where $X(t) \in \mathbf{R}^n$ is the state vector of the master system, $0 < p < 1$, $D_t^p$ is the Caputo fractional derivative of order $p$ and $f : \mathbf{R}^n \to \mathbf{R}^n$.

As the slave system, we consider the following controlled system

$$D_t^q Y(t) = BY(t) + g(Y(t)) + U, \qquad (10)$$

where $Y(t) \in \mathbf{R}^m$ is the state vector of the slave system, $0 < q < 1$, $D_t^q$ is the Caputo fractional derivative of order $q$, $B \in \mathbf{R}^{m \times m}$, and $g : \mathbf{R}^m \to \mathbf{R}^m$ are the linear part and the nonlinear part of the slave system, respectively, and $U = (u_i)_{1 \leq i \leq m}$ is a vector controller.

### 4.2. $\Lambda - \phi$ generalized synchronization in dimension m

In this case, the error system between the master system (9) and the slave system (10), is defined as

$$e(t) = \Lambda(t) Y(t) - \phi(X(t)), \qquad (11)$$

where $\Lambda(t) = (\Lambda_{ij}(t))_{m \times m}$ and $\phi : \mathbf{R}^n \to \mathbf{R}^m$.

The error system (11) can be derived as

$$\dot{e}(t) = \dot{\Lambda}(t) Y(t) + \Lambda(t) \dot{Y}(t) - \mathbf{D}\phi(X(t)) \dot{X}(t), \qquad (12)$$

where $\mathbf{D}\phi(X(t)) = (\partial \phi_i / \partial x_j)_{m \times n}$.

**Remark 4.1:** We can not use the fractional derivative operator to Equation (11), because the fractional derivative of the product of two functions implies having an infinite sum, which includes fractional-order and integer-order derivatives of the functions [35].

The error system (12) can be written as

$$\dot{e}(t) = (B - C) e(t) + \Lambda(t) \dot{Y}(t) + R, \qquad (13)$$

where $C \in \mathbf{R}^{m \times m}$ is a control constant matrix to be selected later and

$$R = (C - B) e(t) + \dot{\Lambda}(t) Y(t) - \mathbf{D}\phi(X(t)) \dot{X}(t). \qquad (14)$$

Hence, we have the following result.

**Theorem 4.1:** *The master system* (9) *and the slave system* (10) *are globally $\Lambda - \phi$ synchronized in dimension m, if the following conditions are satisfied:*

(i)   $\Lambda(t)$ *is an invertible matrix and $\Lambda^{-1}(t)$ its inverse matrix.*

(ii)   $U = -BY(t) - g(Y(t)) + J^{1-q}(-\Lambda^{-1}(t) \times R)$.

(iii)   $(C - B)^T + (C - B)$ *is a positive definite matrix.*

**Proof:** Substituting the control law (ii) into Equation (10), the slave system can be described as

$$D_t^q Y(t) = J^{1-q} \left( -\Lambda^{-1}(t) \times R \right). \qquad (15)$$

Applying the Laplace transform to (15) and letting $\mathbf{F}(s) = \mathbf{L}(Y(t))$, we obtain,

$$s^q \mathbf{F}(s) - s^{q-1} Y(0) = s^{q-1} \mathbf{L} \left( -\Lambda^{-1}(t) \times R \right), \qquad (16)$$

multiplying both the left-hand and right-hand sides of (16) by $s^{1-q}$ and applying the inverse Laplace transform to the result, we get the following equation

$$\dot{Y}(t) = -\Lambda^{-1}(t) \times R. \qquad (17)$$

Now, the error system (13) can be described as follows

$$\dot{e}(t) = (B - C) e(t). \qquad (18)$$

Construct the candidate Lyapunov function in the form $V(e(t)) = e^T(t) e(t)$, we obtain,

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t) e(t) + e^T(t) \dot{e}(t) \\ &= e^T(t) (B - C)^T e(t) + e^T(t) (B - C) e(t) \\ &= -e^T(t) \left[ (C - B)^T + (C - B) \right] e(t). \end{aligned}$$

By using (iii), we get $\dot{V}(e(t)) < 0$. Thus, from the Lyapunov stability theory, it is immediate that $lim_{t \to +\infty} \| e(t) \| = 0$. So, the zero solution of the error system (18) is globally asymptotically stable, and therefore, the master system (9) and the slave system (10) are globally synchronized in dimension $m$. ∎

### 4.3. $\Lambda - \phi$ generalized synchronization in dimension d

In this case, we assume that the synchronization dimension $d < m$. The error system between the master system (9) and the slave system (10) is considered as

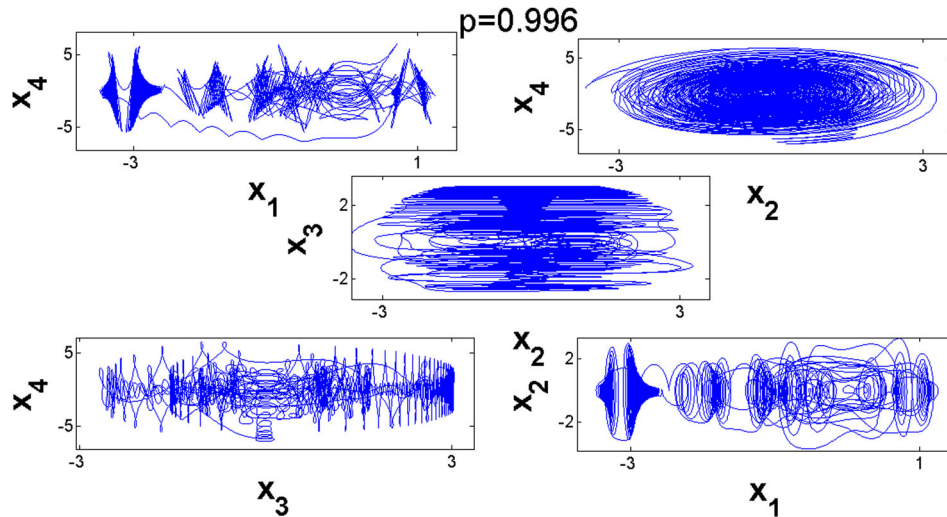$$e(t) = \Lambda(t) Y(t) - \phi(X(t)), \qquad (19)$$

where $\Lambda(t) = (\Lambda_{ij}(t))_{d \times m}$ and $\phi : \mathbf{R}^n \to \mathbf{R}^d$. The error system (19) can be described as

$$\begin{aligned} \dot{e}(t) = &-\text{diag}(l_1, l_2, \ldots, l_d) e(t) + \Lambda_1(t) \dot{Y}_1(t) \\ &+ \Lambda_2(t) \dot{Y}_2(t) + T, \end{aligned} \qquad (20)$$

where $(l_i)_{1 \leq i \leq d}$ are positive control constants, $\dot{Y}_1(t) = (\dot{y}_1(t), \ldots, \dot{y}_d(t))^T$, $\dot{Y}_2(t) = (\dot{y}_{d+1}(t), \ldots, \dot{y}_m(t))^T$, $\Lambda_1(t) = (\Lambda_{ij}(t))_{d \times d}$, $\Lambda_2(t) = (\Lambda_{ij}(t))_{d \times (m-d)}$ and

$$\begin{aligned} T = &\text{diag}(l_1, l_2, \ldots, l_3) e(t) + \dot{\Lambda}(t) Y(t) \\ &- \mathbf{D}\phi(X(t)) \dot{X}(t), \end{aligned} \qquad (21)$$

where, in this case, $\mathbf{D}\phi(X(t)) = (\partial \phi_i / \partial x_j)_{d \times n}$.

**Figure 1.** Attractors of the fractional system (29) with $p = 0.996$ and initial conditions $(0.1, 0.1, 0.1, 0.1)$ in different 2D projections.

**Theorem 4.2:** *The master system* (9) *and the slave system* (10) *are globally* $\Lambda - \phi$ *synchronized in dimension* $d$ *under the following control law*

$$(u_1, u_2, \ldots, u_d)^T = -B_1 - G_1 - J^{1-q}\left(\Lambda_1^{-1}(t) \times T\right),$$
$$\tag{22}$$

*and*

$$\left(u_{d+1}, u_{d+2}, \ldots, u_m\right)^T = -B_2 - G_2, \tag{23}$$

*where* $B_1 = (b_{ij})_{d \times m}$, $B_2 = (b_{ij})_{(m-d) \times m}$, $G_1 = (g_i)_{1 \leq i \leq d}$ *and* $G_2 = (g_i)_{d+1 \leq i \leq m}$.

***Proof:*** By inserting the control law (22)–(23) into Equation (10), we can rewrite the slave system as follows

$$\left(D_t^q y_1(t), \ldots, D_t^q y_d(t)\right)^T = J^{1-q}\left(-\Lambda_1^{-1}(t) \times T\right),$$
$$\tag{24}$$

and

$$D_t^q y_i(t) = 0, \quad i = d+1, \ldots, m. \tag{25}$$

By applying the fractional derivative of order $1-q$ to both the left and right sides of Equations (24) and (25), we obtain

$$\dot{Y}_1(t) = (\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_d)^T$$
$$= D_t^{1-q}\left(\left(D_t^q y_1(t), \ldots, D_t^q y_d(t)\right)^T\right)$$
$$= D_t^{1-q} J^{1-q}\left(-\Lambda_1^{-1}(t) \times T\right)$$
$$= -\Lambda_1^{-1}(t) \times T, \tag{26}$$

and also we get

$$\dot{y}_i(t) = 0, \quad i = d+1, \ldots, m. \tag{27}$$

Now, by using Equations (26) and (27), the error system (20) can be written as

$$\dot{e}_i(t) = -l_i e_i(t), \quad 1 \leq i \leq d, \tag{28}$$

it is immediate that all solutions of error system (28) go to zero as $t \to +\infty$. Therefore, the master system (9)

and the slave system (10) are globally $\Lambda - \phi$ generalized synchronized in dimension $d$. ∎

## 5. Application to new 4-D and 5-D fractional hyperchaotic systems

In this section, we will present some numerical simulations for $\Lambda - \phi$ generalized synchronization to verify and illustrate the effectiveness of the theoretical analysis in Section 4. As the master system we consider the following 4-D fractional hyperchaotic system

$$D^p x_1 = x_2,$$
$$D^p x_2 = -x_1 + x_2 x_3 + \alpha x_1 x_3 x_4,$$
$$D^p x_3 = 1 - x_2^2, \tag{29}$$
$$D^p x_4 = x_3 + \beta x_1 x_3 + \gamma x_1 x_2 x_3,$$

where $x_1, x_2, x_3$ and $x_4$ are states. This system, as shown in [36], exhibits hyperchaotic behaviours when $(\alpha, \beta, \gamma) = (8, -2.5, -30)$ and $p = 0.996$. Using Grunwald–Letnikov approximation method, attractors in 2-D and 3-D, of the master system (29) are shown in Figures 1 and 2. In addition, Lyapunov exponents (LE) are plotted in Figure 3.

The slave system is defined as

$$D^q y_1 = -a y_1 + y_2 y_3 + u_1,$$
$$D^q y_2 = -b y_2 + y_5 + u_2,$$
$$D^q y_3 = -c y_3 + g y_4 + y_1 y_2 + u_3, \tag{30}$$
$$D^q y_4 = d y_4 - h y_3 + u_4,$$
$$D^q y_5 = e y_5 - y_2 y_1^2 + u_5,$$

where $y_i$ and $u_i, (i = 1, 2, 3, 4, 5)$ are states and controllers, respectively. This system, as shown in [37], exhibits hyperchaotic behaviour when $q = 0.95$, $(a, b, c, d, e, g, h) = (10, 60, 20, 15, 40, 50, 10)$ and $(u_1, u_2, u_3,$
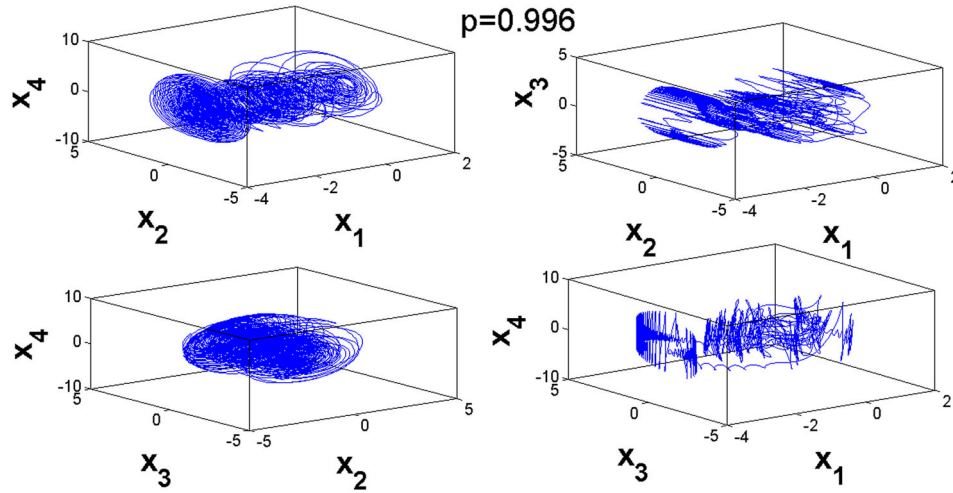
**Figure 2.** Attractors of the fractional system (29) with $p = 0.996$ and initial conditions $(0.1, 0.1, 0.1, 0.1)$ in different 3D projections.
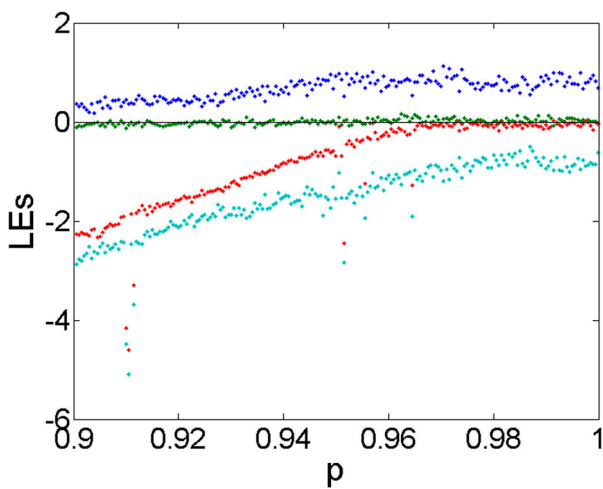


**Figure 3.** Lyapunov exponents of fractional system (29) as a function of $p \in [0.9, 1]$.

$u_4, u_5) = (0, 0, 0, 0, 0)$. Compare system (30) with system (10), one can have

$$B = \begin{pmatrix} -10 & 0 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 & 1 \\ 0 & 0 & -20 & 50 & 0 \\ 0 & 0 & -10 & 15 & 0 \\ 0 & 0 & 0 & 0 & 40 \end{pmatrix},$$

$$g = \begin{pmatrix} y_2 y_3 \\ 0 \\ y_1 y_2 \\ 0 \\ -y_2 y_1^2 \end{pmatrix}.$$

Different views of attractor of the uncontrolled system (30) and Lyapunov exponents are shown in Figures 4–6, respectively.

According to $\Lambda - \phi$ generalized synchronization approach, the error system between the master system (29) and the slave system (30) can be described

by

$$e(t) = \Lambda(t) \times \left(y_1, y_2, y_3, y_4, y_5\right)^T - \phi(x_1, x_2, x_3, x_4). \tag{31}$$

By selecting the scaling matrix $\Lambda(t)$ and the scaling function $\phi$ in different dimensions, we get in the following different synchronization results.

### 5.1. $\Lambda - \phi$ generalized synchronization in 5-D

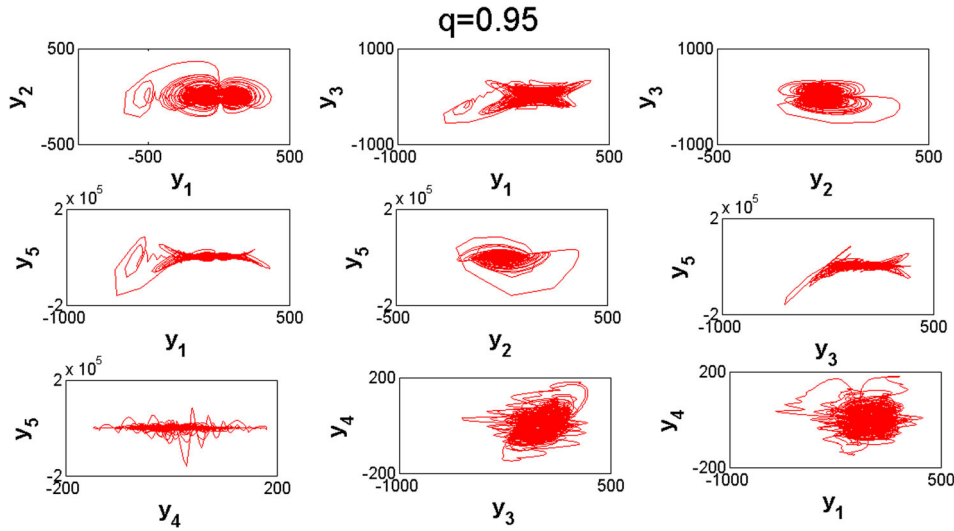In this case, the scaling matrix $\Lambda(t)$ and the scaling function $\phi$ are selected as

$$\Lambda(t) = \begin{pmatrix} t+1 & 0 & 0 & 0 & 0 \\ 0 & t^2+2 & 0 & 0 & 0 \\ 0 & 0 & \exp(t) & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \tag{32}$$
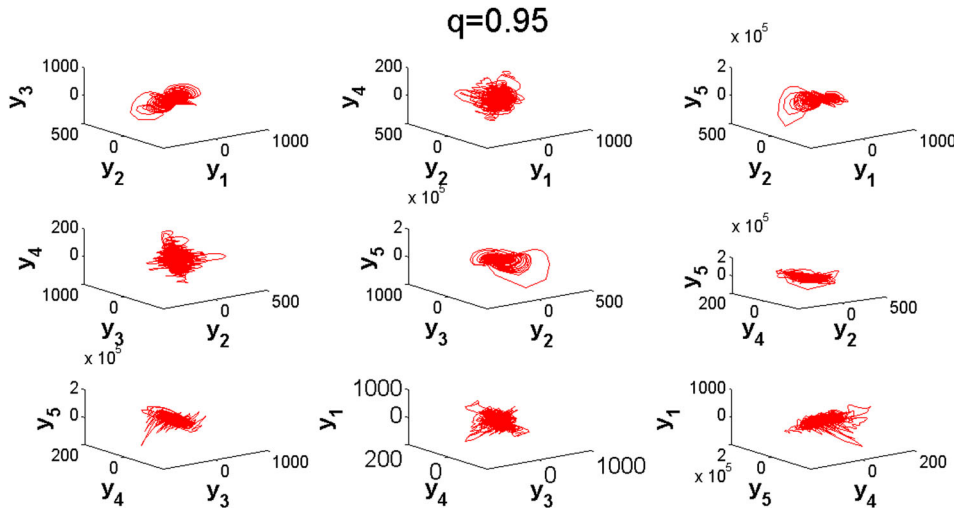
and

$$\phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_1 x_2 + x_3 x_4 \end{pmatrix}. \tag{33}$$

Using the notations presented in subsection 4.2, we get

$$\Lambda^{-1}(t) = \begin{pmatrix} \frac{1}{t+2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2+2} & 0 & 0 & 0 \\ 0 & 0 & \exp(-t) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \quad \text{and}$$

**Figure 4.** Phase portraits of the fractional system (30) with $u_1 = u_2 = u_3 = u_4 = u_5 = 0$, $q = 0.95$ and initial conditions $(1, 1, 1, 1, 1)$ in different 2D projections.



**Figure 5.** Phase portraits of the fractional system (30) with $u_1 = u_2 = u_3 = u_4 = u_5 = 0$, $q = 0.95$ and initial conditions $(1, 1, 1, 1, 1)$ in different 3D projections.

$$\mathbf{D}\phi\left(X(t)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_2 & x_1 & x_4 & x_3 \end{pmatrix},$$

and the control matrix $C$ can be chosen as

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & -10 & 16 & 0 \\ 0 & 0 & 0 & 0 & 41 \end{pmatrix}. \tag{34}$$
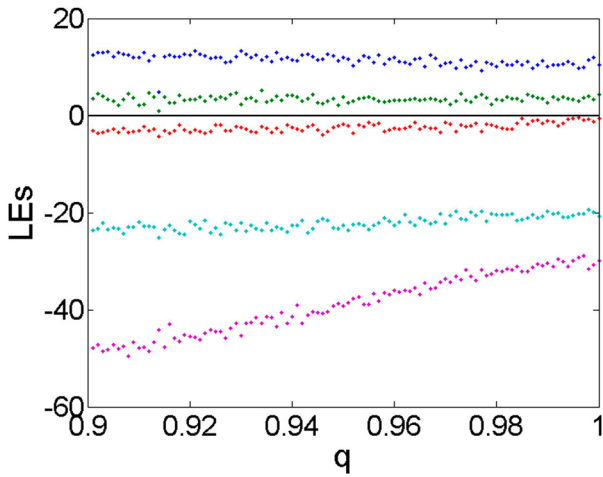
It is easy to show that $(C - B)^T + (C - B)$ is a positive definite matrix. Then, the controllers $u_1, u_2, u_3, u_4$ and $u_5$ are designed as follows
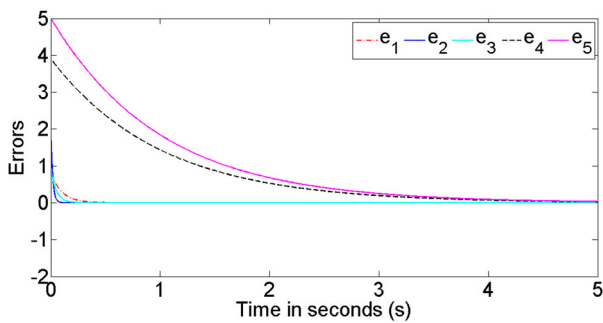
$$u_1 = -ay_1 + y_2y_3 - J^{0.05}\left[\frac{1}{t+1}\left(10e_1 + y_1 - \dot{x}_1\right)\right],$$

$$u_2 = -by_2 + y_5 - J^{0.05}\left[\frac{1}{t^2+2}\left(60e_2 + 2ty_2 - \dot{x}_2\right)\right],$$

$$u_3 = -cy_3 + gy_4 + y_1y_2$$
$$- J^{0.05}\left[\exp\left(-t\right)\left(20e_3 + \exp\left(t\right)y_3 - \dot{x}_3\right)\right],$$

$$u_4 = dy_4 - hy_3 - J^{0.05}\frac{1}{4}\left(e_4 - \dot{x}_4\right),$$

$$u_5 = ey_5 - y_2y_1^2$$
$$- J^{0.05}\frac{1}{5}\left(e_5 - x_2\dot{x}_1 - x_1\dot{x}_2 - x_4\dot{x}_3 - x_3\dot{x}_4\right). \tag{35}$$

The error system, in this case, can be described as follows

$$\dot{e}_1 = -10e_1,$$

**Figure 6.** Lyapunov exponents of fractional system (30) as a function of $q \in [0.94, 1]$.



**Figure 7.** Time-history of the synchronization errors $e_1, e_2, e_3, e_4$ and $e_5$.

$$\dot{e}_2 = -60e_2,$$
$$\dot{e}_3 = -20e_3, \qquad (36)$$
$$\dot{e}_4 = -e_4,$$
$$\dot{e}_5 = -e_5,$$

and the state variables of the errors $e_1, e_2, e_3, e_4$ and $e_5$ versus time are depicted in Figure 7.

### 5.2. $\Lambda - \phi$ generalized synchronization in 4-D

In this case, $\Lambda(t)$ and $\phi$ are chosen as

$$\Lambda(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & t+1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & t \\ 0 & 0 & 0 & \exp t & \cos t \end{pmatrix}, \qquad (37)$$

and

$$\phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 x_2 x_3 x_4 \end{pmatrix}. \qquad (38)$$

According to the notations described in subsection 4.3, the matrix $\Lambda_1(t)$ is given by

$$\Lambda_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \exp t \end{pmatrix}. \qquad (39)$$

So,

$$\Lambda_1^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \exp(-t) \end{pmatrix} \quad \text{and}$$

$$\mathbf{D}\phi(X(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_2 x_3 x_4 & x_1 x_3 x_4 & x_1 x_2 x_4 & x_1 x_2 x_3 \end{pmatrix}.$$

The control constants $(l_i)_{1 \le i \le 4}$ can be chosen as

$$(l_1, l_2, l_3, l_4) = (1, 2, 3, 4).$$

Based on Equations (22) and (23), the controllers $u_1, u_2, u_3, u_4$ and $u_5$ are constructed as follows

$$u_1 = ay_1 - y_2 y_3 - J^{0.05}(e_1 + y_5 - \dot{x}_1),$$

$$u_2 = by_2 - y_5 - J^{0.05}\left(e_2 - \frac{1}{2}\dot{x}_2\right),$$

$$u_3 = cy_3 - gy_4 - y_1 y_2 - J^{0.05}\left(e_3 + \frac{1}{3}y_5 - \frac{1}{3}\dot{x}_3\right),$$

$$\begin{aligned} u_4 = &-dy_4 + hy_3 - J^{0.05}\big[\exp(-t)\big(4e_4 + y_4 \exp t \\ &+ \cos t y_5 - \dot{x}_1 x_2 x_3 x_4 + \dot{x}_2 x_1 x_3 x_4 \\ &+ \dot{x}_3 x_1 x_2 x_4 + \dot{x}_4 x_1 x_2 x_3\big)\big], \end{aligned}$$

$$u_5 = -ey_5 + y_2 y_1^2. \qquad (40)$$

The error system, in this case, can be described as follows

$$\begin{aligned} \dot{e}_1 &= -e_1, \\ \dot{e}_2 &= -2e_2, \\ \dot{e}_3 &= -3e_3, \\ \dot{e}_4 &= -4e_4. \end{aligned} \qquad (41)$$
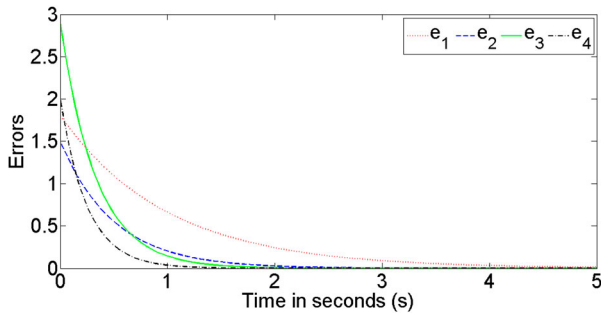
Figure 8 displays the time evolution of the errors $e_1, e_2, e_3$ and $e_4$.

### 5.3. $\Lambda - \phi$ generalized synchronization in 3-D

In this case, we take $\Lambda(t)$ and $\phi$ as follows

$$\Lambda(t) = \begin{pmatrix} 3 & 0 & 0 & 4 & t^2 \\ 0 & 1 & 0 & 3 & \sin t \\ 0 & 0 & t+1 & 5 & t \end{pmatrix}, \text{ and}$$

$$\phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 \\ x_2 \\ x_3^2 + x_4^2 \end{pmatrix}.$$

**Figure 8.** Time-history of the synchronization errors $e_1, e_2, e_3$ and $e_4$.



**Figure 9.** Time-history of the synchronization errors $e_1$, and $e_2$.

So,

$$\Lambda_1(t) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t+1 \end{pmatrix},$$

$$\Lambda_1^{-1}(t) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t+1} \end{pmatrix} \text{ and}$$

$$\mathbf{D}\phi(X(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2x_3 & 2x_4 \end{pmatrix}.$$

In this case, the control constants $(l_i)_{1 \le i \le 3}$ can be chosen as: $(l_1, l_2, l_3) = (0.1, 0.2, 0.3)$ and the control law is given by

$$u_1 = ay_1 - y_2y_3 - \frac{1}{3}J^{0.05}(0.1e_1 + y_52t - \dot{x}_1),$$

$$u_2 = by_2 - y_5 - J^{0.05}\left(e_2 - \frac{1}{2}\dot{x}_2\right),$$

$$u_3 = cy_3 - gy_4 - y_1y_2 - J^{0.05}\frac{1}{t+1} \qquad (42)$$
$$\times (0.3e_3 + y_3 + y_5 - 2x_3\dot{x}_3 - 2x_4\dot{x}_4),$$

$$u_4 = -dy_4 + hy_3.$$

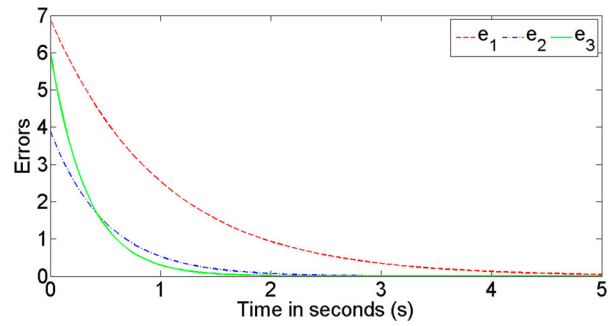$$u_5 = -ey_5 + y_2y_1^2.$$

The error system will be

$$\dot{e}_1 = -e_1,$$
$$\dot{e}_2 = -2e_2, \qquad (43)$$
$$\dot{e}_3 = -3e_3,$$

and the error function evolution, in this case, is shown in Figure 9.

### 5.4. $\Lambda - \phi$ generalized synchronization in 2-D

In this case, $\Lambda(t)$ and $\phi$ are given by

$$\Lambda(t) = \begin{pmatrix} 1 & 0 & \ln(t+1) & 4 & t \\ 0 & 2 & 3 & \frac{1}{t+1} & t^5 \end{pmatrix}, \text{ and}$$

$$\phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1x_2 \\ x_3 + x_4 \end{pmatrix}.$$

So,

$$\Lambda_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \Lambda_1^{-1}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and}$$

$$\mathbf{D}\phi(X(t)) = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Consequently, by taking $(l_1, l_2) = (\frac{1}{2}, 2)$, the controllers are obtained in the following form

$$u_1 = ay_1 - y_2y_3 - \frac{1}{3}J^{0.05}$$
$$\times \left(\frac{1}{2}e_1 + \frac{1}{t+1}y_3 + y_5 - x_2\dot{x}_1 - \dot{x}_2x_1\right),$$

$$u_2 = by_2 - y_5 - \frac{1}{2}J^{0.05} \qquad (44)$$
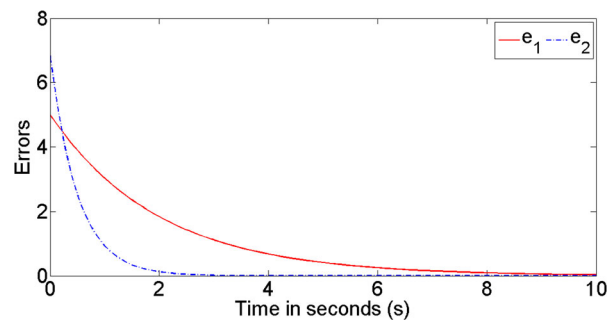$$\times (-2e_2 + 4t^3y_4 + 5t^4y_5 - \dot{x}_3 - \dot{x}_4),$$

$$u_3 = cy_3 - gy_4 - y_1y_2,$$

$$u_4 = -dy_4 + hy_3,$$

$$u_5 = -ey_5 + y_2y_1^2.$$

In this case, the error functions can be written as

$$\dot{e}_1 = -\frac{1}{2}e_1,$$
$$\dot{e}_2 = -2e_2, \qquad (45)$$

and the numerical results are plotted in Figure 10.



**Figure 10.** Time-history of the synchronization errors $e_1, e_2$ and $e_3$.

## 6. Concluding remarks

In this paper, the problem of $\Lambda - \phi$ generalized synchronization with index $d$ has been investigated between different dimensional fractional hyperchaotic systems. The novelty relies on the fact that the approach combines two different synchronization types, i.e. the inverse matrix projective synchronization (based on a matrix $\Lambda$) and the generalized synchronization (based on a functional relationship $\phi$). The technique has exploited nonlinear controllers and stability theory integer-order systems in order to synchronize $n$-dimensional master system and $m$-dimensional slave system. The approach has proved to be effective in achieving synchronized dynamics not only when the synchronization index $d$ equals $m$, but even if the synchronization index $d$ is less than $m$. This represents, in the authors' opinion, a remarkable finding of the present paper. Finally, the $\Lambda - \phi$ generalized synchronization was successfully applied between 4-D fractional hyperchaotic systems (as the master system) and 5-D (as the slave system), with the aim to highlight the capabilities of the new scheme conceived herein. The Grunwald–Letnikov approximation method has been used to solve the fractional-order hyperchaotic systems.

As a concluding remark, we would observe that the basic idea of the present paper, i.e. the combination of two different synchronization types in order to create a novel synchronization scheme, can be further generalized. Namely, by considering two different synchronization types as "building blocks", several new synchronization schemes can be obtained using the technique developed herein. Consequently, the approach illustrating herein can be considered as a "methodology" to create new synchronization schemes starting from two well-established synchronization types.

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