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EXTREMAL BEHAVIOUR OF ±1-VALUED COMPLETELY MULTIPLICATIVE FUNCTIONS IN FUNCTION FIELDS

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Abstract. We investigate the classical Pólya and Turán conjectures in the context of rational function fields over finite fields $\mathbb{F}_q$. Related to these two conjectures we investigate the sign of truncations of Dirichlet L-functions at point $s = 1$ corresponding to quadratic characters over $\mathbb{F}_q[t]$, prove a variant of a theorem of Landau for arbitrary sets of monic, irreducible polynomials over $\mathbb{F}_q[t]$ and calculate the mean value of certain variants of the Liouville function over $\mathbb{F}_q[t]$.

1. Introduction

Let $\lambda(n) = (-1)^{\Omega(n)}$ denote the Liouville function, where $\Omega(n)$ is the number of prime factors of integer n counted with multiplicity. Pólya ([7]) conjectured that $\sum_{n\leq x}\lambda(n) \leq 0$ for all $x \geq 2$ and showed that this conjecture implies the Riemann hypothesis. Similarly, Turán ([11]) showed that the Riemann hypothesis can be proven assuming a related conjecture that $\sum_{n\leq x}\frac{\lambda(n)}{n} \geq 0$ for all $x \geq 0$. Haselgrove ([3]) showed that both of these conjectures are false, despite the extensive numerical data suggesting their validity.

The fact that the Turán conjecture does not hold was used in [2] to exhibit negative values of truncations to the Dirichlet L-functions at $s = 1$ associated to quadratic characters. In particular, the authors prove following theorem.

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THEOREM 1.1 ([2]). *There exists a positive constant c such that for all large x*

$$-\frac{1}{(\log\log x)^{\frac{3}{5}}} \leq \min_{\substack{\chi \text{ a quadratic} \\ \text{character}}} \sum_{n \leq x} \frac{\chi(n)}{n} \leq -\frac{c}{\log x}.$$

Results from [2] can also be used in relation to partial sums of the Möbius function over multiplicative semigroups of the natural numbers $\mathbb{N} = \{1, 2, ...\}$ generated by arbitrary sets of primes. Let $\mathscr{P}$ be any set of primes (finite of infinite) and let $\langle \mathscr{P} \rangle$ denote the multiplicative semigroup generated by $\mathscr{P}$. That is the set of natural numbers, all of whose prime factors lie in $\mathscr{P}$. In [10] Tao proves that elementary bound

$$\left| \sum_{\substack{n \in \langle \mathscr{P} \rangle \\ n \leq x}} \frac{\mu(n)}{n} \right| \leq 1 \tag{1.1}$$

holds for all sets of primes $\mathscr{P}$ and $x \geq 0$. An important remark he makes is that the lower bound of -1 can be improved by Theorem 2 of [2] to $\left(1 - 2\log(1+\sqrt{e}) + 4\int_1^{\sqrt{e}} \frac{\log t}{t+1} dt\right) \log 2 + o(1) = -0.4553 \cdots + o(1)$, the value that is optimal except for the $o(1)$ term.

Another important result proved in [10] is the Landau's theorem for arbitrary set of primes (also see [9] for the proof of this theorem in the more general context of number fields).

THEOREM 1.2 ([10]). *Let $\mathscr{P}$ be any set of primes. Then*

$$\sum_{n \in \langle \mathscr{P} \rangle} \frac{\mu(n)}{n} = \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p}\right).$$

For a given set of primes $\mathscr{P}$ one can also consider the generalization $\lambda_{\mathscr{P}}$ of the Liouville function λ. It is defined by $\lambda_{\mathscr{P}}(n) = (-1)^{\Omega_{\mathscr{P}}(n)}$, where $\Omega_{\mathscr{P}}(n)$ is the number of prime factors of integer n coming from $\mathscr{P}$, counted with multiplicity. Various properties of this function have been studied in [1]. In particular, the authors calculate its mean value.

THEOREM 1.3 ([1]). *Let $\mathscr{P}$ be an arbitrary set of primes. Then the mean value $M_{\mathscr{P}}$ of $\lambda_{\mathscr{P}}$ exists and*

$$M_{\mathscr{P}} = \begin{cases} \prod_{p \in \mathscr{P}} \frac{p-1}{p+1} & \text{if } \sum_{p \in \mathscr{P}} \frac{1}{p} < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we are interested in studying the mentioned problems in the context of rational function fields over finite fields. Let q be an odd prime power and let $\mathbb{F}_q[t]$ denote the set of polynomials in variable t over finite field $\mathbb{F}_q$. We use $\boldsymbol{d}(N)$ to denote the degree of polynomial $N \in \mathbb{F}_q[t]$ and define the norm of N as $|N| = q^{\boldsymbol{d}(N)}$ if $N \neq 0$ and $|0| = 0$. We denote by $\mathcal{M}$ the set of monic polynomials in $\mathbb{F}_q[t]$ and for any non-negative integer x we define $\mathcal{M}_{\leq x} = \{N \in \mathcal{M} \mid \boldsymbol{d}(N) \leq x\}$. We call monic irreducible polynomials in $\mathbb{F}_q[t]$ prime and denote the set of all prime polynomials by $\mathcal{P}$.

The Liouville function is defined as $\lambda(N) = (-1)^{\Omega(N)}$, where $\Omega(N)$ stands for the number of prime factors of $N \in \mathbb{F}_q[t]$, counted with multiplicity. Note that, similarly as the analogous function defined over the set of integers $\mathbb{Z}$, the function $\Omega(N)$ is completely additive, so $\lambda(N)$ is completely multiplicative[1].

We now state the variants of the Pólya and the Turán conjectures in the context of $\mathbb{F}_q[t]$ as following:

Conjecture 1.4 (Variant of the Pólya conjecture). *For all integers $x \geq 2$ we have*

$$\sum_{N \in \mathcal{M}_{\leq x}} \lambda(N) \leq 0.$$

Conjecture 1.5 (Variant of the Turán conjecture). *For all integers $x \geq 0$ we have*

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|} \geq 0.$$

Several function field analogues of the Pólya conjecture were studied by Humphries in his master thesis [4]. In particular, he shows that that the variant of the Pólya conjecture over $\mathbb{F}_q[t]$, as stated above, does not hold (see [4, Proposition 6.8]). In this paper we provide the solution of the Turán conjecture. Towards the solution we prove the more general result, that can also be used to solve the Pólya conjecture.

Theorem 1.6. *Let $\alpha \neq 1/2$ be a real number and x a non-negative integer. Then*

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^{\alpha}} = \begin{cases} \frac{1-q^{1-\alpha}}{1-q^{1-2\alpha}}\left(1 - q^{x(1/2-\alpha)}\right) + q^{x(1/2-\alpha)}, & \text{if } x \text{ is even,} \\ \frac{1-q^{1-\alpha}}{1-q^{1-2\alpha}}\left(1 - q^{(x+1)(1/2-\alpha)}\right), & \text{if } x \text{ is odd.} \end{cases}$$

Furthermore,

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^{1/2}} = \begin{cases} \frac{x}{2}(1 - q^{1/2}) + 1, & \text{if } x \text{ is even,} \\ \frac{x+1}{2}(1 - q^{1/2}), & \text{if } x \text{ is odd.} \end{cases}$$

Putting $\alpha = 0$ in the previous theorem gives the mentioned result of Humphries, while after putting $\alpha = 1$ we obtain the solution of the Turán

[1] A function $f : \mathcal{M} \to \mathbb{R}$ is completely additive if $f(MN) = f(M) + f(N)$ for all $M, N \in \mathcal{M}$. If $f(MN) = f(M)f(N)$ holds for all $M, N \in \mathcal{M}$, then f is called completely multiplicative.

conjecture. Since for all integers $x \geq 0$ the function

$$\alpha \to \frac{1-q^{1-\alpha}}{1-q^{1-2\alpha}}\left(1-q^{(x+1)(1/2-\alpha)}\right)$$

is non-negative when $\alpha \geq 1$, the more general result also follows.

COROLLARY 1.7. *The variant of the Turán conjecture holds over $\mathbb{F}_q[t]$. More generally, if $\alpha \geq 1$, then*

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^{\alpha}} \geq 0$$

for all positive integers x.

In the second part of this paper we are studying the truncations to $L(1,\chi)$, where χ now stands for a quadratic character defined over $\mathbb{F}_q[t]$ (necessary definitions are provided in the next section). Since the Turán conjecture holds in this case, one can expect that such truncations are always non-negative. That is exactly what we prove, along with the upper bound on the minumum of truncations to $L(1,\chi)$, in similarity to Theorem 1.1.

THEOREM 1.8. *If x is an odd positive integer then*

$$\min_{\substack{\chi \text{ a quadratic} \\ \text{character}}} \sum_{N \in \mathcal{M}_{\leq x}} \frac{\chi(N)}{|N|} = 0$$

and if x is an even positive integer then

$$0 \leq \min_{\substack{\chi \text{ a quadratic} \\ \text{character}}} \sum_{N \in \mathcal{M}_{\leq x}} \frac{\chi(N)}{|N|} \leq q^{-x/2}.$$

Next we investigate the partial sums of the Möbius function over multiplicative semigroups of $\mathbb{F}_q[t]$ generated by sets of prime polynomials. The Möbius function for $N \in \mathbb{F}_q[t]$ is defined as $\mu(N) = (-1)^k$ if $N = cP_1P_2 \dots P_k$ for $c \in \mathbb{F}_q^{\times}$ and P_j distinct, prime polynomials in $\mathbb{F}_q[t]$, and $\mu(N) = 0$ otherwise. We denote by $\langle \mathcal{A} \rangle$ the multiplicative semigroup generated by a set (finite of infinite) of prime polynomials $\mathcal{A}$ over $\mathbb{F}_q[t]$. That is the set that contains constant polynomial 1 and the monic polynomials in $\mathbb{F}_q[t]$ whose prime factors all lie in $\mathcal{A}$.

We first provide the elementary bound, similar to (1.1). Note that, in complete agreement to the previous results, we get that the partial sums are always non-negative.

PROPOSITION 1.9 (Elementary bound). *For any set of primes $\mathcal{A}$ in $\mathbb{F}_q[t]$ and non-negative integer x, we have*

$$0 \leq \sum_{\substack{N \in \langle \mathcal{A} \rangle \\ \boldsymbol{d}(N) \leq x}} \frac{\mu(N)}{|N|} \leq 1.$$

The lower bound of 0 in the previous result is the best possible, as it is easy to find a set of primes $\mathcal{A}$ and a non-negative integer x such that $\sum_{\substack{N\in\langle\mathcal{A}\rangle \\ \boldsymbol{d}(N)\le x}} \frac{\mu(N)}{|N|} = 0$. For example, take $\mathcal{A}$ to be the set $\mathcal{P}$ of all primes and $x \ge 1$ an arbitrary integer. Then $\langle\mathcal{A}\rangle = \mathcal{M}$, leaving us with $\sum_{\substack{N\in M \\ \boldsymbol{d}(N)\le x}} \frac{\mu(N)}{|N|}$. Since by the Möbius inversion formula

$$\sum_{\substack{D|N \\ D\in\mathcal{M}}} \mu(D) = \begin{cases} 1, & \text{if } N = 1, \\ 0, & \text{if } N \in \mathcal{M}\setminus\{1\}, \end{cases}$$

we can apply Proposition 4.1 (see Section 4 for details) to get that this sum is equal to 0.

The second result we provide is the Landau's theorem for an arbitrary set of primes over $\mathbb{F}_q[t]$.

THEOREM 1.10. *Let $\mathcal{A}$ be any set of primes in $\mathbb{F}_q[t]$. Then*

$$\sum_{N\in\langle\mathcal{A}\rangle} \frac{\mu(N)}{|N|} = \prod_{P\in\mathcal{A}} \left(1 - \frac{1}{|P|}\right). \tag{1.2}$$

Finally, we investigate the mean value of the generalized Liouville funcion $\lambda_{\mathcal{A}}$ for a given set of prime polynomials $\mathcal{A}$ over $\mathbb{F}_q[t]$. Let $\Omega_{\mathcal{A}}(N)$ denote the number of prime factors of $N \in \mathbb{F}_q[t]$ coming from $\mathcal{A}$ counting multiplicity. The Liouville funcion for $\mathcal{A}$ is defined as

$$\lambda_{\mathcal{A}}(N) = (-1)^{\Omega_{\mathcal{A}}(N)}.$$

Since $\Omega_{\mathcal{A}}(N)$ is completely additive, this function is completely multiplicative. It can be alternatively defined as the completely multiplicative function such that $\lambda_{\mathcal{A}}(P) = -1$ if prime $P \in \mathcal{A}$ and $\lambda_{\mathcal{A}}(P) = 1$ if $P \notin \mathcal{A}$.

The mean value of a function $f : \mathcal{M} \to \mathbb{R}$ is defined as

$$\lim_{n\to\infty} \frac{1}{q^n} \sum_{N\in\mathcal{M}_n} f(N),$$

where $\mathcal{M}_n = \{N \in \mathcal{M} \mid \boldsymbol{d}(N) = n\}$, provided that the limit exists.

THEOREM 1.11. *Let $\mathcal{A}$ be any set of prime polynomials in $\mathbb{F}_q[t]$. Then the mean value of $\lambda_{\mathcal{A}}$ exists,*

$$\lim_{n\to\infty} \frac{1}{q^n} \sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N) = \begin{cases} \prod_{P\in\mathcal{A}} \frac{|P|-1}{|P|+1}, & \textit{if } \sum_{P\in\mathcal{A}} \frac{1}{|P|} < \infty, \\ 0, & \textit{otherwise.} \end{cases}$$

Closely related to the calculation of the mean value of $\lambda_{\mathcal{A}}$, we provide a method to calculate the exact value of $\sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$ for a given finite set of primes $\mathcal{A}$. This is quite specific to the context of $\mathbb{F}_q[t]$ we are considering and the analogous method over $\mathbb{Z}$ is unknown. The main reason for this difference is the fact that in the function field setting the generating Dirichlet series for $\lambda_{\mathcal{A}}$ can be expressed as a rational function multiplied by the zeta function over $\mathbb{F}_q[t]$. One can see this expression after inserting the change of variables $u = q^{-s}$ in (6.1) (see Section 6 for details). Since the zeta function over $\mathbb{F}_q[t]$ is also a rational function in variable u (see (2.1)), it is not hard to find to the coefficient of the generating Dirichlet series for $\lambda_{\mathcal{A}}$ with power u^n for a fixed non-negative integer n, which containts the information about the sum $\sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$ we are interested in.

Theorem 1.12. *Let $\mathcal{A} = \{P_1, P_2, \ldots, P_k\}$ be a finite set of prime polynomials and n a non-negative integer. Denote $L_n = \sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$ and for all $1 \le \ell \le k$ and $1 \le i_1 < i_2 < \cdots < i_\ell \le k$ define positive integers*

$$\boldsymbol{d}_{i_1,i_2,\ldots i_\ell} = \boldsymbol{d}(P_{i_1}) + \boldsymbol{d}(P_{i_2}) + \cdots + \boldsymbol{d}(P_{i_\ell}).$$

Order the elements of the set $\{\boldsymbol{d}_{i_1,i_2,\ldots i_\ell} \mid 1 \le \ell \le k, 1 \le i_1 < i_2 < \cdots < i_\ell \le k\}$ in the ascending order as

$$\boldsymbol{D}_1 < \boldsymbol{D}_2 < \cdots < \boldsymbol{D}_s$$

and define the integers α_j and β_j, $1 \le j \le s$ as

$$\alpha_j = \#\{1 \le \ell \le k, 1 \le i_1 < i_2 < \cdots < i_\ell \le k \mid \boldsymbol{d}_{i_1,i_2,\ldots i_\ell} = \boldsymbol{D}_j\}$$
$$\beta_j = \sum_{\substack{1\le \ell\le k \\ 1\le i_1<i_2<\cdots<i_\ell\le k \\ \boldsymbol{d}_{i_1,i_2,\ldots i_\ell}=\boldsymbol{D}_j}} (-1)^\ell.$$

Then, depending on n we have the following linear recurrence relations for L_n:

$$L_n = q^n, \ \ 0 \le n < \boldsymbol{D}_1,$$
$$L_n + \alpha_1 L_{n-\boldsymbol{D}_1} = q^n + \beta_1 q^{n-\boldsymbol{D}_1}, \ \ \boldsymbol{D}_1 \le n < \boldsymbol{D}_2,$$
$$\vdots$$
$$L_n + \alpha_1 L_{n-\boldsymbol{D}_1} + \cdots + \alpha_j L_{n-\boldsymbol{D}_j} = q^n + \beta_1 q^{n-\boldsymbol{D}_1} + \cdots + \beta_j q^{n-\boldsymbol{D}_j}, \quad \boldsymbol{D}_j \le n < \boldsymbol{D}_{j+1},$$
$$\vdots$$
$$L_n + \alpha_1 L_{n-\boldsymbol{D}_1} + \cdots + \alpha_s L_{n-\boldsymbol{D}_s} = q^n + \beta_1 q^{n-\boldsymbol{D}_1} + \cdots + \beta_s q^{n-\boldsymbol{D}_s}, \ \ \boldsymbol{D}_s \le n.$$

As an immediate consequence of the previous theorem, we can see that L_n depends only on the degrees of polynomials in $\mathcal{A}$. Furthermore, the obtained system of recurrence relations in not difficult to solve in practice. First $s-1$ recurrences are easily solved, as they are valid only in the limited ranges. For the final relation, note that $\boldsymbol{D}_s$ is always equal to $\boldsymbol{d}(P_1) + \boldsymbol{d}(P_2) + \cdots + \boldsymbol{d}(P_k)$, allowing us to rewrite this relation as

$$L_n + \alpha_1 L_{n-\boldsymbol{D}_1} + \cdots + \alpha_s L_{n-\boldsymbol{D}_s} = q^n \prod_{i=1}^{k} \left(1 - \frac{1}{|P_k|}\right).$$

This recurrence has a partial solution $q^n \prod_{i=1}^{k} \left(\frac{|P_k|-1}{|P_k|+1}\right)$ and the characteristic equation of the corresponding homogenous equation is

$$\prod_{i=1}^{k} \left(X^{\boldsymbol{d}(P_k)} + 1\right) = 0.$$

From these two facts one can obtain the formula for $\sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$ in the range $\boldsymbol{D}_s \leq n$. Exact calculations in two concrete cases were carried out at the end of Section 7.

2. Background on function fields

For a non-negative integer n we define

$$\mathcal{M}_{>n} = \{N \in \mathcal{M} \mid \boldsymbol{d}(N) > n\}.$$

The zeta function of $\mathbb{F}_q[t]$ is defined by

$$\mathcal{Z}(s) = \sum_{N\in\mathcal{M}} \frac{1}{|N|^s} = \prod_{P\in\mathcal{P}} \left(1 - \frac{1}{|P|^s}\right)^{-1}, \ \Re(s) > 1.$$

The number of polynomials in $\mathcal{M}_n$ is q^n, so it follows

$$\mathcal{Z}(s) = \frac{1}{1 - q^{1-s}}, \tag{2.1}$$

providing the analytic continuation for $\mathcal{Z}(s)$ to the whole complex plane, with a simple pole at $s = 1$.

The Dirichlet's theorem on prime polynomials in arithmetic progressions states that given two coprime polynomials A and M in $\mathbb{F}_q[t]$, $\boldsymbol{d}(M) \geq 1$, there exist infinitely many prime polynomials P such that $P \equiv A \pmod{M}$.

If P is a prime polynomial in $\mathbb{F}_q[t]$, then the Legendre symbol $\left(\frac{N}{P}\right)$ is defined as

$$\left(\frac{N}{P}\right) = \begin{cases} 1, & \text{if } N \text{ is square modulo } P, P \nmid N, \\ -1, & \text{if } N \text{ is not square modulo } P, P \nmid N, \\ 0, & \text{if } P | N. \end{cases}$$

The Jacobi symbol is defined by extending the Legendre symbol multiplicatively; if $Q = P_1^{e_1} P_2^{e_2} \dots P_k^{e_k}$ is the prime factorisation of polynomial $Q \in \mathcal{M}$, then $\left(\frac{N}{Q}\right) = \prod_{i=1}^{k} \left(\frac{N}{P_i}\right)^{e_i}$. This symbol satisfies the reciprocity formula analogous to the classical law of quadratic reciprocity: if M and N are two non-zero, relatively prime polynomials in $\mathbb{F}_q[t]$, then

$$\left(\frac{M}{N}\right) = \left(\frac{N}{M}\right)(-1)^{\frac{(q-1)}{2}\boldsymbol{d}(M)\boldsymbol{d}(N)}. \tag{2.2}$$

The quadratic character χ_D corresponding to a polynomial $D \in \mathcal{M}$ is defined as

$$\chi_D(N) = \left(\frac{D}{N}\right), \quad \text{for all } N \in \mathcal{M}.$$

The L-function associated to a quadratic character χ_D is defined by

$$L(s, \chi_D) = \sum_{N \in \mathcal{M}} \frac{\chi_D(N)}{|N|^s} = \prod_{P \in \mathcal{P}} \left(1 - \chi_D(P)|P|^{-s}\right)^{-1}.$$

This function is initially defined for $\Re(s) > 1$, but it can be analytically continued to an entire function, as it is a polynomial in q^{-s} (see [8, Proposition 4.3] for details).

3. Proof of Theorem 1.6

The generating Dirichlet series for the Liouville function can be written as

$$\sum_{N \in \mathcal{M}} \frac{\lambda(N)}{|N|^s} = \sum_{n=0}^{\infty} \left(\sum_{N \in \mathcal{M}_n} \lambda(N)\right) u^n, \tag{3.1}$$

where $u = q^{-s}$. On the other hand, since the Liouville function is completely multiplicative, that Dirichlet series has the Euler product expression

$$\prod_{P \in \mathcal{P}} \left(1 + \frac{1}{|P|^s}\right)^{-1} = \prod_{P \in \mathcal{P}} \frac{1 - \frac{1}{|P|^s}}{1 - \frac{1}{|P|^{2s}}} = \frac{\mathcal{Z}(2s)}{\mathcal{Z}(s)} = \frac{1 - qu}{1 - qu^2},$$

after using (2.1) and inserting the change of variables $u = q^{-s}$. Expressing $\frac{1}{1-qu^2}$ as power series yields

$$(1 - qu) \sum_{n=0}^{\infty} q^n u^{2n} = \sum_{n=0}^{\infty} q^n u^{2n} - \sum_{n=0}^{\infty} q^{n+1} u^{2n+1}.$$

Comparing this to (3.1) gives us

$$\sum_{N \in \mathcal{M}_n} \lambda(N) = \begin{cases} q^{n/2}, & \text{if } n \text{ is even,} \\ -q^{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.2}$$

We now turn to

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^\alpha} = \sum_{n=0}^{x} \left(\sum_{N \in \mathcal{M}_n} \lambda(N)\right) q^{-n\alpha}.$$

Using (3.2), it follows that the sum over even $0 \le n \le x$ is

$$1 + q^{1-2\alpha} + q^{2(1-2\alpha)} + \cdots + q^{[x/2](1-2\alpha)}.$$

Similarly, the sum over odd $0 \le n \le x$ is

$$-q^{1-\alpha}\left(1 + q^{1-2\alpha} + q^{2(1-2\alpha)} + \cdots + q^{[(x-1)/2](1-2\alpha)}\right).$$

If $\alpha \neq \frac{1}{2}$ we get

$$\begin{aligned}\sum_{N \in \mathcal{M}_{\le x}} \frac{\lambda(N)}{|N|^\alpha} &= \frac{1 - q^{([x/2]+1)(1-2\alpha)}}{1 - q^{1-2\alpha}} - q^{1-\alpha}\frac{1 - q^{([(x-1)/2]+1)(1-2\alpha)}}{1 - q^{1-2\alpha}} \\ &= \begin{cases} \frac{1-q^{1-\alpha}}{1-q^{1-2\alpha}}\left(1 - q^{x(1/2-\alpha)}\right) + q^{x(1/2-\alpha)}, & \text{if } x \text{ is even} \\ \frac{1-q^{1-\alpha}}{1-q^{1-2\alpha}}\left(1 - q^{(x+1)(1/2-\alpha)}\right), & \text{if } x \text{ is odd.} \end{cases}\end{aligned}$$

If $\alpha = 1/2$ we get

$$\sum_{N \in \mathcal{M}_{\le x}} \frac{\lambda(N)}{|N|^{1/2}} = [x/2] + 1 - q^{1/2}([(x-1)/2] + 1)$$

and the result follows.

4. Positivity of truncations to $L(1, \chi)$

We begin with an auxiliary result, after which we turn to the proof of Theorem 1.8.

Proposition 4.1. *Let $f : \mathcal{M} \to \mathbb{R}$ be a function and put $g(N) = \sum_{\substack{D \mid N \\ D \in \mathcal{M}}} f(D)$ for all $N \in \mathcal{M}$. Then for all positive integers n we have*

$$\sum_{N \in \mathcal{M}_{\le n}} \frac{f(N)}{|N|} = \frac{1}{q^n} \sum_{N \in \mathcal{M}_n} g(N).$$

Proof. We have

$$\begin{aligned}\sum_{N \in \mathcal{M}_n} g(N) &= \sum_{N \in \mathcal{M}_n} \sum_{\substack{D \mid N \\ D \in \mathcal{M}}} f(D) \\ &= \sum_{D \in \mathcal{M}_{\le n}} f(D) \sum_{N \in \mathcal{M}_{n - d(D)}} 1 \\ &= q^n \sum_{D \in \mathcal{M}_{\le n}} \frac{f(D)}{|D|}\end{aligned}$$

and the result follows. ∎

Previous proposition has an important immediate consequence. Suppose that $\sum_{N\in\mathcal{M}} \frac{|f(N)|}{|N|} < \infty$. By Proposition 4.1, for any positive integer n we can write

$$\frac{1}{q^n}\sum_{N\in\mathcal{M}_n} g(N) = \sum_{D\in\mathcal{M}} \frac{f(D)}{|D|} + O\left(\sum_{D\in\mathcal{M}_{>n}} \frac{|f(D)|}{|D|}\right).$$

Convergence of $\sum_{N\in\mathcal{M}} \frac{|f(N)|}{|N|}$ assures that the error term vanishes as $n\to\infty$, thus showing that the mean values of g exists,

$$\lim_{n\to\infty}\frac{1}{q^n}\sum_{N\in\mathcal{M}_n} g(N) = \sum_{D\in\mathcal{M}} \frac{f(D)}{|D|}. \tag{4.1}$$

This result can be seen as the $\mathbb{F}_q[t]$-analogue of the classical theorem of Wintner. Note that the assumption of multiplicativity of g is not necessary for (4.1). However, under that assumption, f must also be multiplicative, so we can express the mean value of g in the form of Euler product

$$\prod_{P\in\mathcal{P}}\left(1-\frac{1}{|P|}\right)\left(\sum_{j=0}^{\infty}\frac{g(P^j)}{|P|^j}\right).$$

More on this topic will be discussed in Section 6.

Proof of Theorem 1.8. We first note that if $f:\mathcal{M}\to[-1,1]$ is a completely multiplicative function, then $g(N)=\sum_{\substack{D\mid N\\ D\in\mathcal{M}}} f(D)$ is a non-negative multiplicative function. Using this fact and Proposition 4.1 we get that for every quadratic character χ and non-negative integer x

$$\sum_{N\in\mathcal{M}_{\le x}}\frac{\chi(N)}{|N|}\ge 0.$$

By the reciprocity law (2.2), for any fixed non-negative integer x we may find an arithmetic progression modulo $\prod_{\substack{P\in\mathcal{P}\\ \boldsymbol{d}(P)\le x}} P$ such that every prime Q lying in that progression satisfies $\left(\frac{P}{Q}\right)=-1$. Dirichlet's theorem on prime polynomials in arithmetic progressions ensures that such polynomial Q exists, showing that there exists a quadratic character ψ such that $\psi(P)=-1=\lambda(P)$ for all prime polynomials P with $\boldsymbol{d}(P)\le x$. It follows that

$$\sum_{N\in\mathcal{M}_{\le x}}\frac{\psi(N)}{|N|}=\sum_{N\in\mathcal{M}_{\le x}}\frac{\lambda(N)}{|N|}.$$

By Theorem 1.6, we have

$$\min_{\substack{\chi \text{ a quadratic}\\ \text{character}}}\frac{\chi(N)}{|N|}\le\sum_{N\in\mathcal{M}_{\le x}}\frac{\lambda(N)}{|N|}=\begin{cases} q^{-x/2}, & \text{if } x \text{ is even,}\\ 0, & \text{if } x \text{ is odd,}\end{cases}$$

finishing the proof. □

5. Partial sums involving the Möbius function

We begin this section by noting that the generating Dirichlet series of the Möbius function μ can be expressed as

$$\sum_{N\in\mathcal{M}} \frac{\mu(N)}{|N|^s} = \prod_{P\in\mathcal{P}} \left(1 - \frac{1}{|P|^s}\right) = \frac{1}{\mathcal{Z}(s)} = 1 - q^{1-s}.$$

In particular, if follows that

$$\sum_{N\in\mathcal{M}} \frac{\mu(N)}{|N|} = 0, \tag{5.1}$$

which can be seen the $\mathbb{F}_q[t]$-analogue of the classical result of von Mangoldt ([6]).

Proof of Proposition 1.9. Let $\overline{\mathcal{A}}$ be the set of prime polynomials over $\mathbb{F}_q[t]$ not contained in $\mathcal{A}$. Using the Möbius inversion we can write

$$1_{N\in\langle\overline{\mathcal{A}}\rangle} = \sum_{\substack{D|N\\ D\in\langle\mathcal{A}\rangle}} \mu(D) = \sum_{\substack{D|N\\ D\in\mathcal{M}}} \mu(D) 1_{D\in\langle\mathcal{A}\rangle},$$

where $1_{N\in\mathcal{S}}$ is the indicator function of subset $\mathcal{S}$ of $\mathcal{M}$,

$$1_{N\in\mathcal{S}} = \begin{cases} 1, & \text{if } N\in\mathcal{S}\\ 0, & \text{if } N\notin\mathcal{S}. \end{cases}$$

After using Proposition 4.1 we get

$$q^x \sum_{\substack{N\in\langle\mathcal{A}\rangle\\ \boldsymbol{d}(N)\le x}} \frac{\mu(N)}{|N|} = \sum_{\substack{N\in\langle\overline{\mathcal{A}}\rangle\\ \boldsymbol{d}(N)=x}} 1. \tag{5.2}$$

The right side is between 0 and q^x, according to how many monic polynomials of degree x are in $\langle\overline{\mathcal{A}}\rangle$. Thus, the result follows. □

Proof of Theorem 1.10. We distinguish two different cases.

i) Suppose $\sum_{P\in\mathcal{A}} \frac{1}{|P|} < \infty$. Then, from the monotone convergence theorem we get that

$$\sum_{N\in\langle\mathcal{A}\rangle} \frac{1}{|N|} = \prod_{P\in\mathcal{A}} \left(1 + \frac{1}{|P|-1}\right)$$

is absolutely convegent. By the dominated covergence it further follows that

$$\sum_{N\in\langle\mathcal{A}\rangle} \frac{\mu(N)}{|N|} = \prod_{P\in\mathcal{A}} \left(1 - \frac{1}{|P|}\right)$$

is conditionally convergent, giving the desired result.

ii) Suppose that $\sum_{P\in\mathcal{A}}\frac{1}{|P|}=\infty$. In this case we utilize the identity (5.2) for a non-negative integer x,

$$q^x\sum_{\substack{N\in\langle\mathcal{A}\rangle\\ \boldsymbol{d}(N)\le x}}\frac{\mu(N)}{|N|}=\sum_{\substack{N\in\langle\overline{\mathcal{A}}\rangle\\ \boldsymbol{d}(N)=x}}1=\sum_{N\in\mathcal{M}_x}1_{N\in\langle\overline{\mathcal{A}}\rangle}.$$

Since $\sum_{P\in\mathcal{P}}\frac{|1-1_{P\in\langle\overline{\mathcal{A}}\rangle}|}{|P|}=\sum_{P\in\mathcal{A}}\frac{1}{|P|}=\infty$, by Theorem 6.1 (see Section 6 for details) the mean value of $1_{N\in\langle\overline{\mathcal{A}}\rangle}$ is 0, so the right side is $o(q^x)$. If follows that

$$\sum_{\substack{N\in\langle\mathcal{A}\rangle\\ \boldsymbol{d}(N)\le x}}\frac{\mu(N)}{|N|}=o(1),$$

finishing the proof.

□

6. The mean value of the Liouville function $\lambda_{\mathcal{A}}$

The important piece of information in finding the mean value of $\lambda_{\mathcal{A}}$ for a fixed set of prime polynomials $\mathcal{A}$,

$$\lim_{n\to\infty}\frac{1}{q^n}\sum_{N\in\mathcal{M}_n}\lambda_{\mathcal{A}}(N),$$

will be the generating Dirichlet series for $\lambda_{\mathcal{A}}$. Since $\lambda_{\mathcal{A}}$ is completely multiplicative, for $\Re(s)>1$ and any set $\mathcal{A}$ we have

$$\begin{aligned}\sum_{N\in\mathcal{M}}\frac{\lambda_{\mathcal{A}}(N)}{|N|^s}&=\prod_{P\in\mathcal{P}}\left(1-\frac{\lambda_{\mathcal{A}}(P)}{|P|^s}\right)^{-1}\\ &=\prod_{P\in\mathcal{A}}\left(1+\frac{1}{|P|^s}\right)^{-1}\prod_{P\notin\mathcal{A}}\left(1-\frac{1}{|P|^s}\right)^{-1}\\ &=\prod_{P\in\mathcal{A}}\left(\frac{1+\frac{1}{|P|^s}}{1-\frac{1}{|P|^s}}\right)^{-1}\prod_{P\in\mathcal{P}}\left(1-\frac{1}{|P|^s}\right)^{-1}\\ &=\prod_{P\in\mathcal{A}}\frac{|P|^s-1}{|P|^s+1}\mathcal{Z}(s).\end{aligned}\tag{6.1}$$

We can get some immediate information about the mean value of $\lambda_{\mathcal{A}}$ from (4.1). However, in order to get the complete picture, we need the following stronger result. It is a direct consequence of the general Halász type theorem from [5, Theorem 1.4.1].

THEOREM 6.1 (Klurman). *Suppose $f : \mathcal{M} \to \mathbb{R}$ is a multiplicative function taking values in $[-1, 1]$.*

- *If $\sum_{P\in\mathcal{P}} \frac{|1-f(P)|}{|P|}$ converges then the mean value of f exists and is equal to*

$$\prod_{P\in\mathcal{P}} \left(1 - \frac{1}{|P|}\right) \left(\sum_{j=0}^{\infty} \frac{f(P^j)}{|P|^j}\right).$$

- *If $\sum_{P\in\mathcal{P}} \frac{|1-f(P)|}{|P|}$ diverges then the mean value of f is equal to zero.*

Note that $\sum_{P\in\mathcal{P}} \frac{|1-\lambda_{\mathcal{A}}(P)|}{|P|} = 2\sum_{P\in\mathcal{A}} \frac{1}{|P|}$ and, by (6.1),

$$\prod_{P\in\mathcal{P}} \left(1 - \frac{1}{|P|}\right) \left(\sum_{j=0}^{\infty} \frac{\lambda_{\mathcal{A}}(P^j)}{|P|^j}\right) = \prod_{P\in\mathcal{A}} \frac{|P|-1}{|P|+1}.$$

Thus, if $\sum_{P\in\mathcal{A}} \frac{1}{|P|} < \infty$, from Theorem 6.1 it follows that the mean value of $\lambda_{\mathcal{A}}$ exists and is equal to $\prod_{P\in\mathcal{A}} \frac{|P|-1}{|P|+1}$. Similarly, if $\sum_{P\in\mathcal{A}} \frac{1}{|P|} = \infty$, Theorem 6.1 shows that the mean value of $\lambda_{\mathcal{A}}$ is equal to zero. This proves Theorem 1.11.

7. EXACT VALUE OF $\sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$

Identity (6.1) can be used to obtain the exact value of $\sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$ for a given finite set of primes $\mathcal{A} = \{P_1, P_2, \ldots, P_k\}$ and a fixed non-negative integer n. Rewrite (6.1) as

$$\prod_{i=1}^{k} \left(1 + \frac{1}{|P_i|^s}\right) \sum_{N\in\mathcal{M}} \frac{\lambda_{\mathcal{A}}(N)}{|N|^s} = \prod_{i=1}^{k} \left(1 - \frac{1}{|P_i|^s}\right) \mathcal{Z}(s)$$

and insert the change of variables $u = q^{-s}$. The left side becomes

$$\prod_{i=1}^{k} \left(1 + u^{\boldsymbol{d}(P_i)}\right) \sum_{n=0}^{\infty} \sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N) u^{\boldsymbol{d}(N)} = \prod_{i=1}^{k} \left(1 + u^{\boldsymbol{d}(P_i)}\right) \sum_{n=0}^{\infty} L_n u^n,$$

where $L_n = \sum_{N\in\mathcal{M}_n} \lambda_{\mathcal{A}}(N)$. Using (2.1), the right side becomes

$$\prod_{i=1}^{k} \left(1 - u^{\boldsymbol{d}(P_i)}\right) (1-qu)^{-1} = \prod_{i=1}^{k} \left(1 - u^{\boldsymbol{d}(P_i)}\right) \sum_{n=0}^{\infty} q^n u^n.$$

We thus get the equality of power series

$$\prod_{i=1}^{k} \left(1 + u^{\boldsymbol{d}(P_i)}\right) \sum_{n=0}^{\infty} L_n u^n = \prod_{i=1}^{k} \left(1 - u^{\boldsymbol{d}(P_i)}\right) \sum_{n=0}^{\infty} q^n u^n. \tag{7.1}$$

Comparing the corresponding coefficients in this equality will give us the recurrence relation for L_n. In order to do that we expand

$$\prod_{i=1}^{k}\left(1+u^{\boldsymbol{d}(P_i)}\right)=1+\sum_{\substack{1\le\ell\le k\\ 1\le i_1<i_2<\cdots<i_\ell\le k}} u^{\boldsymbol{d}_{i_1,i_2,\dots i_\ell}},$$

where

$$\boldsymbol{d}_{i_1,i_2,\dots i_\ell}=\boldsymbol{d}(P_{i_1})+\boldsymbol{d}(P_{i_2})+\cdots+\boldsymbol{d}(P_{i_\ell}).$$

We order the elements of the set $\{\boldsymbol{d}_{i_1,i_2,\dots i_\ell}\mid 1\le\ell\le k, 1\le i_1<i_2<\cdots<i_\ell\le k\}$ in the ascending order as $\boldsymbol{D}_1<\boldsymbol{D}_2<\cdots<\boldsymbol{D}_s$ and define α_j, $1\le j\le s$ as

$$\alpha_j=\#\{1\le\ell\le k, 1\le i_1<i_2<\cdots<i_\ell\le k\mid \boldsymbol{d}_{i_1,i_2,\dots i_\ell}=\boldsymbol{D}_j\}.$$

If follows that

$$\prod_{i=1}^{k}\left(1+u^{\boldsymbol{d}(P_i)}\right)=1+\sum_{j=1}^{s}\alpha_j u^{\boldsymbol{D}_j}.$$

Similarly,

$$\prod_{i=1}^{k}\left(1-u^{\boldsymbol{d}(P_i)}\right)=1+\sum_{j=1}^{s}\beta_j u^{\boldsymbol{D}_j},$$

where

$$\beta_j=\sum_{\substack{1\le\ell\le k\\ 1\le i_1<i_2<\cdots<i_\ell\le k\\ \boldsymbol{d}_{i_1,i_2,\dots i_\ell}=\boldsymbol{D}_j}}(-1)^\ell.$$

Returning to (7.1) we get

$$\sum_{n=0}^{\infty}L_n u^n+\sum_{j=1}^{s}\sum_{n=0}^{\infty}\alpha_j L_n u^{n+\boldsymbol{D}_j}=\sum_{n=0}^{\infty}q^n u^n+\sum_{j=1}^{s}\sum_{n=0}^{\infty}\beta_j q^n u^{n+\boldsymbol{D}_j}. \tag{7.2}$$

Finally, noting that $\sum_{n=0}^{\infty}\alpha_j L_n u^{n+\boldsymbol{D}_j}=\sum_{n=\boldsymbol{D}_j}^{\infty}\alpha_j L_{n-\boldsymbol{D}_j}u^n$ and $\sum_{n=0}^{\infty}\beta_j q^n u^{n+\boldsymbol{D}_j}=\sum_{n=\boldsymbol{D}_j}^{\infty}\beta_j q^{n-\boldsymbol{D}_j}u^n$ for all $1\le j\le s$ and comparing the coefficients on both sides of (7.2) yields Theorem 1.12.

We end this section with the examples of calculations of $\sum_{N\in\mathcal{M}_n}\lambda_{\mathcal{A}}(N)$ in two concrete cases.

Example 7.1. $\mathcal{A}=\{P\}$ for some prime polynomial P
In this case $s=1$, $D_1=\boldsymbol{d}(P)$, $\alpha_1=1$, $\beta_1=-1$, so Theorem 1.12 gives us

$$L_n=q^n,\ \ 0\le n<\boldsymbol{d}(P),$$
$$L_n+L_{n-\boldsymbol{d}(P)}=q^n-q^{n-\boldsymbol{d}(P)},\ \ \boldsymbol{d}(P)\le n.$$

After solving the obtained recurrence relation we get

$$\sum_{N\in\mathcal{M}_n}\lambda_{\mathcal{A}}(N)=\begin{cases}q^n, & 0\le n<\boldsymbol{d}(P),\\ q^n\left(1-2\dfrac{1-\left(\frac{-1}{|P|}\right)^{\left[\frac{n}{\boldsymbol{d}(P)}\right]}}{|P|+1}\right), & \boldsymbol{d}(P)\le n.\end{cases}$$

Example 7.2. $\mathcal{A}=\{P,Q\}$ for some prime polynomials P and Q of the same degree m.

In this case $s=2$, $D_1=m$, $D_2=2m$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=-2$, $\beta_2=1$, so Theorem 1.12 gives us

$$\begin{aligned}&L_n=q^n,\ \ 0\le n<m,\\ &L_n+2L_{n-m}=q^n-2q^{n-m},\ \ m\le n<2m,\\ &L_n+2L_{n-m}+L_{n-2m}=q^n-2q^{n-m}+q^{n-2m},\ \ 2m\le n.\end{aligned}$$

After solving the obtained recurrence relations we get

$$\sum_{N\in\mathcal{M}_n}\lambda_{\mathcal{A}}(N)=\begin{cases}q^n,\ \ 0\le n<m,\\ q^n-4q^{n-m},\ \ m\le n<2m,\\ q^nV_m+q^{n-m\left[\frac{n}{m}\right]}\left(1-V_m\right)(-1)^{\left[\frac{n}{m}\right]-1}\left(\left[\frac{n}{m}\right]q^m+\left[\frac{n}{m}\right]-1\right)\\ +4q^{n-m\left[\frac{n}{m}\right]}\left(1-V_m\right)\left[\frac{n}{m}\right](-1)^{\left[\frac{n}{m}\right]},\ \ 2m\le n,\end{cases}$$

where $V_m=\left(\frac{q^m-1}{q^m+1}\right)^2$.

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