EXTREMEAL BEHAVIOUR OF ±1-VALUED COMPLETELY
MULTIPLICATIVE FUNCTIONS IN FUNCTION FIELDS

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Abstract. We investigate the classical Pólya and Turán conjectures in the context of rational function fields over finite fields \( \mathbb{F}_q \). Related to these two conjectures we investigate the sign of truncation \( s \) of Dirichlet \( L \)-functions at point \( s = 1 \) corresponding to quadratic characters over \( \mathbb{F}_q[t] \), prove a variant of a theorem of Landau for arbitrary sets of monic, irreducible polynomials over \( \mathbb{F}_q[t] \) and calculate the mean value of certain variants of the Liouville function over \( \mathbb{F}_q[t] \).

1. Introduction

Let \( \lambda(n) = (-1)^{\Omega(n)} \) denote the Liouville function, where \( \Omega(n) \) is the number of prime factors of integer \( n \) counted with multiplicity. Pólya ([7]) conjectured that \( \sum_{n \leq x} \lambda(n) \leq 0 \) for all \( x \geq 2 \) and showed that this conjecture implies the Riemann hypothesis. Similarly, Turán ([11]) showed that the Riemann hypothesis can be proven assuming a related conjecture that \( \sum_{n \leq x} \frac{\lambda(n)}{n} \geq 0 \) for all \( x \geq 0 \). Haselgrove ([3]) showed that both of these conjectures are false, despite the extensive numerical data suggesting their validity.

The fact that the Turán conjecture does not hold was used in [2] to exhibit negative values of truncations to the Dirichlet \( L \)-functions at \( s = 1 \) associated to quadratic characters. In particular, the authors prove following theorem.

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Theorem 1.1 ([2]). There exists a positive constant $c$ such that for all large $x$
\[
\frac{1}{(\log \log x)^{\frac{3}{5}}} \leq \min_{\chi \text{ a quadratic character}} \sum_{n \leq x} \frac{\chi(n)}{n} \leq -\frac{c}{\log x}.
\]

Results from [2] can also be used in relation to partial sums of the M"obius function over multiplicative semigroups of the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ generated by arbitrary sets of primes. Let $\mathcal{P}$ be any set of primes (finite or infinite) and let $\langle \mathcal{P} \rangle$ denote the multiplicative semigroup generated by $\mathcal{P}$. That is the set of natural numbers, all of whose prime factors lie in $\mathcal{P}$. In [10] Tao proves that elementary bound
\[
\sum_{n \in \langle \mathcal{P} \rangle \cap \{1, 2, \ldots, x\}} \mu(n) \leq 1
\]
holds for all sets of primes $\mathcal{P}$ and $x \geq 0$. An important remark he makes is that the lower bound of $-1$ can be improved by Theorem 2 of [2] to $\left(1 - 2\log(1 + \sqrt{e}) + 4 \int_{1}^{\sqrt{e}} \frac{\log t}{t+1} dt\right) \log 2 + o(1) = -0.4553 \ldots + o(1)$, the value that is optimal except for the $o(1)$ term.

Another important result proved in [10] is the Landau’s theorem for arbitrary set of primes (also see [9] for the proof of this theorem in the more general context of number fields).

Theorem 1.2 ([10]). Let $\mathcal{P}$ be any set of primes. Then
\[
\sum_{n \in \langle \mathcal{P} \rangle \cap \{1, 2, \ldots, x\}} \frac{\mu(n)}{n} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right).
\]

For a given set of primes $\mathcal{P}$ one can also consider the generalization $\lambda_{\mathcal{P}}$ of the Liouville function $\lambda$. It is defined by $\lambda_{\mathcal{P}}(n) = (-1)^{\Omega_{\mathcal{P}}(n)}$, where $\Omega_{\mathcal{P}}(n)$ is the number of prime factors of integer $n$ coming from $\mathcal{P}$, counted with multiplicity. Various properties of this function have been studied in [1]. In particular, the authors calculate its mean value.

Theorem 1.3 ([1]). Let $\mathcal{P}$ be an arbitrary set of primes. Then the mean value $M_{\mathcal{P}}$ of $\lambda_{\mathcal{P}}$ exists and
\[
M_{\mathcal{P}} = \begin{cases} \prod_{p \in \mathcal{P}} \frac{p^{1-1} p^{-1}}{p+1} & \text{if } \sum_{p \in \mathcal{P}} \frac{1}{p} < \infty, \\ 0 & \text{otherwise}. \end{cases}
\]

In this paper we are interested in studying the mentioned problems in the context of rational function fields over finite fields. Let $q$ be an odd prime power and let $\mathbb{F}_q[t]$ denote the set of polynomials in variable $t$ over finite field $\mathbb{F}_q$. We use $d(N)$ to denote the degree of polynomial $N \in \mathbb{F}_q[t]$ and define
the norm of $N$ as $|N| = q^{d(N)}$ if $N \neq 0$ and $|0| = 0$. We denote by $\mathcal{M}$ the set of monic polynomials in $\mathbb{F}_q[t]$ and for any non-negative integer $x$ we define $\mathcal{M}_{\leq x} = \{ N \in \mathcal{M} \mid d(N) \leq x \}$. We call monic irreducible polynomials in $\mathbb{F}_q[t]$ prime and denote the set of all prime polynomials by $\mathcal{P}$.

The Liouville function is defined as $\lambda(N) = (-1)^{\Omega(N)}$, where $\Omega(N)$ stands for the number of prime factors of $N \in \mathbb{F}_q[t]$, counted with multiplicity. Note that, similarly as the analogous function defined over the set of integers $\mathbb{Z}$, the function $\Omega(N)$ is completely additive, so $\lambda(N)$ is completely multiplicative$^1$.

We now state the variants of the Pólya and the Turán conjectures in the context of $\mathbb{F}_q[t]$ as following:

**Conjecture 1.4 (Variant of the Pólya conjecture).** For all integers $x \geq 2$ we have

$$\sum_{N \in \mathcal{M}_{\leq x}} \lambda(N) \leq 0.$$ 

**Conjecture 1.5 (Variant of the Turán conjecture).** For all integers $x \geq 0$ we have

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|} \geq 0.$$ 

Several function field analogues of the Pólya conjecture were studied by Humphries in his master thesis [4]. In particular, he shows that the variant of the Pólya conjecture over $\mathbb{F}_q[t]$, as stated above, does not hold (see [4, Proposition 6.8]). In this paper we provide the solution of the Turán conjecture. Towards the solution we prove the more general result, that can also be used to solve the Pólya conjecture.

**Theorem 1.6.** Let $\alpha \neq 1/2$ be a real number and $x$ a non-negative integer. Then

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^\alpha} = \begin{cases} \frac{1}{1-q^{1-\alpha}} (1 - q^{x(1/2-\alpha)}) + q^{x(1/2-\alpha)}, & \text{if } x \text{ is even}, \\ \frac{1}{1-q^{1-\alpha}} (1 - q^{(x+1)(1/2-\alpha)}), & \text{if } x \text{ is odd}. \end{cases}$$

Furthermore,

$$\sum_{N \in \mathcal{M}_{\leq x}} \frac{\lambda(N)}{|N|^{1/2}} = \begin{cases} \frac{1}{2} (1 - q^{1/2}) + 1, & \text{if } x \text{ is even}, \\ \frac{x+1}{2} (1 - q^{1/2}), & \text{if } x \text{ is odd}. \end{cases}$$

Putting $\alpha = 0$ in the previous theorem gives the mentioned result of Humphries, while after putting $\alpha = 1$ we obtain the solution of the Turán

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$^1$A function $f : \mathcal{M} \to \mathbb{R}$ is completely additive if $f(MN) = f(M) + f(N)$ for all $M, N \in \mathcal{M}$. If $f(MN) = f(M)f(N)$ holds for all $M, N \in \mathcal{M}$, then $f$ is called completely multiplicative.
conjecture. Since for all integers \( x \geq 0 \) the function

\[
\alpha \to \frac{1 - q^{1-\alpha}}{1 - q^{1-2\alpha}} \left( 1 - q^{(x+1)(1/2-\alpha)} \right)
\]

is non-negative when \( \alpha \geq 1 \), the more general result also follows.

**Corollary 1.7.** The variant of the Turán conjecture holds over \( \mathbb{F}_q[t] \).

More generally, if \( \alpha \geq 1 \), then

\[
\sum_{N \in M \leq x} \lambda(N) |N|^\alpha \geq 0
\]

for all positive integers \( x \).

In the second part of this paper we are studying the truncations to \( L(1, \chi) \), where \( \chi \) now stands for a quadratic character defined over \( \mathbb{F}_q[t] \) (necessary definitions are provided in the next section). Since the Turán conjecture holds in this case, one can expect that such truncations are always non-negative. That is exactly what we prove, along with the upper bound on the minimum of truncations to \( L(1, \chi) \), in similarity to Theorem 1.1.

**Theorem 1.8.** If \( x \) is an odd positive integer then

\[
\min_{\chi \text{ a quadratic character}} \sum_{N \in M \leq x} \frac{\chi(N)}{|N|^\alpha} = 0
\]

and if \( x \) is an even positive integer then

\[
0 \leq \min_{\chi \text{ a quadratic character}} \sum_{N \in M \leq x} \frac{\chi(N)}{|N|^\alpha} \leq q^{-x/2}.
\]

Next we investigate the partial sums of the Möbius function over multiplicative semigroups of \( \mathbb{F}_q[t] \) generated by sets of prime polynomials. The Möbius function for \( N \in \mathbb{F}_q[t] \) is defined as \( \mu(N) = (-1)^k \) if \( N = cP_1P_2\ldots P_k \) for \( c \in \mathbb{F}_q^\times \) and \( P_j \) distinct, prime polynomials in \( \mathbb{F}_q[t] \), and \( \mu(N) = 0 \) otherwise. We denote by \( <A> \) the multiplicative semigroup generated by a set (finite or infinite) of prime polynomials \( A \) over \( \mathbb{F}_q[t] \). That is the set that contains constant polynomial 1 and the monic polynomials in \( \mathbb{F}_q[t] \) whose prime factors all lie in \( A \).

We first provide the elementary bound, similar to (1.1). Note that, in complete agreement to the previous results, we get that the partial sums are always non-negative.

**Proposition 1.9 (Elementary bound).** For any set of primes \( A \) in \( \mathbb{F}_q[t] \) and non-negative integer \( x \), we have

\[
0 \leq \sum_{N \in <A>: d(N) \leq x} \mu(N) |N| \leq 1.
\]
The lower bound of 0 in the previous result is the best possible, as it is easy to find a set of primes \( A \) and a non-negative integer \( x \) such that
\[
\sum_{N \in (A)} \frac{\mu(N)}{|N|} \leq x \left(1 - \frac{1}{|P|}\right).
\]
Since by the Möbius inversion formula
\[
\sum_{D | N} \mu(D) = \begin{cases} 1, & \text{if } N = 1, \\ 0, & \text{if } N \in M \setminus \{1\}, \end{cases}
\]
we can apply Proposition 4.1 (see Section 4 for details) to get that this sum is equal to 0.

The second result we provide is the Landau’s theorem for an arbitrary set of primes over \( \mathbb{F}_q[t] \).

**Theorem 1.10.** Let \( A \) be any set of primes in \( \mathbb{F}_q[t] \). Then
\[
\sum_{N \in (A)} \frac{\mu(N)}{|N|} = \prod_{P \in A} \left(1 - \frac{1}{|P|}\right).
\]

Finally, we investigate the mean value of the generalized Liouville function \( \lambda_A \) for a given set of prime polynomials \( A \) over \( \mathbb{F}_q[t] \). Let \( \Omega_A(N) \) denote the number of prime factors of \( N \in \mathbb{F}_q[t] \) coming from \( A \) counting multiplicity. The Liouville function for \( A \) is defined as
\[
\lambda_A(N) = (-1)^{\Omega_A(N)}.
\]
Since \( \Omega_A(N) \) is completely additive, this function is completely multiplicative. It can be alternatively defined as the completely multiplicative function such that \( \lambda_A(P) = -1 \) if prime \( P \in A \) and \( \lambda_A(P) = 1 \) if \( P \notin A \).

The mean value of a function \( f : M \to \mathbb{R} \) is defined as
\[
\lim_{n \to \infty} \frac{1}{q^n} \sum_{N \in M_n} f(N),
\]
where \( M_n = \{N \in M \mid d(N) = n\} \), provided that the limit exists.

**Theorem 1.11.** Let \( A \) be any set of prime polynomials in \( \mathbb{F}_q[t] \). Then the mean value of \( \lambda_A \) exists,
\[
\lim_{n \to \infty} \frac{1}{q^n} \sum_{N \in M_n} \lambda_A(N) = \begin{cases} \prod_{P \in A} \frac{|P|-1}{|P|+1}, & \text{if } \sum_{P \in A} \frac{1}{|P|} < \infty, \\ 0, & \text{otherwise}. \end{cases}
\]

Closely related to the calculation of the mean value of \( \lambda_A \), we provide a method to calculate the exact value of \( \sum_{N \in M_n} \lambda_A(N) \) for a given finite set of primes \( A \). This is quite specific to the context of \( \mathbb{F}_q[t] \) we are considering and the analogous method over \( \mathbb{Z} \) is unknown. The main reason for this difference
is the fact that in the function field setting the generating Dirichlet series for $\lambda_A$ can be expressed as a rational function multiplied by the zeta function over $\mathbb{F}_q[t]$. One can see this expression after inserting the change of variables $u = q^{-s}$ in (6.1) (see Section 6 for details). Since the zeta function over $\mathbb{F}_q[t]$ is also a rational function in variable $u$ (see (2.1)), it is not hard to find to the coefficient of the generating Dirichlet series for $\lambda_A$ with power $u^n$ for a fixed non-negative integer $n$, which contains the information about the sum $\sum_{N \in M_n} \lambda_A(N)$ we are interested in.

**Theorem 1.12.** Let $A = \{P_1, P_2, \ldots, P_k\}$ be a finite set of prime polynomials and $n$ a non-negative integer. Denote $L_n = \sum_{N \in M_n} \lambda_A(N)$ and for all $1 \leq \ell \leq k$ and $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ define positive integers

$$d_{i_1, i_2, \ldots, i_\ell} = d(P_{i_1}) + d(P_{i_2}) + \cdots + d(P_{i_\ell}).$$

Order the elements of the set $\{d_{i_1, i_2, \ldots, i_\ell} \mid 1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k\}$ in the ascending order as

$$D_1 < D_2 < \cdots < D_s$$

and define the integers $\alpha_j$ and $\beta_j$, $1 \leq j \leq s$ as

$$\alpha_j = \#\{1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k \mid d_{i_1, i_2, \ldots, i_\ell} = D_j\}$$

$$\beta_j = \sum_{1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k, d_{i_1, i_2, \ldots, i_\ell} = D_j} (-1)^\ell.$$

Then, depending on $n$ we have the following linear recurrence relations for $L_n$:

$$L_n = q^n, \quad 0 \leq n < D_1,$$

$$L_n + \alpha_1 L_{n-D_1} = q^n + \beta_1 q^{n-D_1}, \quad D_1 \leq n < D_2,$$

$$\vdots$$

$$L_n + \alpha_1 L_{n-D_1} + \cdots + \alpha_j L_{n-D_j} = q^n + \beta_1 q^{n-D_1} + \cdots + \beta_j q^{n-D_j},$$

$$D_j \leq n < D_{j+1},$$

$$\vdots$$

$$L_n + \alpha_1 L_{n-D_1} + \cdots + \alpha_s L_{n-D_s} = q^n + \beta_1 q^{n-D_1} + \cdots + \beta_s q^{n-D_s}, \quad D_s \leq n.$$

As an immediate consequence of the previous theorem, we can see that $L_n$ depends only on the degrees of polynomials in $A$. Furthermore, the obtained system of recurrence relations is not difficult to solve in practice. First $s - 1$ recurrences are easily solved, as they are valid only in the limited ranges. For the final relation, note that $D_s$ is always equal to $d(P_1) + d(P_2) + \cdots + d(P_k)$,
allowing us to rewrite this relation as

\[ L_n + \alpha_1 L_{n-D_1} + \cdots + \alpha_s L_{n-D_s} = q^n \prod_{i=1}^{k} \left( 1 - \frac{1}{|P_i|} \right). \]

This recurrence has a partial solution \( q^n \prod_{i=1}^{k} \left( \frac{|P_i| - 1}{|P_i| + 1} \right) \) and the characteristic equation of the corresponding homogenous equation is

\[ \prod_{i=1}^{k} \left( X^{d(P_i)} + 1 \right) = 0. \]

From these two facts one can obtain the formula for \( \sum_{N \in \mathcal{M}_n} \lambda_d(N) \) in the range \( D_s \leq n \). Exact calculations in two concrete cases were carried out at the end of Section 7.

2. Background on function fields

For a non-negative integer \( n \) we define

\[ \mathcal{M}_{>n} = \{ N \in \mathcal{M} \mid d(N) > n \} . \]

The zeta function of \( \mathbb{F}_q[t] \) is defined by

\[ \zeta(s) = \sum_{N \in \mathcal{M}} \frac{1}{|N|^s} = \prod_{P \in \mathcal{P}} \left( 1 - \frac{1}{|P|^s} \right)^{-1}, \quad \Re(s) > 1. \]

The number of polynomials in \( \mathcal{M}_n \) is \( q^n \), so it follows

\[ (2.1) \quad \zeta(s) = \frac{1}{1 - q^{1-s}}, \]

providing the analytic continuation for \( \zeta(s) \) to the whole complex plane, with a simple pole at \( s = 1 \).

The Dirichlet’s theorem on prime polynomials in arithmetic progressions states that given two coprime polynomials \( A \) and \( M \) in \( \mathbb{F}_q[t] \), \( d(M) \geq 1 \), there exist infinitely many prime polynomials \( P \) such that \( P \equiv A \pmod{M} \).

If \( P \) is a prime polynomial in \( \mathbb{F}_q[t] \), then the Legendre symbol \( \left( \frac{N}{P} \right) \) is defined as

\[ \left( \frac{N}{P} \right) = \begin{cases} 1, & \text{if } N \text{ is square modulo } P, P \nmid N, \\ -1, & \text{if } N \text{ is not square modulo } P, P \nmid N, \\ 0, & \text{if } P \mid N. \end{cases} \]

The Jacobi symbol is defined by extending the Legendre symbol multiplicatively; if \( Q = P_1^{e_1}P_2^{e_2} \cdots P_k^{e_k} \) is the prime factorisation of polynomial \( Q \in \mathcal{M} \), then \( \left( \frac{N}{Q} \right) = \prod_{i=1}^{k} \left( \frac{N}{P_i} \right)^{e_i} \). This symbol satisfies the reciprocity formula analogous to the classical law of quadratic reciprocity: if \( M \) and \( N \) are two non-zero,
relatively prime polynomials in $\mathbb{F}_q[t]$, then
\begin{equation}
\left( \frac{M}{N} \right) = \left( \frac{N}{M} \right) (-1)^{\frac{(q-1)d(M)d(N)}{2}}.
\end{equation}

The quadratic character $\chi_D$ corresponding to a polynomial $D \in \mathcal{M}$ is defined as
\[ \chi_D(N) = \left( \frac{D}{N} \right), \quad \text{for all } N \in \mathcal{M}. \]

The $L$-function associated to a quadratic character $\chi_D$ is defined by
\[ L(s, \chi_D) = \sum_{N \in \mathcal{M}} \frac{\chi_D(N)}{|N|^s} = \prod_{P \in \mathcal{P}} \left( 1 - \chi_D(P)|P|^{-s} \right)^{-1}. \]
This function is initially defined for $\Re(s) > 1$, but it can be analytically continued to an entire function, as it is a polynomial in $q^{-s}$ (see [8, Proposition 4.3] for details).

3. Proof of Theorem 1.6

The generating Dirichlet series for the Liouville function can be written as
\begin{equation}
\sum_{N \in \mathcal{M}} \frac{\lambda(N)}{|N|^s} = \sum_{n=0}^{\infty} \left( \sum_{N \in \mathcal{M}_n} \lambda(N) \right) u^n,
\end{equation}
where $u = q^{-s}$. On the other hand, since the Liouville function is completely multiplicative, that Dirichlet series has the Euler product expression
\[ \prod_{P \in \mathcal{P}} \left( 1 + \frac{1}{|P|^s} \right)^{-1} = \prod_{P \in \mathcal{P}} \frac{1 - \frac{1}{|P|^s}}{1 - \frac{1}{|P|^s}} = \frac{Z(2s)}{Z(s)} = \frac{1 - qu}{1 - qu^2}, \]
after using (2.1) and inserting the change of variables $u = q^{-s}$. Expressing $\frac{1}{1 - qu^2}$ as power series yields
\[ (1 - qu) \sum_{n=0}^{\infty} q^n u^{2n} = \sum_{n=0}^{\infty} q^n u^{2n} - \sum_{n=0}^{\infty} q^{n+1} u^{2n+1}. \]
Comparing this to (3.1) gives us
\begin{equation}
\sum_{N \in \mathcal{M}_n} \lambda(N) = \begin{cases} 
q^{n/2}, & \text{if } n \text{ is even}, \\
-q^{(n+1)/2}, & \text{if } n \text{ is odd}.
\end{cases}
\end{equation}

We now turn to
\[ \sum_{N \in \mathcal{M} \leq x} \frac{\lambda(N)}{|N|^s} = \sum_{n=0}^{x} \left( \sum_{N \in \mathcal{M}_n} \lambda(N) \right) q^{-ns}. \]
Using (3.2), it follows that the sum over even $0 \leq n \leq x$ is
\[
1 + q^{1 - 2\alpha} + q^{2(1 - 2\alpha)} + \ldots + q^{\lfloor x/2 \rfloor (1 - 2\alpha)}.
\]
Similarly, the sum over odd $0 \leq n \leq x$ is
\[
-q^{1 - \alpha} \left( 1 + q^{1 - 2\alpha} + q^{2(1 - 2\alpha)} + \ldots + q^{\lfloor (x - 1)/2 \rfloor (1 - 2\alpha)} \right).
\]
If $\alpha \neq \frac{1}{2}$ we get
\[
\sum_{N \in M \leq x} \frac{\lambda(N)}{|N|^\alpha} = \frac{1 - q^{\lfloor (x/2) + 1 \rfloor (1 - 2\alpha)}}{1 - q^{1 - 2\alpha}} - q^{1 - \alpha} \frac{1 - q^{\lfloor (x - 1)/2 \rfloor (1 - 2\alpha)}}{1 - q^{1 - 2\alpha}}
\]
\[
= \begin{cases}
\frac{1 - q^{1 - \alpha}}{1 - q^{1/2}} \left( 1 - q^{(1/2 - \alpha)} \right) + q^{(1/2 - \alpha)}, & \text{if } x \text{ is even} \\
\frac{1 - q^{1 - \alpha}}{1 - q^{1/2}} \left( 1 - q^{(x + 1)(1/2 - \alpha)} \right), & \text{if } x \text{ is odd}
\end{cases}
\]
If $\alpha = 1/2$ we get
\[
\sum_{N \in M \leq x} \frac{\lambda(N)}{|N|^{1/2}} = \lfloor x/2 \rfloor + 1 - q^{1/2} \lfloor (x - 1)/2 \rfloor + 1
\]
and the result follows.

4. Positivity of truncations to $L(1, \chi)$

We begin with an auxiliary result, after which we turn to the proof of Theorem 1.8.

**Proposition 4.1.** Let $f : \mathcal{M} \to \mathbb{R}$ be a function and put $g(N) = \sum_{D \mid N} f(D)$ for all $N \in \mathcal{M}$. Then for all positive integers $n$ we have
\[
\sum_{N \in \mathcal{M} \leq n} \frac{f(N)}{|N|} = \frac{1}{q^n} \sum_{N \in \mathcal{M} \leq n} g(N).
\]

**Proof.** We have
\[
\sum_{N \in \mathcal{M} \leq n} g(N) = \sum_{N \in \mathcal{M} \leq n} \sum_{D \mid N} f(D)
\]
\[
= \sum_{D \in \mathcal{M} \leq n} f(D) \sum_{N \in \mathcal{M} \leq n : D \mid N} 1
\]
\[
= q^n \sum_{D \in \mathcal{M} \leq n} \frac{f(D)}{|D|}
\]
and the result follows.
Previous proposition has an important immediate consequence. Suppose that $\sum_{N \in M} |f(N)| < \infty$. By Proposition 4.1, for any positive integer $n$ we can write
\[
\frac{1}{q^n} \sum_{N \in M_n} g(N) = \sum_{D \in M} \frac{f(D)}{|D|} + O \left( \sum_{D \in M_{>n}} \frac{|f(D)|}{|D|} \right).
\]

Convergence of $\sum_{N \in M} |f(N)|$ assures that the error term vanishes as $n \to \infty$, thus showing that the mean values of $g$ exists,
\[
(4.1) \lim_{n \to \infty} \frac{1}{q^n} \sum_{N \in M_n} g(N) = \sum_{D \in M} \frac{f(D)}{|D|}.
\]

This result can be seen as the $F_q[t]$-analogue of the classical theorem of Wintner. Note that the assumption of multiplicativity of $g$ is not necessary for (4.1). However, under that assumption, $f$ must also be multiplicative, so we can express the mean value of $g$ in the form of Euler product
\[
\prod_{p \in \mathcal{P}} \left(1 - \frac{1}{|p|}\right) \left(\sum_{j=0}^{\infty} \frac{g(p^j)}{|p|^j}\right).
\]

More on this topic will be discussed in Section 6.

**Proof of Theorem 1.8.** We first note that if $f : M \to [-1, 1]$ is a completely multiplicative function, then $g(N) = \sum_{D | N} f(D)$ is a non-negative multiplicative function. Using this fact and Proposition 4.1 we get that for every quadratic character $\chi$ and non-negative integer $x$
\[
\sum_{N \in M_{\leq x}} \chi(N) \geq 0.
\]

By the reciprocity law (2.2), for any fixed non-negative integer $x$ we may find an arithmetic progression modulo $\prod_{D | P} \chi(p)$ such that every prime $Q$ lying in that progression satisfies $\left(\frac{P}{Q}\right) = -1$. Dirichlet’s theorem on prime polynomials in arithmetic progressions ensures that such polynomial $Q$ exists, showing that there exists a quadratic character $\psi$ such that $\psi(P) = -1 = \lambda(P)$ for all prime polynomials $P$ with $d(P) \leq x$. It follows that
\[
\sum_{N \in M_{\leq x}} \psi(N) = \sum_{N \in M_{\leq x}} \chi(N).
\]

By Theorem 1.6, we have
\[
\min_{\chi \text{ a quadratic character}} \frac{\chi(N)}{|N|} \leq \sum_{N \in M_{\leq x}} \frac{\lambda(N)}{|N|} = \begin{cases} q^{-x/2}, & \text{if } x \text{ is even}, \\ 0, & \text{if } x \text{ is odd}, \end{cases}
\]
finishing the proof.

5. Partial sums involving the Möbius function

We begin this section by noting that the generating Dirichlet series of the Möbius function $\mu$ can be expressed as

$$
\sum_{N \in \mathcal{M}} \frac{\mu(N)}{|N|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{|P|^s}\right) = \frac{1}{Z(s)} = 1 - q^{1-s}.
$$

In particular, if follows that

$$
(5.1) \quad \sum_{N \in \mathcal{M}} \frac{\mu(N)}{|N|} = 0,
$$

which can be seen the $\mathbb{F}_q[t]$-analogue of the classical result of von Mangoldt ([6]).

**Proof of Proposition 1.9.** Let $\mathcal{A}$ be the set of prime polynomials over $\mathbb{F}_q[t]$ not contained in $\mathcal{A}$. Using the Möbius inversion we can write

$$
1_{N \in \langle \mathcal{A} \rangle} = \sum_{D | N} \mu(D) = \sum_{D | N} \mu(D)1_{D \in \langle \mathcal{A} \rangle},
$$

where $1_{N \in S}$ is the indicator function of subset $S$ of $\mathcal{M}$,

$$
1_{N \in S} = \begin{cases} 
1, & \text{if } N \in S \\
0, & \text{if } N \notin S.
\end{cases}
$$

After using Proposition 4.1 we get

$$
(5.2) \quad q^x \sum_{\substack{N \in \langle \mathcal{A} \rangle \\ d(N) \leq x}} \frac{\mu(N)}{|N|} = \sum_{\substack{N \in \langle \mathcal{A} \rangle \\ d(N)=x}} 1.
$$

The right side is between 0 and $q^x$, according to how many monic polynomials of degree $x$ are in $\langle \mathcal{A} \rangle$. Thus, the result follows.

**Proof of Theorem 1.10.** We distinguish two different cases.

i) Suppose $\sum_{P \in \mathcal{A}} 1/|P| < \infty$. Then, from the monotone convergence theorem we get that

$$
\sum_{N \in \langle \mathcal{A} \rangle} \frac{1}{|N|} = \prod_{P \in \mathcal{A}} \left(1 + \frac{1}{|P| - 1}\right)
$$

is absolutely convergent. By the dominated convergence it further follows that

$$
\sum_{N \in \langle \mathcal{A} \rangle} \frac{\mu(N)}{|N|} = \prod_{P \in \mathcal{A}} \left(1 - \frac{1}{|P|}\right)
$$
is conditionally convergent, giving the desired result.

ii) Suppose that \( \sum_{P \in A} \frac{1}{|P|} = \infty \). In this case we utilize the identity (5.2) for a non-negative integer \( x \),

\[
q^x \sum_{N \in (A)} \frac{\mu(N)}{|N|} = \sum_{N \in (A)} \frac{1}{d(N) \leq x} \sum_{N \in \mathcal{M}} 1_{N \in (A)}.
\]

Since \( \sum_{P \in P} \frac{|1 - 1_{P \in \mathcal{P}}|}{|P|} = \sum_{P \in A} \frac{1}{|P|} = \infty \), by Theorem 6.1 (see Section 6 for details) the mean value of \( 1_{N \in (A)} \) is 0, so the right side is \( o(q^x) \).

If follows that

\[
\sum_{N \in (A)} \frac{\mu(N)}{|N|} d(N) \leq x = o(1),
\]

finishing the proof.

\[\Box\]

6. The mean value of the Liouville function \( \lambda_A \)

The important piece of information in finding the mean value of \( \lambda_A \) for a fixed set of prime polynomials \( A \),

\[
\lim_{n \to \infty} \frac{1}{q^n} \sum_{N \in M_n} \lambda_A(N),
\]

will be the generating Dirichlet series for \( \lambda_A \). Since \( \lambda_A \) is completely multiplicative, for \( \Re(s) > 1 \) and any set \( A \) we have

\[
\sum_{N \in M} \frac{\lambda_A(N)}{|N|^s} = \prod_{P \in P} \left( 1 - \frac{\lambda_A(P)}{|P|^s} \right)^{-1} \prod_{P \notin A} \left( 1 - \frac{1}{|P|^s} \right)^{-1} \prod_{P \in A} \left( 1 + \frac{1}{|P|^s} \right)^{-1} \prod_{P \notin \mathcal{P}} \left( 1 - \frac{1}{|P|^s} \right)^{-1} = \prod_{P \in A} \frac{|P|^s - 1}{|P|^s + 1} \zeta(s).
\]

We can get some immediate information about the mean value of \( \lambda_A \) from (4.1). However, in order to get the complete picture, we need the following stronger result. It is a direct consequence of the general Halász type theorem from [5, Theorem 1.4.1].
Theorem 6.1 (Klurman). Suppose \( f : \mathcal{M} \to \mathbb{R} \) is a multiplicative function taking values in \([-1, 1]\).

- If \( \sum_{P \in \mathcal{P}} \frac{1 - |f(P)|}{|P|} \) converges then the mean value of \( f \) exists and is equal to
  \[
  \prod_{P \in \mathcal{P}} \left( 1 - \frac{1}{|P|} \right) \left( \sum_{j=0}^{\infty} \frac{f(P^j)}{|P|^j} \right).
  \]

- If \( \sum_{P \in \mathcal{P}} \frac{1 - |f(P)|}{|P|} \) diverges then the mean value of \( f \) is equal to zero.

Note that \( \sum_{P \in \mathcal{P}} \frac{|1 - \lambda_A(P)|}{|P|} = 2 \sum_{P \in \mathcal{A}} \frac{1}{|P|} \) and, by (6.1),
\[
\prod_{P \in \mathcal{P}} \left( 1 - \frac{1}{|P|} \right) \left( \sum_{j=0}^{\infty} \frac{\lambda_A(P^j)}{|P|^j} \right) = \prod_{P \in \mathcal{A}} \frac{|P| - 1}{|P| + 1}.
\]

Thus, if \( \sum_{P \in \mathcal{A}} \frac{1}{|P|} < \infty \), from Theorem 6.1 it follows that the mean value of \( \lambda_A \) exists and is equal to \( \prod_{P \in \mathcal{A}} \frac{|P| - 1}{|P| + 1} \). Similarly, if \( \sum_{P \in \mathcal{A}} \frac{1}{|P|} = \infty \), Theorem 6.1 shows that the mean value of \( \lambda_A \) is equal to zero. This proves Theorem 1.11.

7. Exact value of \( \sum_{N \in \mathcal{M}_n} \lambda_A(N) \)

Identity (6.1) can be used to obtain the exact value of \( \sum_{N \in \mathcal{M}_n} \lambda_A(N) \) for a given finite set of primes \( A = \{P_1, P_2, \ldots, P_k\} \) and a fixed non-negative integer \( n \). Rewrite (6.1) as
\[
\prod_{i=1}^{k} \left( 1 + \frac{1}{|P_i|^s} \right) \sum_{N \in \mathcal{M}} \frac{\lambda_A(N)}{|N|^s} = \prod_{i=1}^{k} \left( 1 - \frac{1}{|P_i|^s} \right) \mathcal{Z}(s)
\]
and insert the change of variables \( u = q^{-s} \). The left side becomes
\[
\prod_{i=1}^{k} \left( 1 + u^{d(P_i)} \right) \sum_{N \in \mathcal{M}_n} \lambda_A(N) u^{d(N)} = \prod_{i=1}^{k} \left( 1 + u^{d(P_i)} \right) \sum_{n=0}^{\infty} L_n u^n,
\]
where \( L_n = \sum_{N \in \mathcal{M}_n} \lambda_A(N) \). Using (2.1), the right side becomes
\[
\prod_{i=1}^{k} \left( 1 - u^{d(P_i)} \right) \left( 1 - qu \right)^{-1} = \prod_{i=1}^{k} \left( 1 - u^{d(P_i)} \right) \sum_{n=0}^{\infty} q^{n} u^n.
\]

We thus get the equality of power series
\[
\prod_{i=1}^{k} \left( 1 + u^{d(P_i)} \right) \sum_{n=0}^{\infty} L_n u^n = \prod_{i=1}^{k} \left( 1 - u^{d(P_i)} \right) \sum_{n=0}^{\infty} q^{n} u^n.
\]
Comparing the corresponding coefficients in this equality will give us the recurrence relation for \( L_n \). In order to do that we expand

\[
\prod_{i=1}^{k} \left( 1 + u^{d(P_i)} \right) = 1 + \sum_{1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} u^{d_{i_1,i_2,\ldots,i_\ell}},
\]

where

\[
d_{i_1,i_2,\ldots,i_\ell} = d(P_{i_1}) + d(P_{i_2}) + \cdots + d(P_{i_\ell}).
\]

We order the elements of the set \( \{ d_{i_1,i_2,\ldots,i_\ell} \mid 1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k \} \) in the ascending order as \( D_1 < D_2 < \cdots < D_s \) and define \( \alpha_j, 1 \leq j \leq s \) as

\[
\alpha_j = \# \{ 1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k \mid d_{i_1,i_2,\ldots,i_\ell} = D_j \}.
\]

If follows that

\[
\prod_{i=1}^{k} \left( 1 + u^{d(P_i)} \right) = 1 + \sum_{j=1}^{s} \alpha_j u^{D_j}.
\]

Similarly,

\[
\prod_{i=1}^{k} \left( 1 - u^{d(P_i)} \right) = 1 + \sum_{j=1}^{s} \beta_j u^{D_j},
\]

where

\[
\beta_j = \sum_{1 \leq \ell \leq k, 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k, d_{i_1,i_2,\ldots,i_\ell} = D_j} (-1)^\ell.
\]

Returning to (7.1) we get

\[
(7.2) \quad \sum_{n=0}^{\infty} L_n u^n + \sum_{j=1}^{s} \sum_{n=0}^{\infty} \alpha_j L_n u^{n+D_j} = \sum_{n=0}^{\infty} q^n u^n + \sum_{j=1}^{s} \sum_{n=0}^{\infty} \beta_j q^n u^{n+D_j}.
\]

Finally, noting that \( \sum_{n=0}^{\infty} \alpha_j L_n u^{n+D_j} = \sum_{n=0}^{\infty} \sum_{j=1}^{s} \alpha_j L_{n-D_j} u^n \) and \( \sum_{n=0}^{\infty} \beta_j q^n u^{n+D_j} = \sum_{n=0}^{\infty} \sum_{j=1}^{s} \beta_j q^{n-D_j} u^n \) for all \( 1 \leq j \leq s \) and comparing the coefficients on both sides of (7.2) yields Theorem 1.12.

We end this section with the examples of calculations of \( \sum_{N \in \mathcal{M}_n} \lambda_A(N) \) in two concrete cases.

**Example 7.1.** \( A = \{ P \} \) for some prime polynomial \( P \)

In this case \( s = 1, D_1 = d(P), \alpha_1 = 1, \beta_1 = -1, \) so Theorem 1.12 gives us

\[
L_n = q^n, \quad 0 \leq n < d(P),
\]

\[
L_n + L_{n-d(P)} = q^n - q^{n-d(P)}, \quad d(P) \leq n.
\]
After solving the obtained recurrence relation we get
\[
\sum_{N \in M_n} \lambda_A(N) = \begin{cases} 
q^n, & 0 \leq n < d(P), \\
q^n \left(1 - 2^{1 - \left(\frac{m}{|P|+1}\right)\left[\frac{m}{|P|}\right]}\right), & d(P) \leq n.
\end{cases}
\]

**Example 7.2.** \(A = \{P, Q\}\) for some prime polynomials \(P\) and \(Q\) of the same degree \(m\).

In this case \(s = 2\), \(D_1 = m\), \(D_2 = 2m\), \(\alpha_1 = 2\), \(\alpha_2 = 1\), \(\beta_1 = -2\), \(\beta_2 = 1\), so Theorem 1.12 gives us
\[
L_n = q^n, \quad 0 \leq n < m,
\]
\[
L_n + 2L_{n-m} = q^n - 2q^{n-m}, \quad m \leq n < 2m,
\]
\[
L_n + 2L_{n-m} + L_{n-2m} = q^n - 2q^{n-m} + q^{n-2m}, \quad 2m \leq n.
\]

After solving the obtained recurrence relations we get

\[
\sum_{N \in M_n} \lambda_A(N) = \begin{cases} 
q^n, & 0 \leq n < m, \\
q^n - 4q^{n-m}, & m \leq n < 2m, \\
q^nV_m + q^{n-m}\left[\frac{m}{|P|}\right] (1 - V_m) \left(\frac{m}{|P|}\right)^{-1} \left(\frac{m}{|P|}q^m + \left[\frac{m}{|P|}\right] - 1\right) \\
+4q^{n-m}\left[\frac{m}{|P|}\right] (1 - V_m) \left[\frac{m}{|P|}\right] (1 - \left[\frac{m}{|P|}\right]) - 1, & 2m \leq n,
\end{cases}
\]

where \(V_m = \left(\frac{q^m - 1}{q^m + 1}\right)^2\).

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