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A RESULT RELATED TO DERIVATIONS ON UNITAL SEMIPRIME RINGS

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Abstract. The purpose of this paper is to prove the following result. Let $n \geq 3$ be some fixed integer and let R be a $(n+1)!2^{n-2}$-torsion free semiprime unital ring. Suppose there exists an additive mapping $D\colon R \to R$ satisfying the relation

$$\begin{aligned} &2^{n-2}D(x^n) \\ &\quad = \left(\sum_{i=0}^{n-2} \binom{n-2}{i} x^i D(x^2) x^{n-2-i}\right) + (2^{n-2}-1)(D(x)x^{n-1} + x^{n-1}D(x)) \\ &\quad\quad + \sum_{i=1}^{n-2} \left(\sum_{k=2}^{i} (2^{k-1}-1)\binom{n-k-2}{i-k} + \sum_{k=2}^{n-1-i} (2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right) \\ &\quad\quad\quad x^i D(x) x^{n-1-i} \end{aligned}$$

for all $x \in R$. In this case D is a derivation. The history of this result goes back to a classical result of Herstein, which states that any Jordan derivation on a 2-torsion free prime ring is a derivation.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$.

An additive mapping $D\colon R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A derivation is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping $D\colon R \to R$ is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Every derivation is a

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Jordan derivation. The converse is in general not true. A classical result of Herstein ([13]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein theorem can be found in [3]. Cusack ([6]) has generalized Herstein theorem to 2-torsion free semiprime rings (see [4] for an alternative proof). Beidar, Brešar, Chebotar and Martindale ([2]) have fairly generalized Herstein theorem.

Brešar ([5]) has proved the following result (see [16] for a generalization).

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $D\colon R \to R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all $x, y \in R$. In this case D is a derivation.

An additive mapping D, which maps an arbitrary ring R into itself and satisfies the relation (1) for all pairs $x, y \in R$, is called a Jordan triple derivation. One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring is a Jordan triple derivation, which means that Theorem 1 generalizes Cusack's generalization of Herstein theorem.

The above result represents a motivation for many other results (for example [10, 14, 17]). Vukman ([18]) conjectured that in case there exists an additive mapping $D\colon R \to R$, where R is a 2-torsion free semiprime ring, satisfying the relation

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx) \tag{2}$$

for all $x, y \in R$, then D is a derivation. Putting x for y in relations (1) and (2) leads to

$$D(x^3) = D(x)x^2 + xD(x)x + x^2D(x), \tag{3}$$

$$2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2) \tag{4}$$

for all $x \in R$. Recently, M. Fošner and the authors ([11]) proved the following result regarding the relation (4), which is related to the Vukman's conjecture mentioned above.

Theorem 2. *Let R be a 2-torsion free prime ring and let $D\colon R \to R$ be an additive mapping satisfying the relation*

$$2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2)$$

for all $x \in R$. In this case D is a derivation.

The relation (3) leads to the relation

$$D(x^n) = \sum_{i=1}^{n} x^{i-1}D(x)x^{n-i},$$

which was studied on prime rings by Beidar, M. Brešar, Chebotar and Martindale ([2]) (see also [15] for the result regarding the above relation on unital semiprime rings). It is our aim in this paper to obtain and study the relation, which generalizes the relation (4).

Putting $x, x^2, x^3, \dots, x^{n-2}$ for y in (2) leads to the following system of relations, respectively.

$$2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2), \tag{5}$$

$$4D(x^4) = 2D(x^3)x + 2x^3D(x) + 2D(x)x^3 + 2xD(x^3), \tag{6}$$

$$8D(x^5) = 4D(x^4)x + 4x^4D(x) + 4D(x)x^4 + 4xD(x^4), \tag{7}$$

$$16D(x^6) = 8D(x^5)x + 8x^5D(x) + 8D(x)x^5 + 8xD(x^5), \tag{8}$$

$$\vdots$$

$$\begin{aligned} 2^{n-2}D(x^n) &= 2^{n-3}D(x^{n-1})x + 2^{n-3}x^{n-1}D(x) + 2^{n-3}D(x)x^{n-1} \\ &\quad + 2^{n-3}xD(x^{n-1}). \end{aligned} \tag{9}$$

Considering (5) in (6) leads to

$$\begin{aligned} 4D(x^4) &= 3D(x)x^3 + xD(x)x^2 + x^2D(x)x + 3x^3D(x) + D(x^2)x^2 \\ &\quad + 2xD(x^2)x + x^2D(x^2). \end{aligned}$$

Putting the above relation in the relation (7) we obtain

$$\begin{aligned} 8D(x^5) &= 7D(x)x^4 + 4xD(x)x^3 + 2x^2D(x)x^2 + 4x^3D(x)x + 7x^4D(x) \\ &\quad + D(x^2)x^3 + 3xD(x^2)x^2 + 3x^2D(x^2)x + x^3D(x^2). \end{aligned}$$

Considering the above relation in (8) we obtain

$$\begin{aligned} 16D(x^6) &= 15D(x)x^5 + 11xD(x)x^4 + 6x^2D(x)x^3 + 6x^3D(x)x^2 + 11x^4D(x)x \\ &\quad + 15x^5D(x) + D(x^2)x^4 + 4xD(x^2)x^3 + 6x^2D(x^2)x^2 + 4x^3D(x^2)x \\ &\quad + x^4D(x^2). \end{aligned}$$

We see that in the above relations the coefficients of terms including $D(x^2)$ follow as $(1,1)$, $(1,2,1)$, $(1,3,3,1)$, $(1,4,6,4,1)$ and therefore form a Pascal triangle. The coefficients of terms including $D(x)$ follow as $(1,0,1)$, $(3,1,1,3)$, $(7,4,2,4,7)$, $(15,11,6,6,11,15)$. Is there any specific algorithm that can foretell the coefficients of terms including $D(x)$ as we proceed with the above procedure? It turns out that the answer is positive. We will now present the results regarding the above speculations.

Pellicer and Alvo ([1]) delivered the following definition.

Definition 3. *The modified Pascal triangle $P(m,n)$ generated by sequences $\{a_m\}$ and $\{b_n\}$ is determined by relations:*

(i) $P(m,0) = a_m$,
(ii) $P(0,n) = b_n$,
(iii) $P(m,n) = P(m,n-1) + P(m-1,n)$

for all $m, n \in \mathbb{N}$.

Below we can see the scheme of the modified Pascal triangle, which is generated by the sequences $\{a_m\}$ and $\{b_n\}$.

$$\begin{array}{ccccccc}
 & P(0,1) & P(0,2) & P(0,3) & P(0,4) & P(0,5) & b_n \;\rightarrow \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) & P(1,4) & \ddots & \\
P(2,0) & P(2,1) & P(2,2) & P(2,3) & \ddots & & \\
P(3,0) & P(3,1) & P(3,2) & \ddots & & & \\
P(4,0) & P(4,1) & \ddots & & & & \\
P(5,0) & \ddots & & & & & \\
a_m & & & & & & \\
\downarrow & & & & & &
\end{array}$$

In the same paper Pellicer and Alvo proved the following theorem, which states that given the generating recursive additive pattern of a modified Pascal triangle and given the border sequences, the inside triangle is uniquely determined.

THEOREM 4. *Given a modified Pascal triangle* P *it holds that*

$$P(m,n) = \sum_{k=1}^{m} a_k \binom{m+n-k-1}{m-k} + \sum_{k=1}^{n} b_k \binom{m+n-k-1}{n-k}$$

for all $m, n \in \mathbb{N}$.

We can now proceed with the work regarding the system of relations (5), (6), (7), (8), (9). We have already mentioned that the coefficients for all terms including $D(x^2)$ form a Pascal triangle. According to the theory above, the coefficients for all terms including $D(x)$ form the following modified Pascal triangle, generated by the sequences

$$\{a_m\} = \{b_n\} = (0, 1, 3, 7, 15, \ldots, 2^{i-1} - 1, \ldots),\ i = 1, 2, 3 \ldots.$$

$$\begin{array}{ccccccc}
 & 0 & 1 & 3 & 7 & 15 & 31 \;\cdots \\
0 & 0 & 1 & 4 & 11 & 26 & \ddots \\
1 & 1 & 2 & 6 & 17 & \ddots & \\
3 & 4 & 6 & 12 & \ddots & & \\
7 & 11 & 17 & \ddots & & & \\
15 & 26 & \ddots & & & & \\
31 & \ddots & & & & & \\
\vdots & & & & & &
\end{array}.$$

According to Theorem 4, the induction and (9) lead to

$$\begin{aligned}
&2^{n-2}D(x^n)\\
&=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+\sum_{i=0}^{n-1}P(i,n-1-i)x^iD(x)x^{n-1-i}\\
&=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+P(0,n-1)D(x)x^{n-1}+P(n-1,0)x^{n-1}D(x)\\
&\quad+\sum_{i=1}^{n-2}\left(\sum_{k=1}^{i}a_k\binom{i+n-1-i-k-1}{i-k}+\sum_{k=1}^{n-1-i}b_k\binom{i+n-1-i-k-1}{n-1-i-k}\right)x^iD(x)x^{n-1-i}\\
&=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+(2^{n-2}-1)(D(x)x^{n-1}+x^{n-1}D(x))\\
&\quad+\sum_{i=1}^{n-2}\left(\sum_{k=1}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}+\sum_{k=1}^{n-1-i}(2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right)x^iD(x)x^{n-1-i}\\
&=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+(2^{n-2}-1)(D(x)x^{n-1}+x^{n-1}D(x))\\
&\quad+\sum_{i=1}^{n-2}\left(\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}+\sum_{k=2}^{n-1-i}(2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right)x^iD(x)x^{n-1-i}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&2^{n-2}D(x^n)\\
&=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+(2^{n-2}-1)(D(x)x^{n-1}+x^{n-1}D(x))\\
&\quad+\sum_{i=1}^{n-2}\left(\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}+\sum_{k=2}^{n-1-i}(2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right)x^iD(x)x^{n-1-i}.
\end{aligned}$$

Let us note that $\sum_{k=2}^{1}f(k)=0$, for any function f.

We have to remark that the additive solutions of $\sum_{i=0}^{n}c_ix^iD(x^{n-i})=0$ for fixed constants $c_i\in R$, where R is a commutative ring, was partially characterized by Ebanks ([7]) and Ebanks et al. ([8]). Later, the problem was fully solved by Gselmann et al. ([12]) and Ebanks ([9]) independently using different approach.

It is our aim in this paper to prove the following result.

Theorem 5. *Let $n\geq 3$ be some fixed integer, let R be a $(n+1)!2^{n-2}$-torsion free unital semiprime ring and let $D\colon R\to R$ be an additive mapping* *satisfying the relation*

$$2^{n-2}D(x^n)$$
$$=\sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i}+(2^{n-2}-1)(D(x)x^{n-1}+x^{n-1}D(x))$$
$$+\sum_{i=1}^{n-2}\left(\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}+\sum_{k=2}^{n-1-i}(2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right)x^iD(x)x^{n-1-i}$$

for all $x\in R$. In this case D is a derivation.

For the seek of completeness, we may mention that for commutative, $(n+1)!2^{n-2}$-torsion free rings the above equation can be reduced to $(n-2)x^{n-1}D(x)+x^{n-2}D(x^2)-D(x^n)=0$. By [8], D has to be a derivation of order at most 2. Since every derivation (of order 1) is clearly a solution of this equation it is enough to verify that the equation for derivations of order 2 does not hold which can be checked by direct calculation.

For the proof of the above theorem, we will need the following lemma.

Lemma 6. *The relation*

$$\sum_{i=2}^{n-2}\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}=(n-4)2^{n-3}+1$$

holds for all integers $n\geq 3$.

Proof. Considering $\sum_{k=0}^{n}\binom{n}{k}=2^n$ in the next calculation proves the lemma.

$$\sum_{i=2}^{n-2}\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}$$
$$=\sum_{k=2}^{2}(2^{k-1}-1)\binom{n-k-2}{2-k}+\sum_{k=2}^{3}(2^{k-1}-1)\binom{n-k-2}{3-k}+\cdots$$
$$+\sum_{k=2}^{n-3}(2^{k-1}-1)\binom{n-k-2}{n-k-3}+\sum_{k=2}^{n-2}(2^{k-1}-1)\binom{n-k-2}{n-k-2}$$
$$=(2^1-1)\left(\binom{n-4}{0}+\binom{n-4}{1}+\cdots+\binom{n-4}{n-4}\right)$$
$$+(2^2-1)\left(\binom{n-5}{0}+\binom{n-5}{1}+\cdots+\binom{n-5}{n-5}\right)+\cdots$$
$$+(2^{n-4}-1)\left(\binom{1}{0}+\binom{1}{1}\right)+(2^{n-3}-1)\binom{0}{0}$$
$$=(2-1)\,2^{n-4}+\left(2^2-1\right)2^{n-5}+\cdots+\left(2^{n-4}-1\right)2^1+\left(2^{n-3}-1\right)2^0$$
$$=(n-3)\,2^{n-3}-\left(1+2+2^2+2^3+\cdots+2^{n-5}+2^{n-4}\right)$$
$$=(n-3)\,2^{n-3}-\left(2^{n-3}-1\right)$$

$$= (n-4)\,2^{n-3} + 1.$$

∎

We are now in the position to prove Theorem 5.

PROOF OF THEOREM 5. We have the relation

$$\begin{aligned}
&2^{n-2}D(x^n) \\
&= \sum_{i=0}^{n-2}\binom{n-2}{i}x^iD(x^2)x^{n-2-i} + (2^{n-2}-1)(D(x)x^{n-1}+x^{n-1}D(x)) \\
&\quad+\sum_{i=1}^{n-2}\left(\sum_{k=2}^{i}(2^{k-1}-1)\binom{n-k-2}{i-k}+\sum_{k=2}^{n-1-i}(2^{k-1}-1)\binom{n-k-2}{n-i-k-1}\right)x^iD(x)x^{n-1-i}.
\end{aligned} \tag{10}$$

We will denote the identity element of the ring R by e. Putting e for x in the above relation we obtain

$$\begin{aligned}
2^{n-2}D(e) &= \sum_{i=0}^{n-2}\binom{n-2}{i}D(e) + 2(2^{n-2}-1)D(e) \\
&\quad+2\Big(\sum_{k=2}^{2}(2^{k-1}-1)\binom{n-k-2}{2-k}+\sum_{k=2}^{3}(2^{k-1}-1)\binom{n-k-2}{3-k}+\cdots \\
&\quad+\sum_{k=2}^{n-3}(2^{k-1}-1)\binom{n-k-2}{n-k-3}+\sum_{k=2}^{n-2}(2^{k-1}-1)\binom{n-k-2}{n-k-2}\Big)D(e).
\end{aligned}$$

Lemma 6 reduces the above relation to

$$2^{n-2}D(e) = 2^{n-2}D(e) + 2(2^{n-2}-1)D(e) + 2\left((n-4)\,2^{n-3}+1\right)D(e).$$

Further calculation leads to

$$(n-2)2^{n-2}D(e) = 0$$

and considering the torsion freeness, we obtain

$$D(e) = 0. \tag{11}$$

Let y be an arbitrary element from $Z(R)$. Putting $x+y$ for x in the relation (10) we obtain

$$\begin{aligned}
&2^{n-2}\sum_{i=0}^{n}\binom{n}{i}D(x^{n-i}y^i) \\
&= \binom{n-2}{0}D(x^2+2xy+y^2)\left(\sum_{i=0}^{n-2}\binom{n-2}{i}x^{n-2-i}y^i\right) \\
&\quad+\binom{n-2}{1}(x+y)\,D(x^2+2xy+y^2)\left(\sum_{i=0}^{n-3}\binom{n-3}{i}x^{n-3-i}y^i\right)
\end{aligned}$$

$$
\begin{aligned}
&+\cdots\\
&+\tbinom{n-2}{n-3}\left(\sum_{i=0}^{n-3}\tbinom{n-3}{i}x^{n-3-i}y^i\right)D(x^2+2xy+y^2)\,(x+y)\\
&+\tbinom{n-2}{n-2}\left(\sum_{i=0}^{n-2}\tbinom{n-2}{i}x^{n-2-i}y^i\right)D(x^2+2xy+y^2)\\
&+(2^{n-2}-1)\sum_{i=0}^{n-1}\tbinom{n-1}{i}(D(x+y)x^{n-1-i}y^i+x^{n-1-i}y^iD(x+y))\\
&+\left(\sum_{k=2}^{1}(2^{k-1}-1)\tbinom{n-k-2}{1-k}+\sum_{k=2}^{n-2}(2^{k-1}-1)\tbinom{n-k-2}{n-k-2}\right)\\
&\quad\cdot(x+y)D(x+y)\left(\sum_{j=0}^{n-2}\tbinom{n-2}{j}x^{n-2-j}y^j\right)\\
&\left(\sum_{k=2}^{2}(2^{k-1}-1)\tbinom{n-k-2}{2-k}+\sum_{k=2}^{n-3}(2^{k-1}-1)\tbinom{n-k-2}{n-k-3}\right)\\
&\quad\cdot(x^2+2xy+y^2)D(x+y)\left(\sum_{j=0}^{n-3}\tbinom{n-3}{j}x^{n-3-j}y^j\right)\\
&+\cdots\\
&+\left(\sum_{k=2}^{n-3}(2^{k-1}-1)\tbinom{n-k-2}{n-k-3}+\sum_{k=2}^{2}(2^{k-1}-1)\tbinom{n-k-2}{2-k}\right)\\
&\quad\cdot\left(\sum_{j=0}^{n-3}\tbinom{n-3}{j}x^{n-3-j}y^j\right)D(x+y)(x^2+2xy+y^2)\\
&+\left(\sum_{k=2}^{n-2}(2^{k-1}-1)\tbinom{n-k-2}{n-k-2}+\sum_{k=2}^{1}(2^{k-1}-1)\tbinom{n-k-2}{1-k}\right)\\
&\quad\cdot\left(\sum_{j=0}^{n-2}\tbinom{n-2}{j}x^{n-2-j}y^j\right)D(x+y)(x+y).
\end{aligned}
$$

Using (10) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1}f_i\,(x,y)=0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replacing x by $x+2y,\ x+3y,\ \ldots,\ x+(n-1)y$ in turn in the relation (10) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(x,y),\ i=1,2,...,n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular, if y is replaced with the identity element e and considering (11), we obtain

$$\begin{aligned}
&f_{n-2}(x,e) \\
&= 2^{n-2}\tbinom{n}{n-2}D(x^2) - \tbinom{n-2}{0}\left(\tbinom{n-2}{n-2}D(x^2) + 2\tbinom{n-2}{n-3}D(x)x\right) \\
&\quad - \tbinom{n-2}{1}\left(\tbinom{n-3}{n-3}D(x^2) + 2\tbinom{n-3}{n-4}D(x)x + 2\tbinom{1}{0}xD(x)\right) \\
&\quad - \tbinom{n-2}{2}\left(\tbinom{n-4}{n-4}D(x^2) + 2\tbinom{n-4}{n-5}D(x)x + 2\tbinom{2}{1}xD(x)\right) \\
&\quad - \cdots \\
&\quad - \tbinom{n-2}{n-4}\left(\tbinom{2}{2}D(x^2) + 2\tbinom{2}{1}D(x)x + 2\tbinom{n-4}{n-5}xD(x)\right) \\
&\quad - \tbinom{n-2}{n-3}\left(\tbinom{1}{1}D(x^2) + 2\tbinom{1}{0}D(x)x + 2\tbinom{n-3}{n-4}xD(x)\right) \\
&\quad - \tbinom{n-2}{n-2}\left(\tbinom{2}{2}D(x^2) + 2\tbinom{n-2}{n-3}xD(x)\right) \\
&\quad - (2^{n-2}-1)\tbinom{n-1}{n-2}\left(D(x)x + xD(x)\right) \\
&\quad - \left(\sum_{k=2}^{1}(2^{k-1}-1)\tbinom{n-k-2}{1-k} + \sum_{k=2}^{n-2}(2^{k-1}-1)\tbinom{n-k-2}{n-k-2}\right) \\
&\qquad \cdot \left(\tbinom{n-2}{n-3}D(x)x + \tbinom{1}{0}xD(x)\right) \\
&\quad - \left(\sum_{k=2}^{2}(2^{k-1}-1)\tbinom{n-k-2}{2-k} + \sum_{k=2}^{n-3}(2^{k-1}-1)\tbinom{n-k-2}{n-k-3}\right) \\
&\qquad \cdot \left(\tbinom{n-3}{n-4}D(x)x + \tbinom{2}{1}xD(x)\right) \\
&\quad - \left(\sum_{k=2}^{3}(2^{k-1}-1)\tbinom{n-k-2}{3-k} + \sum_{k=2}^{n-4}(2^{k-1}-1)\tbinom{n-k-2}{n-k-4}\right) \\
&\qquad \cdot \left(\tbinom{n-4}{n-5}D(x)x + \tbinom{3}{2}xD(x)\right) \\
&\quad - \cdots
\end{aligned}$$

$$
\begin{aligned}
&-\left(\sum_{k=2}^{n-4}(2^{k-1}-1)\binom{n-k-2}{n-k-4}+\sum_{k=2}^{3}(2^{k-1}-1)\binom{n-k-2}{3-k}\right)\\
&\quad\cdot\left(\binom{3}{2}D(x)x+\binom{n-4}{n-5}xD(x)\right)\\
&-\left(\sum_{k=2}^{n-3}(2^{k-1}-1)\binom{n-k-2}{n-k-3}+\sum_{k=2}^{2}(2^{k-1}-1)\binom{n-k-2}{2-k}\right)\\
&\quad\cdot\left(\binom{2}{1}D(x)x+\binom{n-3}{n-4}xD(x)\right)\\
&-\left(\sum_{k=2}^{n-2}(2^{k-1}-1)\binom{n-k-2}{n-k-2}+\sum_{k=2}^{1}(2^{k-1}-1)\binom{n-k-2}{1-k}\right)\\
&\quad\cdot\left(\binom{1}{0}D(x)x+\binom{n-2}{n-3}xD(x)\right).
\end{aligned}
$$

As the sums in the above relation are pairwise equal, the above relation reduces to

$$
\begin{aligned}
&2^{n-2}\binom{n}{n-2}D(x^2)\\
&=2^{n-2}D(x^2)\\
&\quad+2\left(\binom{n-2}{0}(n-2)+\binom{n-2}{1}(n-3)+\cdots+\binom{n-2}{n-4}2+\binom{n-2}{n-3}\right)\\
&\quad\quad\cdot(D(x)x+xD(x))\\
&\quad+\left(2^{n-2}-1\right)(n-1)(D(x)x+xD(x))\\
&\quad+(n-1)\left(\sum_{k=2}^{1}(2^{k-1}-1)\binom{n-k-2}{1-k}+\sum_{k=2}^{2}(2^{k-1}-1)\binom{n-k-2}{2-k}\right.\\
&\quad+\cdots\\
&\quad\left.+\sum_{k=2}^{n-3}(2^{k-1}-1)\binom{n-k-2}{n-k-3}+\sum_{k=2}^{n-2}(2^{k-1}-1)\binom{n-k-2}{n-k-2}\right)(D(x)x+xD(x)).
\end{aligned}
$$

Using Lemma 6 in the above relation and some calculation leads to

$$
\begin{aligned}
&2^{n-2}\left(\binom{n}{n-2}-1\right)D(x^2)\\
&=2\sum_{k=0}^{n-3}\binom{n-2}{k}(n-2-k)(D(x)x+xD(x))\\
&\quad+(n-1)(2^{n-2}-1)(D(x)x+xD(x))\\
&\quad+(n-1)\left((n-4)2^{n-3}+1\right)(D(x)x+xD(x)).
\end{aligned}
\tag{12}
$$

Since

$$\begin{aligned}\sum_{k=0}^{n-3}\binom{n-2}{k}(n-2-k) &= \sum_{k=0}^{n-2}\binom{n-2}{k}(n-2-k)\\ &= \sum_{k=0}^{n-2}\binom{n-2}{k}(n-2)-\sum_{k=0}^{n-2}\binom{n-2}{k}k\\ &= (n-2)2^{n-2}-(n-2)2^{n-3}=(n-2)2^{n-3},\end{aligned}$$

the relation (12) can be rewritten as

$$\begin{aligned}&2^{n-3}(n+1)(n-2)D(x^2)\\ &=\big((n-2)2^{n-2}+(n-1)(2^{n-2}+(n-4)2^{n-3})\big)(D(x)x+xD(x))\\ &=((n-2)2^{n-2}+(n-1)(n-2)2^{n-3})(D(x)x+xD(x))\\ &=2^{n-3}(n+1)(n-2)(D(x)x+xD(x)).\end{aligned}$$

We therefore have

$$2^{n-3}(n+1)(n-2)D(x^2)=2^{n-3}(n+1)(n-2)(D(x)x+xD(x)).$$

Since R is $(n+1)!2^{n-2}$-torsion free, the above relation reduces to

$$D(x^2)=D(x)x+xD(x)$$

for all $x\in R$. In other words, D is a Jordan derivation on R. According to Cusack's generalization of Herstein theorem, one can conclude that D is a derivation, which completes the proof of the theorem. □

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