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UNITARY DUAL OF p-ADIC GROUP $SO(7)$ WITH SUPPORT ON MINIMAL PARABOLIC SUBGROUP

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Abstract. In this paper, the unitary dual of p-adic group $SO(7)$ with support on minimal parabolic subgroup is determined. In explicit determination of the unitary dual the external approach is used, which represents the basic approach for finding the unitary dual, and consists of two main steps: a complete description of the non-unitary dual and the extraction of the classes of unitarizable representations among the obtained irreducible subquotients. We expect that our results will provide deeper insight into the structure of the unitary dual in the general case.

1. Introduction

The unitary dual determination problem of a reductive algebraic group over a local non-archimedean field F is one of the most important problems in representation theory. Denote by $\widetilde{G}$ set of all irreducible smooth representations of a reductive p-adic group G, and by $\widehat{G}$ subset of all unitarizable classes. Then $\widetilde{G}$ is called non-unitary dual of G, and $\widehat{G}$ unitary dual of G. Thus, the unitary dual determination problem is the problem of finding the subset $\widehat{G}$ of $\widetilde{G}$. Unitary dual of general linear group of arbitrary rank over a local non-archimedean field was found by Tadić in [18]. In that work the external approach is used to this end, in which the first step is to give a complete description of the non-unitary dual using Langlands classification. The second step is to find classes of unitarizable representations among irreducible subquotients.

Since then, for some groups of rank two unitary duals were found, e.g. the group $Sp(4, F)$ in [17], the group G_2 in [15], the group $U(2, 2)$ in [8], the

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Hermitian quaternionic group of split rank 2 in [6], and the group $SO(5, F)$ in [9]. However, not much is known regarding the solution of the unitarizability problem for classical groups in the general case. As the natural next step leading towards its solution, one that would provide deeper insight into the wanted structure, studying induced representations of classical groups of rank three is imposed. Thusly, apart from enabling the study of induced representations of groups of small rank, one could also get a better insight into the structure of the unitary dual in the general case.

Reducibility of parabolically induced representations falls amongst more prominent problems in representation theory. One critical utilisation of reducibility is at the determination of the unitary dual of classical p-adic groups. From newer works of Arthur, Mœglin and Waldspurger it is now known that, in the local field of characteristic zero, cuspidal reducibility points come from $\frac{1}{2}\mathbb{Z}$, and the proof, for the fields of characteristic 0, can be found in [13, Théorème 3.1.1.], while for the fields of positive characteristic in [4, Theorem 7.8.] in the case of classical groups.

Today, classifications of tempered representations and irreducible square integrable representations from [14] and [21] can be used in describing the non-unitary dual, along with the known Jacquet modules methods from [3] and [19], intertwining operator methods and R-groups from [5].

In the way established in [16] and further developed in [11], using Tadić's structural formula from [19], as a consequence of the Bernstein and Zelevinsky Geometric lemma, for calculating Jacquet modules, and using Langlands classification, we obtain the complete description of irreducible subquotients in every examined case of induced representations of p-adic group $SO(7)$.

In order to get the complete description of the unitary dual of the p-adic group $SO(7)$ with the support on the minimal parabolic subgroup, we identify unitarizable classes. That was achieved using primarily Tadić's result about the unitarizability of representations of classical p-adic groups of corank three from [22], as well as unitary duals of p-adic general linear groups from [18] and unitary dual of p-adic group $SO(5)$ from [9].

This paper is organized as follows. In Section 2 we recall the notations and the well-known facts and properties. In Section 3 technical results that will be frequently used are given. In Sections 4, 5, 6 and 7 all unitarizable subquotients are given and the proofs are provided for the most complicated cases. The results are divided according to the number of quadratic characters that occur in the induced representation of the group $SO(7, F)$, and are also further distinguished by the number of isomorphic characters. In the case of three quadratic characters an additional separation dependant on the number of $\frac{1}{2}$ that occur in the exponents is made.

2. Preliminaries

Let F be a local non-archimedean field of characteristic 0. Odd special orthogonal group $SO(2n+1,F)$ is the group of all $(2n+1)\times(2n+1)$ matrices g with elements from F whose determinant is equal to 1 and which satisfy ${}^{\tau}gg = I_{2n+1}$, where I_{2n+1} is the identity matrix in $GL(2n+1,F)$ and ${}^{\tau}g$ denotes the transpose matrix of matrix g with respect to the second diagonal. The set of unitary characters of $F^{\times}$ is denoted by $\widehat{F^{\times}}$.

Fix minimal parabolic subgroup P_{min} of the $SO(2n+1,F)$ which consists of all upper triangular matrices in the group $SO(2n+1,F)$. Let P denote a standard parabolic subgroup of the $SO(2n+1,F)$, i.e. a parabolic subgroup which contains the minimal parabolic subgroup. Maximal split torus A consists of all diagonal matrices in the group.

Let $\alpha = (n_1, n_2, \ldots, n_k)$ be an ordered partition of m, where $0 < m \leq n$. Then

$$(n_1, n_2, \ldots, n_k, 2(n-m)+1, n_k, \ldots, n_2, n_1)$$

is an ordered partition of $2n+1$. It is easy to show that there exists a bijection amongst the set of all ordered partitions of all m for which $0 < m \leq n$ and the set of all proper standard parabolic subgroups of the $SO(2n+1,F)$. The parabolic subgroup P_α has Levi decomposition $P_\alpha = M_\alpha N_\alpha$. Especially, notice that M_α is naturally isomorphic to

$$GL(n_1,F)\times GL(n_2,F)\times\cdots\times GL(n_k,F)\times SO(2(n-m)+1,F).$$

Moreover, for $m \leq n$ is $M_{(m)} \cong GL(m,F)\times SO(2(n-m)+1,F)$.

For a connected reductive group G defined over F, Grothendieck group of the category of all smooth finite length representations of $G(F)$ is denoted by $R(G(F))$.

Let π_i be the representation of the group $GL(n_i,F)$, for $1 \leq i \leq k$, and let σ be the representation of the group $SO(2(n-m)+1,F)$. Then $\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k\otimes\sigma$ is the representation of the Levi factor M_α of the parabolic subgroup P_α of the $SO(2n+1,F)$. We can trivially extend it along N_α to the representation of P_α. The obtained representation is also denoted by the $\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k\otimes\sigma$. Then

$$\pi_1\times\pi_2\times\cdots\times\pi_k\rtimes\sigma = \mathrm{Ind}_{P_\alpha}^{SO(2n+1,F)}(\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k\otimes\sigma)$$

is a parabolically induced representation of the group $SO(2n+1,F)$ by representation $\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k\otimes\sigma$ from the group P_α which gives us a group homomorphism $R(M_\alpha)\to R(SO(2n+1,F))$.

Let us state several well-known facts regarding parabolically induced representations of odd special orthogonal groups. Let π_1 be a smooth representation of the group $GL(n_1,F)$, π_2 be a smooth representation of the group $GL(n_2,F)$ and σ be a smooth representation of the group $SO(2n+1,F)$.

Then from the induction in stages follows that

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma.$$

Further, if smooth representations π of $GL(m, F)$ and σ of $SO(2n+1, F)$ are of finite length, then parabolically induced representations $\pi \rtimes \sigma$ and $\widetilde{\pi} \rtimes \sigma$ have the same composition series. Also,

$$\widetilde{\pi \rtimes \sigma} \cong \widetilde{\pi} \rtimes \widetilde{\sigma}.$$

Denote by

$$R = \bigoplus_{n \geqslant 0} R_n \text{ and } S = \bigoplus_{n \geqslant 0} R(SO(2n+1, F)),$$

where $R_n = R(GL(n, F))$. For representations π of $GL(m, F)$ and σ of $SO(2n+1, F)$ we have association $(\pi, \sigma) \mapsto \pi \rtimes \sigma$, that is defined with the parabolic induction. Then, in a natural way, one gets the mapping $\rtimes\colon R \times S \to S$ and S is then R-module. Further, $\rtimes$ can bilinearly extend to $R \otimes S$. Now for irreducible representation σ of the group $SO(2n+1, F)$ we can consider $\text{s.s.}(s_{(k)}(\sigma)) \in R_k \otimes R(SO(2(n-k)+1, F))$, where $s_{(k)}(\sigma)$ denotes (normalized) Jacquet module of the representation σ with respect to standard parabolic subgroup whose Levi factor is $M_{(k)}$, for $k \leq n$. Define

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k)}(\sigma)) \in R \otimes S.$$

Mapping μ^* extends to additive mapping $\mu^*\colon S \to R \otimes S$. Especially, it can be shown that mapping $R(SO(2n+1, F)) \to R(M_{(k)})$ is Grothendieck groups homomorphism.

We use the subrepresentation version of the Langlands classification in this paper. Namely, non-tempered representation $\pi \in \text{Irr}(SO(2n+1, F))$ can be written as the unique irreducible (Langlands) subrepresentation of the induced representation of the form

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau,$$

where $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square integrable representations such that

$$e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0,$$

and $\tau \in \text{Irr}(SO(2n'+1, F))$ is a tempered representation. Thus, we can write

$$\pi \cong L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau).$$

We say that the irreducible representation σ of $SO(2n+1, F)$ is strongly positive or strongly positive discrete series if for every embedding

$$\sigma \hookrightarrow \nu^{a_1}\rho_1 \times \nu^{a_2}\rho_2 \times \cdots \times \nu^{a_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in \mathrm{Irr}(GL(n_i, F))$, for $i = 1, 2, \ldots, k$, is a cuspidal unitarizable representation and $\sigma_{cusp} \in \mathrm{Irr}(SO(2n' + 1, F))$ a cuspidal representation, we have $a_i > 0$, for all i.

Now we provide the definition of the Aubert dual and the main properties from [1] and [2]. Let G be a connected reductive p-adic group over local non-archimedean field F. Fix maximal split torus A_{min} of G and minimal parabolic subgroup P_{min} which contains A_{min}. Further, let $W = W(G/A_{min})$ be a Weyl group of G with respect to A_{min}. Denote by Σ the set of roots of G with respect to fixed minimal parabolic subgroup and let Δ denote the basis of Σ, which is determined with the choice of the minimal parabolic subgroup P_{min}. Let P_Θ be standard parabolic subgroup of G which corresponds to Θ, where Θ is the subset of Δ. Also, let M_Θ be standard Levi subgroup of G which corresponds to Θ.

Like earlier, $\mathrm{Ind}_M^G(\sigma)$ denotes normalized parabolically induced representation of G, which is induced by σ, where P is a parabolic subgroup of G with Levi factor M, and σ representation of M. Also, normalized Jacquet module of σ with respect to the standard parabolic subgroup that has a Levi factor equal to M is denoted by $r_M^G(\sigma)$, where σ is an admissible representation of G of finite length.

Theorem 2.1. *Define the operator on the Grothendieck group of admissible representations of finite length of G with*

$$D_G = \sum_{\Theta \subseteq \Delta} (-1)^{|\Theta|} Ind_{M_\Theta}^G \circ r_{M_\Theta}^G.$$

Operator D_G has the following properties:

(1) *D_G is an involution, that is $D_G^2 = \mathrm{id}$.*
(2) *The image of an irreducible representation is irreducible up to sign.*
(3) *For a standard Levi subgroup $M = M_\Theta$ we have*

$$r_M^G \circ D_G = Ad(w) \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M)}^G,$$

where w is the longest element of the set $\{w \in W : w^{-1}(\Theta) > 0\}$.

(4) *For standard Levi subgroup $M = M_\Theta$ we have*

$$D_G \circ Ind_M^G = Ind_M^G \circ D_M.$$

Now we can define the Aubert dual. If σ is an irreducible representation of the group $SO(2n + 1, F)$, then by $\widehat{\sigma}$ we denote the representation $\pm D_{SO(2n+1,F)}(\sigma)$, where we take the sign $+$ or $-$ such that $\widehat{\sigma}$ is a positive element in the Grothendieck group of admissible representations of finite length of $SO(2n + 1, F)$ and $\widehat{\sigma}$ is called the Aubert dual of σ.

A very interesting conjecture about the Aubert involution states that the Aubert involution preserves unitarity. Hanzer has proved in [7] that the Aubert involution preserves unitarity in the case of strongly positive representations.

In what follows, the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta \rtimes 1_{F^\times}$, where $\zeta \in \widehat{F^\times}$ is a quadratic character and $\frac{1}{2}$ is the unique nonnegative reducibility point, will be denoted by St_ζ.

3. Technical results

In this section a few technical results that will be frequently used are given. A description of irreducible subquotients of particular type is given with the next lemma.

Lemma 3.1. *The induced representation*

$$\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}, \tag{3.1}$$

where $\chi_1, \chi_2, \chi_3 \in \widehat{F^\times}$ and $0 \leq a_1 \leq a_2 \leq a_3$ of the group $SO(7, F)$ contains an irreducible non-tempered subquotient of the form $L(\delta_1 \times \cdots \times \delta_k \rtimes \tau)$ with δ_i being a representation of $GL(2, F)$ for some i if and only if there exist $i, j \in \{1, 2, 3\}$, where $i \neq j$, such that $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$. Then, for $l \in \{1, 2, 3\}$, also $l \neq i$ and $l \neq j$, an irreducible subquotient of (3.1) *has one of the following forms:*

(i) $L(\nu^{-a_l}\chi_l^{-1} \times \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \rtimes 1_{F^\times})$, *if* $a_l \geq a_j - \frac{1}{2}$.

(ii) $L(\delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \nu^{-a_l}\chi_l^{-1} \rtimes 1_{F^\times})$, *if* $a_l < a_j - \frac{1}{2}$.

(iii) $L(\delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \chi_l \rtimes 1_{F^\times})$, *if* $a_l = 0$.

Proof. Let us first suppose that (3.1) contains an irreducible constituent of the form $L(\delta_1 \times \cdots \times \delta_k \rtimes \tau)$ such that δ_i is a representation of $GL(2, F)$ for some i. Then we can conclude that $k \leq 2$ and $i \in \{1, 2\}$. If $i = 1$, then $\mu^*(\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times})$ contains an irreducible constituent of the form $\delta_1 \otimes \delta$, and the structural formula implies $\delta_1 \leq \nu^{\pm a_i}\chi_i^{\pm 1} \times \nu^{\pm a_j}\chi_j^{\pm 1}$, for $1 \leq i, j \leq 3$, $i \neq j$. This implies that $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$. If $i = 2$, using the structural formula again, we deduce that $\tau \cong 1_{F^\times}$ and there are $1 \leq i, j \leq 3$, $i \neq j$, such that $\mu^*(\nu^{a_i}\chi_i \times \nu^{a_j}\chi_j) \geq \delta_2 \otimes 1_{F^\times}$. Again, we obtain that $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$. If $k = 2$ and $i = 2$, note that we are in the case (i). If $k = 2$ and $i = 1$, note that we are in the case (ii). Finally, if $k = 1$, we are in the case (iii).

Let us now prove the converse. For $l \in \{1, 2, 3\}$, also $l \neq i$ and $l \neq j$, let first $a_l \neq 0$. If $a_l \geq a_j - \frac{1}{2}$, let $\delta_1 \cong \nu^{-a_l}\chi_l^{-1}$ and $\delta_2 \cong \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}])$, where $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for mutually unequal $i, j, l \in \{1, 2, 3\}$. Also, let $\tau \cong 1_{F^\times}$ and then we have

$$\begin{aligned} L(\delta_1 \times \delta_2 \rtimes \tau) &\hookrightarrow \nu^{-a_l}\chi_l^{-1} \times \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \rtimes 1_{F^\times} \\ &\leq \nu^{-a_l}\chi_l^{-1} \times \nu^{-a_j+1}\chi_j^{-1} \times \nu^{-a_j}\chi_j^{-1} \rtimes 1_{F^\times}, \end{aligned}$$

which is in the Grothendieck group $R(SO(7,F))$ equal to (3.1).

If $a_l < a_j - \frac{1}{2}$, let $\delta_1 \cong \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}])$ and $\delta_2 \cong \nu^{-a_l}\chi_l^{-1}$, where $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for mutually unequal $i, j, l \in \{1,2,3\}$. Let $\tau \cong 1_{F^\times}$ and we have

$$\begin{aligned} L(\delta_1 \times \delta_2 \rtimes \tau) &\hookrightarrow \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \nu^{-a_l}\chi_l^{-1} \rtimes 1_{F^\times} \\ &\leq \nu^{-a_j+1}\chi_j^{-1} \times \nu^{-a_j}\chi_j^{-1} \times \nu^{-a_l}\chi_l^{-1} \rtimes 1_{F^\times}, \end{aligned}$$

which is in the Grothendieck group $R(SO(7,F))$ equal to (3.1).

If now $a_l = 0$, let $\delta_1 \cong \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}])$, where $a_j = a_i + 1$ and $\chi_i \cong \chi_j$ or $a_j = 1 - a_i$ with $a_j > \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for mutually unequal $i, j \in \{1,2,3\}$, and $\tau \cong \chi_l \rtimes 1_{F^\times}$, for $l \in \{1,2,3\}$, also $l \neq i$ and $l \neq j$. Then we have

$$\begin{aligned} L(\delta_1 \rtimes \tau) &\hookrightarrow \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \chi_l \rtimes 1_{F^\times} \\ &\leq \nu^{-a_j+1}\chi_j^{-1} \times \nu^{-a_j}\chi_j^{-1} \times \chi_l \rtimes 1_{F^\times}, \end{aligned}$$

which is in the Grothendieck group $R(SO(7,F))$ equal to (3.1). ∎

The following lemma gives the description of the irreducible subquotient of the induced representation of the group $SO(7,F)$ which is of the form $L(\delta_1 \times \delta_2 \times \delta_3 \rtimes \tau)$.

Lemma 3.2. *Let $\chi_1, \chi_2, \chi_3 \in \widehat{F^\times}$ and $0 < a_1 \leq a_2 \leq a_3$. The induced representation*

$$\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times} \tag{3.2}$$

has the unique irreducible subquotient of the form $L(\delta_1 \times \delta_2 \times \delta_3 \rtimes \tau)$.

Proof. By the uniqueness of cuspidal support $\{\delta_1, \delta_2, \delta_3\}$ is contragredient permutation of $\{\nu^{a_1}\chi_1, \nu^{a_2}\chi_2, \nu^{a_3}\chi_3\}$ and $\tau \cong 1_{F^\times}$. Now the claim follows from conditions on $\{\delta_1, \delta_2, \delta_3\}$ as Langlands data from above. ∎

4. Irreducible unitarizable subquotients in the case of three quadratic characters

In this section we will give the description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all three characters are quadratic.

4.1. *The case when exactly one $\frac{1}{2}$ appears in the exponent.* All irreducible subquotients of the induced representation whose cuspidal support contains exactly one character of the form $\nu^a\zeta$ such that $\zeta \in \widehat{F^\times}$ is a quadratic character and $a = \frac{1}{2}$ are given with the next lemma that is proved using Langlands classification, the Jacquet module method, the classification of tempered representations and also Lemma 3.1 and Lemma 3.2.

Lemma 4.1. *Let* $\zeta_i \in \widehat{F^\times}$ *such that* $\zeta_i^2 \cong 1_{F^\times}$ *and let* $a_i \geq 0$*, for all* $i \in \{1,2,3\}$*, also let there exist exactly one* $i \in \{1,2,3\}$ *such that* $a_i = \frac{1}{2}$*. Without loss of generality, we can take* $a_3 = \frac{1}{2}$ *and let* $0 \leq a_1 \leq a_2$*. Then:*

(a) If $(a_s, \zeta_s) \neq (\frac{3}{2}, \zeta_3)$*, for* $s \in \{1,2\}$*, and* $(a_1, \zeta_1) \neq (a_2 - 1, \zeta_2)$*, then all irreducible subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

- *(i)* $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$*, if* $\frac{1}{2} < a_1$*, or if* $0 < a_1 < \frac{1}{2} < a_2$*, then* $L(\nu^{-a_2}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$*, or if* $a_1 > 0$ *and* $a_2 < \frac{1}{2}$*, then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$*.*
- *(ii)* $L(\nu^{-a_2}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$*, if* $a_1 = 0$ *and* $a_2 > \frac{1}{2}$*, or if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$*, then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$*, or* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$*, if* $a_1 = a_2 = 0$*.*
- *(iii)* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \rtimes 1_{F^\times})$*, if* $a_2 = 1 - a_1$ *with* $a_2 > \frac{1}{2}$ *and* $\zeta_1 \cong \zeta_2$*.*
- *(iv)* $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$*, if* $a_1 > 0$*, or* $L(\nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$*, if* $a_1 = 0$ *and* $a_2 > 0$*, or* $\zeta_1 \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$*, if* $a_1 = a_2 = 0$*.*
- *(v)* $L(\delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \rtimes \mathrm{St}_{\zeta_3})$*, if* $a_2 = 1 - a_1$ *with* $a_2 > \frac{1}{2}$ *and* $\zeta_1 \cong \zeta_2$*.*

(b) If $(a_s, \zeta_s) \neq (\frac{3}{2}, \zeta_3)$*, for* $s \in \{1,2\}$*, and* $(a_1, \zeta_1) = (a_2 - 1, \zeta_2)$*, then all irreducible subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

- *(i)* $L(\nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$*, if* $\frac{1}{2} < a_1$*, or if* $0 < a_1 < \frac{1}{2} < a_2$*, then* $L(\nu^{-a_2}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$*, or if* $a_1 = 0$*, then* $L(\nu^{-1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$*.*
- *(ii)* $L(\delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$*, if* $a_2 > 1$*, or if* $a_2 = 1$*, then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \delta([\nu^{-1}\zeta_2, \zeta_2]) \rtimes 1_{F^\times})$*.*
- *(iii)* $L(\nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$*, if* $a_1 > 0$*, or* $L(\nu^{-1}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$*, if* $a_1 = 0$*.*
- *(iv)* $L(\delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \rtimes \mathrm{St}_{\zeta_3})$*.*

(c) If $(a_s, \zeta_s) = (\frac{3}{2}, \zeta_3)$*, for some* $s \in \{1,2\}$*, and* $(a_1, \zeta_1) \neq (a_2 - 1, \zeta_2)$*, then all irreducible subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

- *(i)* $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$*, if* $\frac{1}{2} < a_1$*, or if* $0 < a_1 < \frac{1}{2}$*, then* $L(\nu^{-\frac{3}{2}}\zeta_3 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$*, or if* $a_1 = 0$*, then* $L(\nu^{-\frac{3}{2}}\zeta_3 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$*.*
- *(ii)* $L(\nu^{-a_j}\zeta_j \times \delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \rtimes 1_{F^\times})$*, if* $a_j \geq 1$ *and* $a_l = \frac{3}{2}$*, for* $\{j,l\} = \{1,2\}$*, or* $L(\delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$*, if* $0 < a_1 < 1$*, or* $L(\delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \times \zeta_1 \rtimes 1_{F^\times})$*, if* $a_1 = 0$*.*

(iii) $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $a_1 > 0$, *or* $L(\nu^{-\frac{3}{2}}\zeta_3 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $a_1 = 0$.

(iv) $L(\nu^{-a_j}\zeta_j \rtimes \sigma_{sp}^{(1)})$, *if* $a_j > 0$ *and* $a_i = \frac{3}{2}$, *for* $\{i, j\} = \{1, 2\}$, *or* $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, *if* $a_1 = 0$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{3}{2}}\zeta_3]) \rtimes 1_{F^\times}$.

(d) *If* $(a_1, \zeta_1) = (\frac{3}{2}, \zeta_3)$ *and* $(a_2, \zeta_2) = (\frac{5}{2}, \zeta_3)$, *then all irreducible subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-\frac{5}{2}}\zeta_3 \times \nu^{-\frac{3}{2}}\zeta_3 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{5}{2}}\zeta_3 \times \delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \rtimes 1_{F^\times})$.

(iii) $L(\delta([\nu^{-\frac{5}{2}}\zeta_3, \nu^{-\frac{3}{2}}\zeta_3]) \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.

(iv) $L(\delta([\nu^{-\frac{5}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \rtimes 1_{F^\times})$.

(v) $L(\nu^{-\frac{5}{2}}\zeta_3 \times \nu^{-\frac{3}{2}}\zeta_3 \rtimes \mathrm{St}_{\zeta_3})$.

(vi) $L(\delta([\nu^{-\frac{5}{2}}\zeta_3, \nu^{-\frac{3}{2}}\zeta_3]) \rtimes \mathrm{St}_{\zeta_3})$.

(vii) $L(\nu^{-\frac{5}{2}}\zeta_3 \rtimes \sigma_{sp}^{(1)})$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the representation* $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{3}{2}}\zeta_3]) \rtimes 1_{F^\times}$.

(viii) *The strongly positive representation* $\sigma_{sp}^{(2)}$ *which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{5}{2}}\zeta_3]) \rtimes 1_{F^\times}$.

Proof. We have the induced representation

$$\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}, \tag{4.1}$$

where $\zeta_i \in \widehat{F^\times}$ is such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1, 2, 3\}$, and $0 \le a_1 \le a_2$.

If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau) \le \nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, we conclude that $k \le 3$. All irreducible subquotients of the induced representation (4.1) for $k = 3$ we get using the Lemma 3.2.

Further, if $k = 2$ and $L(\delta_1 \times \delta_2 \rtimes \tau) \le \nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, where δ_i is an irreducible representation of the group $GL(n_i, F)$, for $i = 1, 2$, we have either $n_1 = 1$ and $n_2 = 2$ or $n_1 = 2$ and $n_2 = 1$ or $n_1 = n_2 = 1$.

If $n_1 = 1$ and $n_2 = 2$, then there are a few possibilities we get because of the premise of the lemma and the definition of non-tempered representations. First, we can have $\delta_1 \cong \nu^{-\frac{1}{2}}\zeta_3$ and $\delta_2 \cong \delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2])$, if $\zeta_1 \cong \zeta_2$ and $a_2 = a_1 + 1$ or $a_2 = 1 - a_1$ with $a_2 > \frac{1}{2}$. However, using the Lemma 3.1 we conclude that $a_2 \le 1$. Hence, since $a_1 \ge 0$, the case when $a_2 = a_1 + 1$ is possible only when $a_1 = 0$ and $a_2 = 1$, and the case when $a_2 = 1 - a_1$ when $\frac{1}{2} < a_2 \le 1$. Now we conclude that $\tau \cong 1_{F^\times}$ and it is easy to conclude that the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\nu^{-\frac{1}{2}}\zeta_3 \times \delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \rtimes 1_{F^\times})$, if $\zeta_1 \cong \zeta_2$, also $a_1 = 0$ and $a_2 = 1$ or $a_2 = 1 - a_1$ with $\frac{1}{2} < a_2 \le 1$. Second, we can have $\delta_1 \cong \nu^{-a_j}\zeta_j$ and $\delta_2 \cong \delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3])$, in the case when $a_j \ne 0$ and $a_l = \frac{3}{2}$, also $\zeta_l \cong \zeta_3$, for mutually unequal $j,l \in \{1,2\}$. Using the Lemma 3.1 we conclude that $a_j \geq 1$. Hence, it follows that $\tau \cong 1_{F^\times}$ and the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\nu^{-a_j}\zeta_j \times \delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \rtimes 1_{F^\times})$, if $a_j \geq 1$ and $a_l = \frac{3}{2}$, also $\zeta_l \cong \zeta_3$, for mutually unequal $j,l \in \{1,2\}$.

If $n_1 = 2$ and $n_2 = 1$, then again we have a few possibilities because of the premise of the lemma and the definition of non-tempered representations. First, we can have $\delta_1 \cong \delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2])$ and $\delta_2 \cong \nu^{-\frac{1}{2}}\zeta_3$, if $\zeta_1 \cong \zeta_2$ and $a_2 = a_1+1$ or $a_2 = 1-a_1$ with $a_2 > \frac{1}{2}$. Using the Lemma 3.1 we conclude that $a_2 > 1$. Thus, since $a_1 \geq 0$, the case when $a_2 = a_1 + 1$ is possible only when $a_1 > 0$, but the case when $a_2 = 1 - a_1$ is not possible. Now we conclude that $\tau \cong 1_{F^\times}$. Therefore, we get that the $L(\delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$, if $a_2 = a_1 + 1$ with $a_2 > 1$ and $\zeta_1 \cong \zeta_2$, is the irreducible subquotient of the induced representation (4.1) of the wanted form. Second, we can have $\delta_1 \cong \delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3])$ and $\delta_2 \cong \nu^{-a_j}\zeta_j$, if $a_l = \frac{3}{2}$ and $a_j \neq 0$, also $\zeta_l \cong \zeta_3$. Furthermore, from the Lemma 3.1 and the premise of the lemma it follows that $a_j < 1$, so $j = 1$ and $l = 2$. Now we can conclude that $\tau \cong 1_{F^\times}$. Then the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, if $0 < a_1 < 1$ and $a_2 = \frac{3}{2}$, also $\zeta_2 \cong \zeta_3$.

If $n_1 = n_2 = 1$, that is δ_1 and δ_2 are irreducible representations of the group $GL(1,F)$, then they are of the form $\nu^{-a}\zeta$, and τ is an irreducible tempered representation of the group $SO(3,F)$. Using the classification of tempered representations we conclude that τ is either the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ or $\tau \cong \zeta_1 \rtimes 1_{F^\times}$, for $a_1 = 0$, because $a_1 \leq a_2$. Now in the first case we get that the $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, for $a_1, a_2 \neq 0$, is the irreducible subquotient of the induced representation (4.1) of the wanted form. Further, in the second case we get that the irreducible subquotient of the induced representation (4.1) of the wanted form, if $a_1 = 0$, is equal either to the $L(\nu^{-a_2}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$, for $a_2 > \frac{1}{2}$, or to the $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, for $0 < a_2 < \frac{1}{2}$.

If $k = 1$ and $L(\delta_1 \rtimes \tau) \leq \nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, where δ_1 is an irreducible representation of the general linear group $GL(n_1,F)$, we have $1 \leq n_1 \leq 3$.

If $n_1 = 3$, then, because of the premise of the lemma and the definition of non-tempered representations, $\delta_1 \cong \delta([\nu^{-\frac{5}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3])$, for $a_1 = \frac{3}{2}$, $a_2 = \frac{5}{2}$ and $\zeta_1 \cong \zeta_2 \cong \zeta_3$. Then $\tau \cong 1_{F^\times}$ and we conclude that the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\delta([\nu^{-\frac{5}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \rtimes 1_{F^\times})$, if $a_1 = \frac{3}{2}$, $a_2 = \frac{5}{2}$ and $\zeta_1 \cong \zeta_2 \cong \zeta_3$.

If $n_1 = 2$, then δ_1 is an irreducible representation of the group $GL(2,F)$, and is then of the form $\delta([\nu^{-a}\zeta, \nu^{-a+1}\zeta])$, while τ is an irreducible tempered representation of the group $SO(3,F)$. Using the classification of tempered representations we conclude that τ is either the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ or $\tau \cong \zeta_1 \rtimes 1_{F^\times}$, for $a_1 = 0$, because $a_1 \leq a_2$. Now, using the Lemma 3.1, we conclude that the irreducible subquotient of the induced representation (4.1) of the wanted form is either equal to the $L(\delta([\nu^{-a_2}\zeta_2, \nu^{-a_2+1}\zeta_2]) \rtimes \mathrm{St}_{\zeta_3})$, if $\zeta_1 \cong \zeta_2$ and $a_2 = a_1 + 1$ or $a_2 = 1 - a_1$ with $a_2 > \frac{1}{2}$, or it is equal to the $L(\delta([\nu^{-\frac{3}{2}}\zeta_3, \nu^{-\frac{1}{2}}\zeta_3]) \times \zeta_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = \frac{3}{2}$, also $\zeta_2 \cong \zeta_3$.

Further, if $n_1 = 1$, then δ_1 is an irreducible representation of the group $GL(1, F)$, and is then of the form $\nu^{-a}\zeta$, while τ is then an irreducible tempered representation of $SO(5, F)$ so we conclude, using the classification of tempered representations, that τ is one of the following representations:

(1) The unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{3}{2}}\zeta_3]) \rtimes 1_{F^\times}$, if $a_i = \frac{3}{2}$ and $\zeta_i \cong \zeta_3$, for some $i \in \{1, 2\}$.
(2) $\tau \cong \zeta_1 \times \zeta_2 \rtimes 1_{F^\times}$, if $a_1 = a_2 = 0$.
(3) $\tau \cong \zeta_1 \rtimes \mathrm{St}_{\zeta_3}$, if $a_1 = 0$, because $a_1 \leq a_2$.

In case (1) we get that the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\nu^{-a_j}\zeta_j \rtimes \sigma_{\mathrm{sp}}^{(1)})$, if $a_i = \frac{3}{2}$ and $a_j \neq 0$, also $\zeta_i \cong \zeta_3$, for mutually unequal $i, j \in \{1, 2\}$, where $\sigma_{\mathrm{sp}}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{3}{2}}\zeta_3]) \rtimes 1_{F^\times}$. Further, in case (2) we get that the irreducible subquotient of the induced representation (4.1) of the wanted form is equal to the $L(\nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$, if $a_1 = a_2 = 0$. Finally, in case (3) we get that the $L(\nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, if $a_1 = 0$ and $a_2 \neq 0$, is the irreducible subquotient of the induced representation (4.1) of the wanted form.

It remains to find all irreducible subquotients when τ is an irreducible tempered representation of the group $SO(7, F)$. Using the classification of tempered representations we conclude that we have the following irreducible subquotients of the induced representation (4.1):

(1) The strongly positive representation $\sigma_{\mathrm{sp}}^{(2)}$ which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{5}{2}}\zeta_3]) \rtimes 1_{F^\times}$, if $a_1 = \frac{3}{2}$ and $a_2 = \frac{5}{2}$, also $\zeta_1 \cong \zeta_2 \cong \zeta_3$.
(2) $\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}$, if $a_1 = 0$ and $a_2 = \frac{3}{2}$, also $\zeta_2 \cong \zeta_3$, because $a_1 \leq a_2$, where $\sigma_{\mathrm{sp}}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_3, \nu^{\frac{3}{2}}\zeta_3]) \rtimes 1_{F^\times}$.
(3) $\zeta_1 \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$, if $a_1 = a_2 = 0$.

With that the proof of the lemma is finished. $\square$

Now, all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which three mutually isomorphic quadratic characters appear and exactly one exponent equals $\frac{1}{2}$ are given with the following theorem.

THEOREM 4.2. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, and let $\zeta_1 \cong \zeta_2 \cong \zeta_3$. Further, let a_1 and a_2 unequal to $\frac{1}{2}$ such that $0 \le a_1 \le a_2$ and let $a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = \frac{3}{2}$, *or* $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 = \frac{3}{2}$.

(ii) $L(\nu^{-a_2}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = a_1 + 1$ *or* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = 1 - a_1$.

(iii) $L(\nu^{-\frac{5}{2}}\zeta_1 \times \nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = \frac{3}{2}$ *and* $a_2 = \frac{5}{2}$.

(iv) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 \le a_2 < \frac{1}{2}$, *or if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *then* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$, *then* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$.

(v) $L(\nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, *if* $0 < a_1 \le a_2 < \frac{1}{2}$, *or* $L(\nu^{-a_2}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, *if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *or* $\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1}$, *if* $a_1 = a_2 = 0$.

(vi) $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = \frac{3}{2}$, *or* $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, *if* $a_1 = 0$ *and* $a_2 = \frac{3}{2}$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

(vii) $L(\delta([\nu^{-a_2}\zeta_1, \nu^{-a_2+1}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$, *if* $a_1 < \frac{1}{2}$ *and* $a_2 = a_1 + 1$ *or* $a_1 < \frac{1}{2}$ *and* $a_2 = 1 - a_1$.

(viii) *The strongly positive representation* $\sigma_{sp}^{(2)}$ *which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{5}{2}}\zeta_1]) \rtimes 1_{F^\times}$, *if* $a_1 = \frac{3}{2}$ *and* $a_2 = \frac{5}{2}$.

PROOF. All irreducible unitarizable subquotients of the induced representation

$$\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \tag{4.2}$$

are found using the Lemma 4.1, [22, Theorem 8.13.], and the appropriate results from [10] and [12]. One can also follow the proof of [22, Theorem 8.13.] for getting all unitarizable subquotients in this case. From [22, Theorem 8.13.] we see at once that the strongly positive representation $\sigma_{\mathrm{sp}}^{(2)}$ which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{5}{2}}\zeta_1]) \rtimes 1_{F^\times}$, if $a_1 = \frac{3}{2}$ and $a_2 = \frac{5}{2}$, is the irreducible unitarizable subquotient of the induced representation (4.2). Further, using [10, Theorem 3.5.], we conclude that $\widehat{\sigma_{\mathrm{sp}}^{(2)}} \cong L(\nu^{-\frac{5}{2}}\zeta_1 \times \nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times})$, if $a_1 = \frac{3}{2}$ and $a_2 = \frac{5}{2}$, is the irreducible unitarizable subquotient of the induced representation (4.2).

Also, using [22, Theorem 8.13.] we can conclude that the subquotient $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)})$, if because of the premise of the theorem $0 < a_1 < \frac{1}{2}$, and if $a_2 = \frac{3}{2}$, as well as the subquotient $\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}$, if $a_1 = 0$ and $a_2 = \frac{3}{2}$, are the irreducible unitarizable subquotients of the induced representation (4.2). Thereby the representation $\sigma_{\mathrm{sp}}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$. Further, because of

$$\begin{aligned} L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}) &\hookrightarrow \nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)} \\ &\hookrightarrow \nu^{-a_1}\zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times} \\ &\hookrightarrow \nu^{-a_1}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{a_1}\zeta_1 \rtimes 1_{F^\times}, \end{aligned}$$

using the Frobenius reciprocity we conclude that the Jacquet module of the $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)})$ with respect to the corresponding parabolic subgroup contains $\nu^{\frac{3}{2}}\zeta_1 \otimes \nu^{\frac{1}{2}}\zeta_1 \otimes \nu^{a_1}\zeta_1 \otimes 1_{F^\times}$. Now, using [10, Lemma 2.2.] we can conclude that the Jacquet module of the $L(\widehat{\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}})$ with respect to the corresponding parabolic subgroup contains $\nu^{-\frac{3}{2}}\zeta_1 \otimes \nu^{-\frac{1}{2}}\zeta_1 \otimes \nu^{-a_1}\zeta_1 \otimes 1_{F^\times}$. Further, from [14, 3.1. Lemma] follows that

$$L(\widehat{\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}}) \hookrightarrow \nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times}.$$

Now we conclude that $L(\widehat{\nu^{-a_1}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}}) \cong L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, if $0 < a_1 < \frac{1}{2}$ and $a_2 = \frac{3}{2}$, is the wanted irreducible unitarizable subquotient of the induced representation (4.2). Also, because of

$$\begin{aligned} \zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)} &\hookrightarrow \zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times} \\ &\hookrightarrow \zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times}, \end{aligned}$$

follows that

$$\widehat{\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}} \hookrightarrow \nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times}.$$

Thus, $\widehat{\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(1)}} \cong L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = \frac{3}{2}$, is the sought after irreducible unitarizable subquotient of the induced representation (4.2).

Now if we use [22, Theorem 8.13.] we can easily conclude that the subquotient $L(\delta([\nu^{-a_2}\zeta_1, \nu^{-a_2+1}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$, if $a_1 < \frac{1}{2}$ and $a_2 = a_1 + 1$ or $a_1 < \frac{1}{2}$ and $a_2 = 1 - a_1$, is the irreducible unitarizable subquotient of the induced representation (4.2). Further, according to [16, Theorem 2.3.], the induced representation $\delta([\nu^{-a_2}\zeta_1, \nu^{-a_2+1}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}$ is irreducible, so we have

$$\begin{aligned} L(\delta([\nu^{-a_2}\zeta_1, \nu^{-a_2+1}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}) &\hookrightarrow \delta([\nu^{-a_2}\zeta_1, \nu^{-a_2+1}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1} \\ &\cong \delta([\nu^{a_2-1}\zeta_1, \nu^{a_2}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1} \\ &\hookrightarrow \nu^{a_2}\zeta_1 \times \nu^{a_2-1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{a_2}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{a_2-1}\zeta_1 \rtimes 1_{F^\times} \end{aligned}$$

and it follows that

$$L(\delta([\nu^{-a_2}\zeta_1, \widehat{\nu^{-a_2+1}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}) \hookrightarrow \nu^{-a_2}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_2+1}\zeta_1 \rtimes 1_{F^\times}.$$

So, $L(\delta([\nu^{-a_2}\zeta_1, \widehat{\nu^{-a_2+1}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}) \cong L(\nu^{-a_2}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, if $0 < a_1 < \frac{1}{2}$ and $a_2 = a_1 + 1$ or $0 < a_1 < \frac{1}{2}$ and $a_2 = 1 - a_1$, is the wanted irreducible unitarizable subquotient of the induced representation (4.2).

Given the assumption of the theorem, all of the remaining irreducible unitarizable subquotients of the induced representation (4.2) we get using the second part of [22, Theorem 8.13.]. This finishes the proof. ∎

The next theorem gives the description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which exactly two quadratic characters that are mutually isomorphic appear and exactly one $\frac{1}{2}$ appears in the exponent.

THEOREM 4.3. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1, 2, 3\}$, and let $\zeta_j \cong \zeta_k \not\cong \zeta_l$ for $\{j, k, l\} = \{1, 2, 3\}$. Further, let a_1 and a_2 unequal to $\frac{1}{2}$ such that $0 \leq a_1 \leq a_2$ and let $a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{3}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = \frac{3}{2}$, *also* $\zeta_2 \cong \zeta_3$, *or* $L(\nu^{-\frac{3}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 = \frac{3}{2}$, *also* $\zeta_2 \cong \zeta_3$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq a_2 < \frac{1}{2}$, *or if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$, *then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$.

(iii) $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $0 < a_1 \leq a_2 < \frac{1}{2}$, *or* $L(\nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *or* $\zeta_1 \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$, *if* $a_1 = a_2 = 0$.

(iv) $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = \frac{3}{2}$, *also* $\zeta_2 \cong \zeta_3$, *or* $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, *if* $a_1 = 0$ *and* $a_2 = \frac{3}{2}$, *also* $\zeta_2 \cong \zeta_3$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_2, \nu^{\frac{3}{2}}\zeta_2]) \rtimes 1_{F^\times}$.

PROOF. The question of unitarity, according to [22, Corollary 9.4.], is now actually the question of the same problem in groups $SO(5, F)$ and $SO(3, F)$

whose unitary duals are known. For finding all irreducible unitarizable subquotients one should use [22, Proposition 7.2.], [22, Section 7.1.], as well as the appropriate results from [10] and Lemma 4.1. ∎

At the end, the theorem that provides the description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all three quadratic characters are mutually nonisomorphic and exactly one $\frac{1}{2}$ appears in the exponent is given.

THEOREM 4.4. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, and let $\zeta_i \not\cong \zeta_j$ for $i \neq j$ and $i,j \in \{1,2,3\}$. Further, let a_1 and a_2 unequal to $\frac{1}{2}$ such that $0 \leq a_1 \leq a_2$ and let $a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq a_2 < \frac{1}{2}$, *or if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$, *then* $L(\nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $0 < a_1 \leq a_2 < \frac{1}{2}$, *or* $L(\nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_3})$, *if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *or* $\zeta_1 \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$, *if* $a_1 = a_2 = 0$.

PROOF. Due to [22, Corollary 9.4.], the question of unitarity is equal to the analogous problem in the group $SO(3,F)$. Thus, using [22, Section 7.1.] we conclude that all irreducible subquotients of the given induced representation in which the exponents a_1 and a_2 are greater than or equal to 0, but less than $\frac{1}{2}$, because of the premise of the theorem, are unitarizable. ∎

4.2. *The case when exactly two $\frac{1}{2}$ appear in the exponents.* First, the following lemma gives all irreducible subquotients of the induced representation whose cuspidal support contains exactly two characters of the form $\nu^{\frac{1}{2}}\zeta$ and $\nu^{\frac{1}{2}}\zeta'$ such that $\zeta, \zeta' \in \widehat{F^\times}$ are quadratic characters.

LEMMA 4.5. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$ and let $a_i \geq 0$, for all $i \in \{1,2,3\}$, also let there exist exactly two $i,j \in \{1,2,3\}$, $i \neq j$, such that $a_i = a_j = \frac{1}{2}$. Without loss of generality, we can take $a_2 = a_3 = \frac{1}{2}$. Then:*

(a) *If* $\nu^{a_1}\zeta_1 \times \nu^{a_s}\zeta_s$ *is irreducible, for all* $s \in \{2,3\}$, *all irreducible subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$, *if* $a_1 > \frac{1}{2}$, *or if* $0 < a_1 < \frac{1}{2}$, *then* $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = 0$, *then* $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_l \rtimes \mathrm{St}_{\zeta_j})$, *if* $a_1 > \frac{1}{2}$, *or* $L(\nu^{-\frac{1}{2}}\zeta_l \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_j})$, *if* $0 < a_1 < \frac{1}{2}$, *for* $\{j,l\} = \{2,3\}$.

(iii) *$L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, if $a_1 > 0$ and $\zeta_2 \not\cong \zeta_3$, or $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, if $a_1 = 0$ and $\zeta_2 \not\cong \zeta_3$, where $\sigma_{sp}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.*

(iv) *$L(\nu^{-\frac{1}{2}}\zeta_j \times \zeta_1 \rtimes \mathrm{St}_{\zeta_i})$, if $a_1 = 0$, for $\{i,j\} = \{2,3\}$.*

(v) *$L(\nu^{-a_1}\zeta_1 \rtimes \tau_1)$ and $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, if $a_1 > 0$ and $\zeta_2 \cong \zeta_3$, or $\zeta_1 \rtimes \tau_1$ and $\zeta_1 \rtimes \tau_2$, if $a_1 = 0$ and $\zeta_2 \cong \zeta_3$, where τ_1 and τ_2 are mutually nonisomorphic irreducible tempered subrepresentations of the $\delta([\nu^{-\frac{1}{2}}\zeta_2, \nu^{\frac{1}{2}}\zeta_2]) \rtimes 1_{F^\times}$.*

(b) *If $\nu^{a_1}\zeta_1 \times \nu^{a_s}\zeta_s$ reduces, for some $s \in \{2,3\}$, then $a_1 = \frac{3}{2}$ and $\zeta_1 \cong \zeta_s$, and all irreducible subquotients of the $\nu^{a_1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) *$L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.*

(ii) *$L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \times \nu^{-\frac{1}{2}}\zeta_l \rtimes 1_{F^\times})$, if there exist j, l such that $\zeta_j \cong \zeta_1$ and $\{j,l\} = \{2,3\}$.*

(iii) *$L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_l \rtimes \mathrm{St}_{\zeta_j})$, for $\{j,l\} = \{2,3\}$.*

(iv) *$L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$, if $\zeta_2 \cong \zeta_3$, or $L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, if $\zeta_2 \not\cong \zeta_3$, where $\sigma_{sp}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.*

(v) *$L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_j})$, if there exist i, j such that $\zeta_i \cong \zeta_1$ and $\{i,j\} = \{2,3\}$.*

(vi) *$L(\nu^{-\frac{1}{2}}\zeta_j \rtimes \sigma_{sp}^{(2)})$, if there exist i, j such that $\zeta_i \cong \zeta_1$ and that $\{i,j\} = \{2,3\}$, where $\sigma_{sp}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the representation $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.*

(vii) *$L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_1)$ and $L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2)$, if $\zeta_2 \cong \zeta_3$, where τ_1 and τ_2 are mutually nonisomorphic irreducible tempered subrepresentations of the $\delta([\nu^{-\frac{1}{2}}\zeta_2, \nu^{\frac{1}{2}}\zeta_2]) \rtimes 1_{F^\times}$.*

(viii) *The strongly positive representation which is the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_i \rtimes \sigma_{sp}^{(2)}$, if $a_1 = \frac{3}{2}$ and $\zeta_1 \cong \zeta_j \not\cong \zeta_i$, for $\{i,j\} = \{2,3\}$, where $\sigma_{sp}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.*

(ix) *Mutually nonisomorphic irreducible tempered subrepresentations, σ_1 and σ_2, of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, which are discrete series, if $\zeta_2 \cong \zeta_3$.*

PROOF. We have the induced representation

$$\nu^{a_1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}, \tag{4.3}$$

where $\zeta_i \in \widehat{F^\times}$ is such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, also $a_1 \geq 0$ and $a_1 \neq \frac{1}{2}$. If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau) \leq \nu^{a_1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, then $k \leq 3$.

All irreducible subquotients of the induced representation (4.3) when $k = 3$, we get using the Lemma 3.2 and the rest of them one gets using the approach used in the proof of the Lemma 4.1; one uses the Langlands classification, the Jacquet module method, the classification of tempered representations and also Lemma 3.1. □

In what follows, we provide all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which three mutually isomorphic quadratic characters appear and exactly two exponents are equal to $\frac{1}{2}$.

THEOREM 4.6. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1, 2, 3\}$, and let $\zeta_1 \cong \zeta_2 \cong \zeta_3$. Further, let $a_1 \geq 0$ and $a_1 \neq \frac{1}{2}$, also let $a_2 = a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

- *(i)* $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = \frac{3}{2}$, *or if* $0 < a_1 < \frac{1}{2}$, *then* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *or* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$.
- *(ii)* $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, *if* $a_1 = \frac{3}{2}$, *or* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, *if* $0 < a_1 < \frac{1}{2}$, *or* $L(\nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, *if* $a_1 = 0$.
- *(iii)* $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{sp}^{(2)})$, *if* $a_1 = \frac{3}{2}$, *where* $\sigma_{sp}^{(2)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.
- *(iv)* $L(\nu^{-a_1}\zeta_1 \rtimes \tau_1)$, *if* $0 < a_1 < \frac{1}{2}$, *or* $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, *if* $0 < a_1 \leq \frac{3}{2}$ *and* $a_1 \neq \frac{1}{2}$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the* $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, *where* τ_2 *is not a subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.
- *(v)* $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$, *if* $a_1 = \frac{3}{2}$.
- *(vi)* $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$, *if* $a_1 = \frac{3}{2}$.
- *(vii) Two mutually nonisomorphic irreducible subrepresentations,* σ_1 *and* σ_2, *of the* $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, *which are discrete series, if* $a_1 = \frac{3}{2}$.
- *(viii)* $\zeta_1 \rtimes \tau_1$ *and* $\zeta_1 \rtimes \tau_2$, *if* $a_1 = 0$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the representation* $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, *where* τ_2 *is not a subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.

PROOF. All irreducible subquotients of the induced representation

$$\nu^{a_1}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \tag{4.4}$$

are determined in the Lemma 4.5. One now needs to extract the unitarizable ones. We primarily use [22, Theorem 8.13.]. Let us mention that one can also follow the proof of [22, Theorem 8.13.] for getting all unitarizable subquotients in this case.

We conclude that $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)})$, if $a_1 = \frac{3}{2}$, where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, is the irreducible unitarizable subquotient of the induced representation (4.4). Also, [22, Theorem 8.13.] gives us the irreducible unitarizable subquotient of the induced representation (4.4) which is equal to the subquotient $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$, if $a_1 = \frac{3}{2}$.

Further, we conclude that irreducible unitarizable subquotients of the induced representation (4.4) are as follows: $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, if $0 < a_1 \leq \frac{3}{2}$ and $a_1 \neq \frac{1}{2}$, also $\zeta_1 \rtimes \tau_2$, if $a_1 = 0$. Thereby τ_2 is the unique irreducible tempered subrepresentation of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.

Now, we conclude that the two mutually nonisomorphic irreducible subrepresentations, σ_1 and σ_2, of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, which are discrete series, if $a_1 = \frac{3}{2}$, are the irreducible unitarizable subquotients of the induced representation (4.4). Also, the $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$, if $a_1 = \frac{3}{2}$, is the irreducible unitarizable subquotient of the induced representation (4.4).

Finally, we can conclude that we have the following irreducible unitarizable subquotients of the induced representation (4.4):

(1) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, if $0 < a_1 < \frac{1}{2}$.
(2) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, if $0 < a_1 < \frac{1}{2}$.
(3) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, if $a_1 = 0$.
(4) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, if $a_1 = 0$.
(5) $L(\nu^{-a_1}\zeta_1 \rtimes \tau_1)$ and $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, if $0 < a_1 < \frac{1}{2}$, where τ_1 and τ_2 are mutually nonisomorphic irreducible tempered subrepresentations of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$.
(6) $\zeta_1 \rtimes \tau_1$ and $\zeta_1 \rtimes \tau_2$, if $a_1 = 0$, where τ_1 and τ_2 are mutually nonisomorphic irreducible tempered subrepresentations of the representation $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

We still need to determine the following Aubert duals:

(i) Aubert duals of all irreducible subquotients of the induced representation $\nu^{\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)}$, which are, according to [16, Theorem 5.1.], the $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)})$, where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, and discrete series σ_1, which is the unique common irreducible subrepresentation of the induced representation $\nu^{\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)}$ and the induced representation $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.
(ii) Aubert duals of all irreducible subquotients of the induced representation $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}$, which are $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$, $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$, $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)})$, where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly

positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, and discrete series σ_1, which is the unique common irreducible subrepresentation of the induced representations $\nu^{\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)}$ and $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

(iii) Aubert duals of all irreducible subquotients of the induced representation $\nu^{a_1}\zeta_1 \rtimes \tau_2$, where τ_2 is an irreducible tempered subrepresentation of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$. They are $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, if $0 < a_1 < \frac{3}{2}$ and $a_1 \neq \frac{1}{2}$, as well as $\zeta_1 \rtimes \tau_2$, if $a_1 = 0$; in the case when $a_1 = \frac{3}{2}$, the discrete series $\sigma_2 \hookrightarrow \nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_2$, while the Aubert duals of the remaining possible subquotients are determined in the other cases.

(iv) Aubert duals of all irreducible subquotients of $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, which are the discrete series $\sigma_1, \sigma_2 \hookrightarrow \delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, as well as the $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$.

Using [16, Theorem 2.1.] we conclude that

$$\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times} = \sigma_1 + \sigma_2 + L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}),$$

in the appropriate Grothendieck group.

Firstly, we determine the Aubert duals $\widehat{\sigma_1}$ and $\widehat{\sigma_2}$. From the classification of tempered representations of the group $SO(7, F)$ we know that

$$\sigma_i \hookrightarrow \nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_i,$$

for $i = 1, 2$, where it is, in the appropriate Grothendieck group,

$$\tau_1 \oplus \tau_2 = \delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}.$$

By [16, Theorem 5.1.] there is a unique $i \in \{1, 2\}$ such that $\tau_i \hookrightarrow \nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$ and we see that $i = 1$ because of (iii). Then $\mu^*(\tau_1)$ contains $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \otimes 1_{F^\times}$ and since such an irreducible constituent appears with multiplicity one in $\mu^*(\tau_1 \oplus \tau_2)$ we get $\mu^*(\tau_2) \ngeq \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \otimes 1_{F^\times}$. Now, we have

$$\mu^*(\sigma_2) \ngeq \nu^{\frac{1}{2}}\zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \otimes 1_{F^\times}.$$

Using [12, Theorem 5.2.], we conclude that the $\widehat{\sigma_2} \cong L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, if $a_1 = \frac{3}{2}$, is the sought after irreducible unitarizable subquotient of the induced representation (4.4). From (ii) we know that

$$\begin{aligned}\sigma_1 &\hookrightarrow \nu^{\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)} \\ &\hookrightarrow \nu^{\frac{1}{2}}\zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times},\end{aligned}$$

and because of that and irreducibility we get that

$$\mu^*(\sigma_1) \geq \nu^{\frac{1}{2}}\zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \otimes 1_{F^\times}.$$

Using [12, Theorem 5.2.], we can now conclude that the Aubert dual $\widehat{\sigma_1} \cong L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times})$, if $a_1 = \frac{3}{2}$, is the wanted irreducible unitarizable subquotient of the induced representation (4.4).

We still need to determine the Aubert dual $L(\delta([\nu^{-\frac{3}{2}}\widehat{\zeta_1, \nu^{\frac{1}{2}}}\zeta_1]) \rtimes 1_{F^\times})$. Since

$$L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}) \hookrightarrow \nu^{\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times},$$

it follows that

$$L(\delta([\nu^{-\frac{3}{2}}\widehat{\zeta_1, \nu^{\frac{1}{2}}\zeta_1}]) \rtimes 1_{F^\times}) \hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times}.$$

Now, by calculating $\mu^*(\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times})$ we get that $\nu^{\frac{3}{2}}\zeta_1 \otimes \tau_1$ and $\nu^{\frac{3}{2}}\zeta_1 \otimes \tau_2$ are the only irreducible constituents of the form $\nu^{\frac{3}{2}}\zeta_1 \otimes \pi$ which $\mu^*(\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times})$ contains, that appear with the multiplicity one. Since

$$\mu^*(\sigma_i) \geq \nu^{\frac{3}{2}}\zeta_1 \otimes \tau_i,$$

it follows that

$$\mu^*(L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})) \not\geq \nu^{\frac{3}{2}}\zeta_1 \otimes \pi.$$

Using [10, Lemma 2.2.], we can conclude that

$$\mu^*(L(\delta([\nu^{-\frac{3}{2}}\widehat{\zeta_1, \nu^{\frac{1}{2}}}\zeta_1]) \rtimes 1_{F^\times})) \not\geq \nu^{-\frac{3}{2}}\zeta_1 \otimes \pi'.$$

Besides, considering that

$$\mu^*(\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}) \not\geq \nu^{-\frac{3}{2}}\zeta_1 \otimes \pi,$$

we conclude that

$$\mu^*(L(\delta([\nu^{-\frac{3}{2}}\widehat{\zeta_1, \nu^{\frac{1}{2}}\zeta_1}]) \rtimes 1_{F^\times})) \not\geq \nu^{\frac{3}{2}}\zeta_1 \otimes \pi'.$$

Further, the Aubert dual of the representation $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$ can be presented as the unique irreducible subrepresentation of the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where $\delta_i \cong \delta([\nu^{-x_i}\zeta_1, \nu^{y_i}\zeta_1])$, for $i = 1, 2, \ldots, k$ (it is possible that $k = 0$), such that $e(\delta_i) \leq e(\delta_{i+1}) < 0$, for $i = 1, 2, \ldots, k-1$, and τ is the appropriate tempered representation. Now we can conclude that $y_i = -\frac{1}{2}$, for some $i \in \{1, 2, \ldots, k\}$, and that $y_i \neq -\frac{3}{2}$, for all $i \in \{1, 2, \ldots, k\}$. From that follows that $1 \leq k \leq 2$.

If $k = 2$, using the definition of non-tempered representations and Lemma 3.1, we can conclude that $\delta_1 \cong \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1])$ and $\delta_2 \cong \nu^{-\frac{1}{2}}\zeta_1$, as well as that $\tau \cong 1_{F^\times}$. Accordingly, we have

$$\begin{aligned} L(\delta(\widehat{[\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]}) \rtimes 1_{F^\times}) &\hookrightarrow \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{-\frac{1}{2}}\zeta_1 \times \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times} \\ &\hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times}, \end{aligned}$$

and we have got a contradiction.

If $k = 1$, then, using the definition of non-tempered representations, we conclude that $\delta_1 \in \{\nu^{-\frac{1}{2}}\zeta_1, \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1])\}$. In case that $\delta_1 \cong \nu^{-\frac{1}{2}}\zeta_1$, we conclude that then $\tau \cong \sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$, and we have

$$\begin{aligned} L(\delta(\widehat{[\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]}) \rtimes 1_{F^\times}) &\hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)} \\ &\hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times} \\ &\hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times} \\ &\cong \nu^{\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times}, \end{aligned}$$

therefore we have got a contradiction again. Thus, we can conclude that $\delta_1 \cong \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1])$ and then that $\tau \cong \mathrm{St}_{\zeta_1}$. Hence, we have

$$L(\delta(\widehat{[\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]}) \rtimes 1_{F^\times}) \hookrightarrow \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}.$$

Now we can conclude that the Aubert dual

$$L(\delta(\widehat{[\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]}) \rtimes 1_{F^\times}) \cong L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}), \tag{4.5}$$

if $a_1 = \frac{3}{2}$, is the wanted irreducible unitarizable subquotient of the induced representation (4.4).

Also, using the Theorem 2.1, we conclude that the Aubert dual

$$L(\delta(\widehat{[\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]}) \rtimes \mathrm{St}_{\zeta_1}) \cong L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}),$$

if $a_1 = \frac{3}{2}$, is the sought after irreducible unitarizable subquotient of the induced representation (4.4).

Further, from the description of discrete series we can conclude that the induced representation $\nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_2$, where τ_2 is the unique irreducible subrepresentation of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$, contains $L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2)$ and the discrete series $\sigma_2 \hookrightarrow \nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_2$. The only remaining representation that the given induced representation can contain is $L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times})$, whose Aubert dual we know. So, it remains to determine the Aubert dual $L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2})$. From the Theorem 2.1, we have

$$L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2}) \hookrightarrow \nu^{\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times}$$
$$\cong \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times}.$$

By calculating the appropriate Jacquet modules we get that $\mu^*(\nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_2)$ contains $\nu^{\frac{3}{2}}\zeta_1 \otimes \tau_2$ and that it is the only irreducible constituent of the form $\nu^{\frac{3}{2}}\zeta_1 \otimes \pi$, that is of multiplicity one. Further, since $\sigma_2 \hookrightarrow \nu^{\frac{3}{2}}\zeta_1 \rtimes \tau_2$, it follows that $\mu^*(\sigma_2) \geq \nu^{\frac{3}{2}}\zeta_1 \otimes \tau_2$. Thus, $\mu^*(L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2))$ does not contain an irreducible constituent of the form $\nu^{\frac{3}{2}}\zeta_1 \otimes \pi$, which, analogous as before, implies that $\mu^*(L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2}))$ does not contain $\nu^{-\frac{3}{2}}\zeta_1 \otimes \pi'$. So, $L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2})$ can be presented as the unique irreducible subrepresentation of the $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where $\delta_i \cong \delta([\nu^{-x_i}\zeta_1, \nu^{y_i}\zeta_1])$, for $i = 1, 2, \ldots, k$, such that $e(\delta_i) \leq e(\delta_{i+1}) < 0$, for $i = 1, 2, \ldots, k-1$, and τ is the appropriate tempered representation. Further, we can conclude that $y_i \neq -\frac{3}{2}$, for all $i \in \{1, 2, \ldots, k\}$, and that $y_i = -\frac{1}{2}$, for some $i \in \{1, 2, \ldots, k\}$. From that follows that $1 \leq k \leq 2$.

If $k = 2$, then, using the definition of non-tempered representations and Lemma 3.1, we can conclude that $\delta_1 \cong \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1])$ and $\delta_2 \cong \nu^{-\frac{1}{2}}\zeta_1$, also that $\tau \cong 1_{F^\times}$. Thus, we have

$$L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2}) \hookrightarrow \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times}$$
$$\cong \nu^{-\frac{1}{2}}\zeta_1 \times \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$$
$$\hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times},$$

and it follows that

$$L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2) \hookrightarrow \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times}.$$

Now, since the representation $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1$ is irreducible, it follows that

$$\mu^*(L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2)) \geq \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \otimes \nu^{\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times}.$$

However,

$$\mu^*(L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2)) \not\geq \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \otimes \nu^{\frac{3}{2}}\zeta_1 \rtimes 1_{F^\times}$$

since

$$\mu^*(\tau_2) \not\geq \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_1 \otimes 1_{F^\times},$$

which we know from the classification of tempered representations of the group $SO(5, F)$.

Hence, $k = 1$. Using the definition of non-tempered representations, analogous as before, we conclude that

$$(\delta_1, \tau) \in \{(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]), \mathrm{St}_{\zeta_1}), (\nu^{-\frac{1}{2}}\zeta_1, \sigma_{\mathrm{sp}}^{(2)})\},$$

where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

Then in the first case we have

$$L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes} \tau_2) \hookrightarrow \delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1},$$

and we can conclude that $L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes} \tau_2) \cong L(\delta([\nu^{-\frac{3}{2}}\zeta_1, \nu^{-\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1})$. However, because of (4.5) and the Theorem 2.1, we conclude that it is not the wanted irreducible unitarizable subquotient of the induced representation (4.4). In the second case we have

$$L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes} \tau_2) \hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)},$$

and we can conclude that the sought after irreducible unitarizable subquotient of the induced representation (4.4) is $L(\widehat{\nu^{-\frac{3}{2}}\zeta_1 \rtimes} \tau_2) \cong L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \sigma_{\mathrm{sp}}^{(2)})$, if $a_1 = \frac{3}{2}$, where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$. Also, using the Theorem 2.1, we conclude that the irreducible unitarizable subquotient of the induced representation (4.4) is the $L(\widehat{\nu^{-\frac{1}{2}}\zeta_1 \rtimes} \sigma_{\mathrm{sp}}^{(2)}) \cong L(\nu^{-\frac{3}{2}}\zeta_1 \rtimes \tau_2)$, if $a_1 = \frac{3}{2}$, where the representation τ_2 is the unique irreducible subrepresentation of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.

At the end, it remains to determine $L(\widehat{\nu^{-a_1}\zeta_1 \rtimes} \tau_2) \cong \widehat{\nu^{-a_1}\zeta_1 \rtimes} \tau_2$, if $0 < a_1 < \frac{3}{2}$ and $a_1 \neq \frac{1}{2}$, and $\widehat{\zeta_1 \rtimes \tau_2}$, if $a_1 = 0$, where τ_2 is the unique irreducible subrepresentation of the $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.

Because of the Theorem 2.1 and irreducibility we have

$$\begin{aligned}
\widehat{\nu^{-a_1}\zeta_1 \rtimes} \tau_2 &\cong \nu^{a_1}\zeta_1 \rtimes L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}) \\
&\cong \nu^{-a_1}\zeta_1 \rtimes L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}) \\
&\hookrightarrow \nu^{-a_1}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1} \\
&\cong \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}.
\end{aligned}$$

Now we can conclude that the wanted irreducible unitarizable subquotient of the induced representation (4.4) is $\widehat{\nu^{-a_1}\zeta_1 \rtimes} \tau_2 \cong L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, if $0 < a_1 < \frac{1}{2}$, while τ_2 is the unique irreducible subrepresentation of the representation $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$, that is not a subrepresentation of the $\nu^{\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}$.

Finally, because of the Theorem 2.1 we have

$$\begin{aligned}
\widehat{\zeta_1 \rtimes \tau_2} &\hookrightarrow \zeta_1 \times L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1}) \\
&\hookrightarrow \zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1} \\
&\cong \nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1},
\end{aligned}$$

so we conclude that the sought after irreducible unitarizable subquotient of the induced representation (4.4) is $\widehat{\zeta_1 \rtimes \tau_2} \cong L(\nu^{-\frac{1}{2}}\zeta_1 \times \zeta_1 \rtimes \mathrm{St}_{\zeta_1})$, if $a_1 = 0$, and with that we are finished with the proof of the theorem. $\square$

All irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which exactly two quadratic characters that are mutually isomorphic appear and exactly two $\frac{1}{2}$ appear in the exponents are given with the next theorem.

THEOREM 4.7. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1, 2, 3\}$, and let $\zeta_j \cong \zeta_k \not\cong \zeta_l$ for $\{j, k, l\} = \{1, 2, 3\}$. Further, let $a_1 \geq 0$ and $a_1 \neq \frac{1}{2}$, also let $a_2 = a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_j \rtimes 1_{F^\times})$, if $a_1 = \frac{3}{2}$ and if there exist i, j such that $\zeta_i \cong \zeta_1$ and $\{i, j\} = \{2, 3\}$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, if $0 < a_1 < \frac{1}{2}$, or if $a_1 = 0$, then $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$.

(iii) $L(\nu^{-\frac{3}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_i})$, if $a_1 = \frac{3}{2}$ and if there exist i, j such that $\zeta_j \cong \zeta_1$ and $\zeta_j \not\cong \zeta_i$, also $\{i, j\} = \{2, 3\}$.

(iv) $L(\nu^{-\frac{1}{2}}\zeta_l \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_j})$, if $0 < a_1 < \frac{1}{2}$, for $\{j, l\} = \{2, 3\}$, or $L(\nu^{-\frac{1}{2}}\zeta_j \times \zeta_1 \rtimes \mathrm{St}_{\zeta_i})$, if $a_1 = 0$, for $\{i, j\} = \{2, 3\}$.

(v) $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, if $0 < a_1 < \frac{1}{2}$ and $\zeta_2 \not\cong \zeta_3$, or $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, if $a_1 = 0$ and $\zeta_2 \not\cong \zeta_3$, where $\sigma_{sp}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.

(vi) $L(\nu^{-\frac{1}{2}}\zeta_j \rtimes \sigma_{sp}^{(2)})$, if $a_1 = \frac{3}{2}$ and if there exist i, j such that $\zeta_i \cong \zeta_1$ and $\{i, j\} = \{2, 3\}$, where $\sigma_{sp}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

(vii) The strongly positive representation τ which is the unique irreducible subrepresentation of the $\nu^{\frac{1}{2}}\zeta_i \rtimes \sigma_{sp}^{(2)}$, if $a_1 = \frac{3}{2}$ and if there exist i, j such that $\zeta_j \cong \zeta_1$ and $\zeta_j \not\cong \zeta_i$, also $\{i, j\} = \{2, 3\}$, where $\sigma_{sp}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{3}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

(viii) $L(\nu^{-a_1}\zeta_1 \rtimes \tau_1)$ and $L(\nu^{-a_1}\zeta_1 \rtimes \tau_2)$, if $0 < a_1 < \frac{1}{2}$ and $\zeta_2 \cong \zeta_3$, or $\zeta_1 \rtimes \tau_1$ and $\zeta_1 \rtimes \tau_2$, if $a_1 = 0$ and $\zeta_2 \cong \zeta_3$, where τ_1 and τ_2 are mutually nonisomorphic irreducible tempered subrepresentations of the $\delta([\nu^{-\frac{1}{2}}\zeta_2, \nu^{\frac{1}{2}}\zeta_2]) \rtimes 1_{F^\times}$.

PROOF. The unitarity question, because of [22, Corollary 9.4.], reduces to the analogous problem in groups $SO(5, F)$ and $SO(3, F)$, whose unitary duals are known. Using Lemma 4.5, [22, Section 7.1.] and [22, Proposition 7.2.], with the appropriate results from [10], all the irreducible unitarizable subquotients in this case can be found. $\square$

Finally, the description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all quadratic characters are mutually nonisomorphic and exactly two $\frac{1}{2}$ appear in the exponents is given.

THEOREM 4.8. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, and let $\zeta_i \not\cong \zeta_j$ for $i \neq j$ and $i,j \in \{1,2,3\}$. Further, let $a_1 \geq 0$ and $a_1 \neq \frac{1}{2}$, also let $a_2 = a_3 = \frac{1}{2}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 < \frac{1}{2}$, *or if* $a_1 = 0$, *then* $L(\nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \times \zeta_1 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_l \times \nu^{-a_1}\zeta_1 \rtimes \mathrm{St}_{\zeta_j})$, *if* $0 < a_1 < \frac{1}{2}$, *for* $\{j,l\} = \{2,3\}$, *or* $L(\nu^{-\frac{1}{2}}\zeta_j \times \zeta_1 \rtimes \mathrm{St}_{\zeta_i})$, *if* $a_1 = 0$, *for* $\{i,j\} = \{2,3\}$.

(iii) $L(\nu^{-a_1}\zeta_1 \rtimes \sigma_{sp}^{(1)})$, *if* $0 < a_1 < \frac{1}{2}$, *or* $\zeta_1 \rtimes \sigma_{sp}^{(1)}$, *if* $a_1 = 0$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.

PROOF. The unitarity question, because of [22, Corollary 9.4.], in this case reduces to the analogous problem in the group $SO(3,F)$. Hence, using [22, Section 7.1.] we conclude that all irreducible subquotients of the given induced representation in which the exponent a_1 is greater than or equal to 0, but less than $\frac{1}{2}$, because of the theorems' premise, are unitarizable. □

4.3. *The case when all three exponents are equal to $\frac{1}{2}$.* In this subsection all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all three exponents are equal to $\frac{1}{2}$ are given. Like before, the results are separated depending on the number of mutually isomorphic quadratic characters and details of proofs are left for the reader.

At the beginning, in the next lemma, all irreducible subquotients in this case are given.

LEMMA 4.9. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$ and let $a_i = \frac{1}{2}$, for all $i \in \{1,2,3\}$. Then all irreducible subquotients of the induced representation $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_i \times \nu^{-\frac{1}{2}}\zeta_j \rtimes \mathrm{St}_{\zeta_l})$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \sigma_{sp}^{(1)})$, *if* $\zeta_j \not\cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_j \times \nu^{\frac{1}{2}}\zeta_l \rtimes 1_{F^\times}$.

(iv) $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \tau_1)$ *and* $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \tau_2)$, *if* $\zeta_j \cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the* $\delta([\nu^{-\frac{1}{2}}\zeta_j, \nu^{\frac{1}{2}}\zeta_j]) \rtimes 1_{F^\times}$.

(v) *The strongly positive representation* $\sigma_{sp}^{(2)}$ *which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, *if* $\zeta_1 \not\cong \zeta_2$, $\zeta_1 \not\cong \zeta_3$ *and* $\zeta_2 \not\cong \zeta_3$.

(vi) $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}$, *if* $\zeta_1 \cong \zeta_2 \cong \zeta_3$.

(vii) *Two mutually nonisomorphic irreducible tempered subrepresentations of the* $\delta([\nu^{-\frac{1}{2}}\zeta_j, \nu^{\frac{1}{2}}\zeta_j]) \rtimes \mathrm{St}_{\zeta_i}$, *if* $\zeta_j \cong \zeta_l$ *and* $\zeta_j \not\cong \zeta_i$, *for* $\{i,j,l\} = \{1,2,3\}$.

PROOF. If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau) \leq \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$, where $\zeta_i \in \widehat{F^\times}$ is such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, then $k \leq 3$.

For $k = 3$, the irreducible subquotient of the induced representation $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$ we get using the Lemma 3.2 and it's equal to $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$. The remaining irreducible subquotients of the given induced representation one gets using the approach used in the proof of the Lemma 4.1. ∎

First the case when all three quadratic characters that appear are mutually isomorphic is considered.

THEOREM 4.10. *Let* $\zeta_i \in \widehat{F^\times}$ *such that* $\zeta_i^2 \cong 1_{F^\times}$ *and* $a_i = \frac{1}{2}$, *for all* $i \in \{1,2,3\}$, *also let* $\zeta_1 \cong \zeta_2 \cong \zeta_3$. *Then all irreducible unitarizable subquotients of the induced representation* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes \mathrm{St}_{\zeta_1})$.

(iii) $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \tau_1)$ *and* $L(\nu^{-\frac{1}{2}}\zeta_1 \rtimes \tau_2)$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the representation* $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes 1_{F^\times}$.

(iv) $\delta([\nu^{-\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_1]) \rtimes \mathrm{St}_{\zeta_1}$.

PROOF. For finding all irreducible unitarizable subquotients of the given induced representation one uses Lemma 4.9, [22, Theorem 8.13.], [10] and [12] with the approach similar to the one used in the previous sections. ∎

After that, all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which exactly two quadratic characters are mutually isomorphic are given with the next theorem.

THEOREM 4.11. *Let* $\zeta_i \in \widehat{F^\times}$ *such that* $\zeta_i^2 \cong 1_{F^\times}$ *and let* $a_i = \frac{1}{2}$, *for all* $i \in \{1,2,3\}$, *also let* $\zeta_j \cong \zeta_k \not\cong \zeta_l$ *for* $\{j,k,l\} = \{1,2,3\}$. *Then all irreducible unitarizable subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_i \times \nu^{-\frac{1}{2}}\zeta_j \rtimes \mathrm{St}_{\zeta_l})$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \sigma_{sp}^{(1)})$, *if* $\zeta_j \not\cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_j \times \nu^{\frac{1}{2}}\zeta_l \rtimes 1_{F^\times}$.

(iv) $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \tau_1)$ *and* $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \tau_2)$, *if* $\zeta_j \cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the* $\delta([\nu^{-\frac{1}{2}}\zeta_j, \nu^{\frac{1}{2}}\zeta_j]) \rtimes 1_{F^\times}$.

(v) Two mutually nonisomorphic irreducible tempered subrepresentations of the representation $\delta([\nu^{-\frac{1}{2}}\zeta_j, \nu^{\frac{1}{2}}\zeta_j]) \rtimes \mathrm{St}_{\zeta_i}$, *if* $\zeta_j \cong \zeta_l$ *and* $\zeta_j \not\cong \zeta_i$, *for* $\{i,j,l\} = \{1,2,3\}$.

Proof. With [22, Corollary 9.4.] and the premise of the theorem it is easy to see that one gets all irreducible unitarizable subquotients of the given induced representation using the third part of [22, Proposition 7.2.] and [22, Section 7.1.] with the Lemma 4.9. ∎

At the end, all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all quadratic characters are mutually nonisomorphic are determined in the following theorem.

Theorem 4.12. *Let* $\zeta_i \in \widehat{F^\times}$ *such that* $\zeta_i^2 \cong 1_{F^\times}$ *and let* $a_i = \frac{1}{2}$, *for all* $i \in \{1,2,3\}$, *also let* $\zeta_i \not\cong \zeta_j$ *for* $i \neq j$ *and* $i,j \in \{1,2,3\}$. *Then all irreducible unitarizable subquotients of the* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta_i \times \nu^{-\frac{1}{2}}\zeta_j \rtimes \mathrm{St}_{\zeta_l})$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\nu^{-\frac{1}{2}}\zeta_i \rtimes \sigma_{sp}^{(1)})$, *for* $\{i,j,l\} = \{1,2,3\}$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_j \times \nu^{\frac{1}{2}}\zeta_l \rtimes 1_{F^\times}$.

(iv) The strongly positive representation $\sigma_{sp}^{(2)}$ *which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.

Proof. All irreducible unitarizable subquotients of the given induced representation are determined using the Lemma 4.9, [22, Corollary 9.4.] and [22, Section 7.1.]. ∎

4.4. *The case when all exponents are unequal to* $\frac{1}{2}$. In the case when all three characters are quadratic it remains to determine all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ when all the exponents are unequal to $\frac{1}{2}$. Details of proofs are left for the reader.

First all irreducible subquotients in this case using the Langlands classification, the Jacquet module method, and the classification of tempered representations have been found. Also, one needs to use Lemma 3.1 and Lemma 3.2.

Lemma 4.13. *Let* $\zeta_i \in \widehat{F^\times}$ *such that* $\zeta_i^2 \cong 1_{F^\times}$ *and let* $a_i \neq \frac{1}{2}$, *for all* $i \in \{1,2,3\}$, *also let* $0 \leq a_1 \leq a_2 \leq a_3$. *Then:*

(a) *If* $(a_s,\zeta_s) \neq (a_t-1,\zeta_t)$, *for all* $s,t \in \{1,2,3\}$, *then all irreducible subquotients of the induced representation* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 > 0$, *or if* $a_1 = 0$ *and* $a_2 > 0$, *then* $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$ *and* $a_3 > 0$, *then* $L(\nu^{-a_3}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$, *or* $\zeta_1 \times \zeta_2 \times \zeta_3 \rtimes 1_{F^\times}$, *if* $a_1 = a_2 = a_3 = 0$.

(ii) $L(\nu^{-a_i}\zeta_i \times \delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \rtimes 1_{F^\times})$, *if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = 1 - a_l$ *with* $a_j > \frac{1}{2}$, *also* $\zeta_j \cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$, *or* $L(\delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \times \nu^{-a_i}\zeta_i \rtimes 1_{F^\times})$, *if* $0 < a_i < a_j - \frac{1}{2}$ *and* $a_j = 1 - a_l$ *with* $a_j > \frac{1}{2}$, *also* $\zeta_j \cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+1}\zeta_3]) \times \zeta_l \rtimes 1_{F^\times})$, *if* $a_l = 0$ *and* $a_3 = 1 - a_j$ *with* $a_3 > \frac{1}{2}$, *also* $\zeta_3 \cong \zeta_j$, *for* $\{j,l\} = \{1,2\}$.

(b) *If* $(a_s,\zeta_s) = (a_t-1,\zeta_t)$, *for some* $s,t \in \{1,2,3\}$, *but no such* s,t,u *exist that* $\{s,t,u\} = \{1,2,3\}$, $(a_s,\zeta_s) = (a_t-1,\zeta_t)$ *and* $(a_t,\zeta_t) = (a_u-1,\zeta_u)$, *then all irreducible subquotients of the induced representation* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 > 0$, *or if* $a_1 = 0$ *and* $a_2 > 0$, *then* $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$, *then* $L(\nu^{-1}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_i}\zeta_i \times \delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \rtimes 1_{F^\times})$, *if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$ *or if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = 1 - a_l$ *with* $a_j > \frac{1}{2}$, *also* $\zeta_j \cong \zeta_l$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \times \nu^{-a_i}\zeta_i \rtimes 1_{F^\times})$, *if* $0 < a_i < a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iv) $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+2}\zeta_3]) \rtimes 1_{F^\times})$, *if* $a_3 = 2 - a_i$ *and* $a_3 = a_j + 1$, *for* $\{i,j\} = \{1,2\}$, *with* $a_3 > 1$ *and* $\zeta_1 \cong \zeta_2$.

(v) $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+1}\zeta_3]) \times \zeta_l \rtimes 1_{F^\times})$, *if* $a_l = 0$ *and* $a_3 = a_j + 1$, *for* $\{j,l\} = \{1,2\}$.

(vi) $\delta([\nu^{-1}\zeta_1, \nu\zeta_1]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = 1$, *also* $\zeta_2 \cong \zeta_3$.

(c) *If* $(a_1,\zeta_1) = (a_2-1,\zeta_2)$ *and* $(a_2,\zeta_2) = (a_3-1,\zeta_3)$, *then all irreducible subquotients of the induced representation* $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 > 0$, *or if* $a_1 = 0$, *then* $L(\nu^{-2}\zeta_1 \times \nu^{-1}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_i}\zeta_j \times \delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \rtimes 1_{F^\times})$, *if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i,j,l\} = \{1,2,3\}$, *or* $L(\delta([\nu^{-a_j}\zeta_j, \nu^{-a_j+1}\zeta_j]) \times \nu^{-a_i}\zeta_j \rtimes 1_{F^\times})$, *if* $0 < a_i < a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+2}\zeta_3]) \rtimes 1_{F^\times})$.

(iv) $L(\delta([\nu^{-2}\zeta_3, \nu^{-1}\zeta_3]) \times \zeta_3 \rtimes 1_{F^\times})$, *if* $a_1 = 0$.

Proof. We have the induced representation

$$\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}, \tag{4.6}$$

where $0 \leq a_1 \leq a_2 \leq a_3$, also $a_i \neq \frac{1}{2}$ and $\zeta_i \in \widehat{F^\times}$ is such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$. If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$ is an irreducible subquotient of the induced representation (4.6), then $k \leq 3$. Now one gets all irreducible subquotients of the induced representation (4.6) using the approach used in the proof of the Lemma 4.1. ∎

Using the previous result, all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in the case when all three quadratic characters that appear are mutually isomorphic are determined in the next theorem.

Theorem 4.14. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$, and let $\zeta_1 \cong \zeta_2 \cong \zeta_3$. Further, let $0 \leq a_1 \leq a_2 \leq a_3$ and $a_i \neq \frac{1}{2}$, for all $i \in \{1,2,3\}$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-a_3}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq a_2 \leq a_3 < \frac{1}{2}$, *or* $L(\nu^{-a_3}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $0 < a_2 \leq a_3 < \frac{1}{2}$, *or* $L(\nu^{-a_3}\zeta_1 \times \zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = 0$ *and* $0 < a_3 < \frac{1}{2}$, *or* $\zeta_1 \times \zeta_1 \times \zeta_1 \rtimes 1_{F^\times}$, *if* $a_1 = a_2 = a_3 = 0$.

(ii) $L(\nu^{-a_3}\zeta_1 \times \nu^{-a_2}\zeta_1 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $a_3 = a_1 + 1$ *and* $a_3 = -a_2 + 2$, *also* $0 < a_1 < \frac{1}{2}$.

(iii) $L(\nu^{-1}\zeta_1 \times \nu^{-1}\zeta_1 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 = a_3 = 1$.

(iv) $L(\delta([\nu^{-a_3}\zeta_1, \nu^{-a_3+2}\zeta_1]) \rtimes 1_{F^\times})$, *if* $a_3 = a_1 + 1$ *and* $a_3 = -a_2 + 2$, *also* $0 < a_1 < \frac{1}{2}$.

(v) $\delta([\nu^{-1}\zeta_1, \nu\zeta_1]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = 1$.

Proof. For finding all the wanted irreducible unitarizable subquotients of the given induced representation one uses Lemma 4.13 and [22, Theorem 8.13.] with the appropriate results from [10] and [12]. ∎

After that, all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in the case when exactly two quadratic characters are mutually isomorphic and in the case when all three quadratic characters are mutually nonisomorphic are given with the following theorem.

Theorem 4.15. *Let $\zeta_i \in \widehat{F^\times}$ such that $\zeta_i^2 \cong 1_{F^\times}$, for all $i \in \{1,2,3\}$ and let $\zeta_j \cong \zeta_k \not\cong \zeta_l$ for $\{j,k,l\} = \{1,2,3\}$ or let $\zeta_i \not\cong \zeta_j$ for $i \neq j$ and $i,j \in \{1,2,3\}$. Further, let $0 \leq a_1 \leq a_2 \leq a_3$, and let $a_i \neq \frac{1}{2}$, for all $i \in \{1,2,3\}$. Then all irreducible unitarizable subquotients of the $\nu^{a_1}\zeta_1 \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \nu^{-a_1}\zeta_1 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq a_2 \leq a_3 < \frac{1}{2}$.

(ii) $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \zeta_1 \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $0 < a_2 \leq a_3 < \frac{1}{2}$.

(iii) $L(\nu^{-a_3}\zeta_3 \times \zeta_1 \times \zeta_2 \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = 0$ *and* $0 < a_3 < \frac{1}{2}$.
(iv) $\zeta_1 \times \zeta_2 \times \zeta_3 \rtimes 1_{F^\times}$, *if* $a_1 = a_2 = a_3 = 0$.

PROOF. All irreducible subquotients of the given induced representation are given in the Lemma 4.13 and unitarizable ones can now be found using [22, Corollary 9.4.], and after that using the third part of [22, Proposition 7.2.] and [22, Section 7.1.]. ∎

5. Irreducible unitarizable subquotients in the case of two quadratic characters

The description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which exactly two quadratic characters appear is given in this section.

Since in the case when the exponent that appears with the character that is not quadratic is greater than 0 we will not have unitarizable subquotients of the given induced representation, which we conclude using the unitary dual of general linear group from [18], we are only considering the case when that exponent is equal to 0.

In the following lemma the description of all irreducible subquotients of the induced representation in this case is given.

LEMMA 5.1. *Let* $\chi, \zeta_2, \zeta_3 \in \widehat{F^\times}$ *such that* $\zeta_2^2 \cong \zeta_3^2 \cong 1_{F^\times}$ *and* $\chi^2 \not\cong 1_{F^\times}$, *also let* $a_1 = 0$ *and* $0 \leq a_2 \leq a_3$. *Then:*

(a) *If* $\nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3$ *is irreducible, then all irreducible subquotients of the induced representation* $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*
  - *(i)* $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \chi \rtimes 1_{F^\times})$, *if* $a_2 > 0$, *or if* $a_2 = 0$ *and* $a_3 > 0$, *then* $L(\nu^{-a_3}\zeta_3 \times \zeta_2 \times \chi \rtimes 1_{F^\times})$, *or* $\chi \times \zeta_2 \times \zeta_3 \rtimes 1_{F^\times}$, *if* $a_2 = a_3 = 0$.
  - *(ii)* $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+1}\zeta_3]) \times \chi \rtimes 1_{F^\times})$, *if* $a_3 = 1 - a_2$ *with* $a_3 > \frac{1}{2}$ *and* $\zeta_2 \cong \zeta_3$.
  - *(iii)* $L(\nu^{-a_i}\zeta_i \times \chi \rtimes \mathrm{St}_{\zeta_j})$, *if* $a_i > 0$ *and* $a_j = \frac{1}{2}$, *for* $\{i, j\} = \{2, 3\}$.
  - *(iv)* $\chi \rtimes \sigma_{sp}^{(1)}$, *if* $a_2 = a_3 = \frac{1}{2}$ *and* $\zeta_2 \not\cong \zeta_3$, *where* $\sigma_{sp}^{(1)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.
  - *(v)* $\chi \rtimes \tau_1$ *and* $\chi \rtimes \tau_2$, *if* $a_2 = a_3 = \frac{1}{2}$ *and* $\zeta_2 \cong \zeta_3$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the* $\delta([\nu^{-\frac{1}{2}}\zeta_2, \nu^{\frac{1}{2}}\zeta_2]) \rtimes 1_{F^\times}$.
  - *(vi)* $\chi \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$, *if* $a_2 = 0$ *and* $a_3 = \frac{1}{2}$.

(b) *If* $\nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3$ *reduces, then* $a_3 - a_2 = 1$ *and* $\zeta_2 \cong \zeta_3$, *and all irreducible subquotients of the induced representation* $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*
  - *(i)* $L(\nu^{-a_3}\zeta_2 \times \nu^{-a_2}\zeta_2 \times \chi \rtimes 1_{F^\times})$, *if* $a_2 > 0$, *or if* $a_2 = 0$, *then* $L(\nu^{-1}\zeta_2 \times \zeta_2 \times \chi \rtimes 1_{F^\times})$.

(ii) $L(\delta([\nu^{-a_3}\zeta_3, \nu^{-a_3+1}\zeta_3]) \times \chi \rtimes 1_{F^\times})$.

(iii) $L(\nu^{-\frac{3}{2}}\zeta_2 \times \chi \rtimes \mathrm{St}_{\zeta_2})$, *if* $a_2 = \frac{1}{2}$.

(iv) $\chi \rtimes \sigma_{sp}^{(2)}$, *if* $a_2 = \frac{1}{2}$, *where* $\sigma_{sp}^{(2)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_2, \nu^{\frac{3}{2}}\zeta_2]) \rtimes 1_{F^\times}$.

Proof. If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau) \leq \chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$, where $\delta_i \cong \delta([\nu^{-x_i}\rho_i, \nu^{y_i}\rho_i])$ and $-x_i+y_i < 0$, for $i = 1, 2, \ldots, k$, then it can be easily shown that $\delta_i \not\cong \nu^{-a_1}\chi^{-1}$, for all $i \in \{1, 2, \ldots, k\}$. Now we conclude that $k < 3$ and using the approach used in the proof of the Lemma 4.1 one now finds all irreducible subquotients of the induced representation $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$. □

First all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which both quadratic characters are mutually isomorphic are given in the following theorem.

Theorem 5.2. *Let* $\chi, \zeta_2, \zeta_3 \in \widehat{F^\times}$ *such that* $\zeta_2^2 \cong \zeta_3^2 \cong 1_{F^\times}$ *and* $\chi^2 \not\cong 1_{F^\times}$, *also let* $\zeta_2 \cong \zeta_3$. *Further, let* $a_1 = 0$ *and* $0 \leq a_2 \leq a_3$. *Then all irreducible unitarizable subquotients of the induced representation* $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\zeta_2 \times \nu^{-a_2}\zeta_2 \times \chi \rtimes 1_{F^\times})$, *if* $0 < a_2 \leq a_3 \leq \frac{1}{2}$, *or if* $a_2 = 0$ *and* $0 < a_3 \leq \frac{1}{2}$, *then* $L(\nu^{-a_3}\zeta_2 \times \zeta_2 \times \chi \rtimes 1_{F^\times})$, *or* $\chi \times \zeta_2 \times \zeta_2 \rtimes 1_{F^\times}$, *if* $a_2 = a_3 = 0$.

(ii) $L(\nu^{-\frac{3}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_2 \times \chi \rtimes 1_{F^\times})$, *if* $a_2 = \frac{1}{2}$ *and* $a_3 = \frac{3}{2}$.

(iii) $L(\nu^{-a_2}\zeta_2 \times \chi \rtimes \mathrm{St}_{\zeta_2})$, *if* $0 < a_2 \leq \frac{1}{2}$ *and* $a_3 = \frac{1}{2}$, *or* $\chi \times \zeta_2 \rtimes \mathrm{St}_{\zeta_2}$, *if* $a_2 = 0$ *and* $a_3 = \frac{1}{2}$.

(iv) $\chi \rtimes \sigma_{sp}^{(2)}$, *if* $a_2 = \frac{1}{2}$ *and* $a_3 = \frac{3}{2}$, *where* $\sigma_{sp}^{(2)}$ *is a strongly positive representation which is the unique irreducible subrepresentation of the* $\delta([\nu^{\frac{1}{2}}\zeta_2, \nu^{\frac{3}{2}}\zeta_2]) \rtimes 1_{F^\times}$.

(v) $\chi \rtimes \tau_1$ *and* $\chi \rtimes \tau_2$, *if* $a_2 = a_3 = \frac{1}{2}$, *where* τ_1 *and* τ_2 *are mutually nonisomorphic irreducible tempered subrepresentations of the representation* $\delta([\nu^{-\frac{1}{2}}\zeta_2, \nu^{\frac{1}{2}}\zeta_2]) \rtimes 1_{F^\times}$.

Proof. Since all irreducible subquotients of the induced representation

$$\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_2 \rtimes 1_{F^\times} \tag{5.1}$$

are given with the Lemma 5.1 it is left to find the unitarizable ones among them. Because of [20, Theorem 4.2], we can conclude that the unitarity problem reduces to the unitarity problem of representations of the general linear group and of representations of the odd special orthogonal group of lower rank.

Now, since $a_1 = 0$, all irreducible unitarizable subquotients of the induced representation (5.1) we get using [22, Proposition 7.2.] and [10].

Using [22, Proposition 7.2.] we can conclude that $\chi \rtimes \sigma_{\mathrm{sp}}^{(2)}$, if $a_2 = \frac{1}{2}$ and $a_3 = \frac{3}{2}$, where $\sigma_{\mathrm{sp}}^{(2)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the $\delta([\nu^{\frac{1}{2}}\zeta_2, \nu^{\frac{3}{2}}\zeta_2]) \rtimes 1_{F^\times}$, is the irreducible unitarizable subquotient of the induced representation (5.1). Further, using [10], we can easily conclude that $L(\nu^{-\frac{3}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_2 \times \chi \rtimes 1_{F^\times})$, if $a_2 = \frac{1}{2}$ and $a_3 = \frac{3}{2}$, is the irreducible unitarizable subquotient of the induced representation (5.1). The rest of the unitarizable subquotients of the induced representation (5.1) one gets directly using the third part of [22, Proposition 7.2.]. ∎

In the next theorem, all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which quadratic characters are mutually nonisomorphic are given.

Theorem 5.3. *Let $\chi, \zeta_2, \zeta_3 \in \widehat{F^\times}$ such that $\zeta_2^2 \cong \zeta_3^2 \cong 1_{F^\times}$ and $\chi^2 \not\cong 1_{F^\times}$, also let $\zeta_2 \not\cong \zeta_3$. Further, let $a_1 = 0$ and $0 \leq a_2 \leq a_3$. Then all irreducible unitarizable subquotients of the induced representation $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-a_3}\zeta_3 \times \nu^{-a_2}\zeta_2 \times \chi \rtimes 1_{F^\times})$, if $0 < a_2 \leq a_3 \leq \frac{1}{2}$, or if $a_2 = 0$ and $0 < a_3 \leq \frac{1}{2}$, then $L(\nu^{-a_3}\zeta_3 \times \zeta_2 \times \chi \rtimes 1_{F^\times})$, or $\chi \times \zeta_2 \times \zeta_3 \rtimes 1_{F^\times}$, if $a_2 = a_3 = 0$.

(ii) $L(\nu^{-a_2}\zeta_2 \times \chi \rtimes \mathrm{St}_{\zeta_3})$, if $0 < a_2 \leq \frac{1}{2}$ and $a_3 = \frac{1}{2}$, or $\chi \times \zeta_2 \rtimes \mathrm{St}_{\zeta_3}$, if $a_2 = 0$ and $a_3 = \frac{1}{2}$.

(iii) $\chi \rtimes \sigma_{sp}^{(1)}$, if $a_2 = a_3 = \frac{1}{2}$, where $\sigma_{sp}^{(1)}$ is a strongly positive representation which is the unique irreducible subrepresentation of the representation $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_3 \rtimes 1_{F^\times}$.

Proof. Using [20, Theorem 4.2], unitary dual of general linear group from [18] and [22, Section 7.1.], alongside with the Lemma 5.1, one can in a similar way as in the previous theorem get all irreducible unitarizable subquotients of the induced representation $\chi \times \nu^{a_2}\zeta_2 \times \nu^{a_3}\zeta_3 \rtimes 1_{F^\times}$. ∎

6. Irreducible unitarizable subquotients in the case of one quadratic character

In this section we find all irreducible unitarizable subquotients of the induced representation of the group $SO(7, F)$ in which exactly one character is quadratic. First all irreducible subquotients of the induced representation of the group $SO(7, F)$ in which exactly one quadratic character appears are given with the following lemma.

Lemma 6.1. *Let $\zeta, \chi_2, \chi_3 \in \widehat{F^\times}$ such that $\zeta^2 \cong 1_{F^\times}$, $\chi_2^2 \not\cong 1_{F^\times}$ and $\chi_3^2 \not\cong 1_{F^\times}$, also let $a_i \geq 0$, for all $i \in \{1, 2, 3\}$, where $0 \leq a_2 \leq a_3$. Then:*

(a) *If* $\nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3$ *is irreducible, then all irreducible subquotients of the induced representation* $\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_1}\zeta \times \nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \rtimes 1_{F^\times})$, *if* $a_1 \geq a_3$ *and* $a_2 > 0$, *or* $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_1}\zeta \times \nu^{-a_2}\chi_2^{-1} \rtimes 1_{F^\times})$, *if* $0 < a_2 \leq a_1 < a_3$, *or* $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, *if* $0 < a_1 < a_2$.

(ii) $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \zeta \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 > 0$, *or* $L(\nu^{-a_3}\chi_3^{-1} \times \zeta \times \chi_2 \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = 0$ *and* $a_3 > 0$, *or* $\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, *if* $a_i = 0$, *for all* $i \in \{1, 2, 3\}$.

(iii) $L(\nu^{-a_1}\zeta \times \nu^{-a_3}\chi_3^{-1} \times \chi_2 \rtimes 1_{F^\times})$, *if* $a_1 \geq a_3 > 0$ *and* $a_2 = 0$, *or* $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_1}\zeta \times \chi_2 \rtimes 1_{F^\times})$, *if* $0 < a_1 < a_3$ *and* $a_2 = 0$, *or* $L(\nu^{-a_1}\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times})$, *if* $a_1 > 0$ *and* $a_2 = a_3 = 0$.

(iv) $L(\nu^{-a_1}\zeta \times \delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \rtimes 1_{F^\times})$, *if* $a_1 \geq a_3 - \frac{1}{2}$ *and* $a_3 = 1 - a_2$ *with* $a_3 > \frac{1}{2}$, *also* $\chi_2 \cong \chi_3^{-1}$, *or if* $0 < a_1 < a_3 - \frac{1}{2}$ *and* $a_3 = 1 - a_2$ *with* $a_3 > \frac{1}{2}$, *also* $\chi_2 \cong \chi_3^{-1}$, *then* $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$.

(v) $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \rtimes \mathrm{St}_\zeta)$, *if* $a_1 = \frac{1}{2}$ *and* $a_2 > 0$, *or if* $a_1 = \frac{1}{2}$, $a_2 = 0$ *and* $a_3 > 0$, *then* $L(\nu^{-a_3}\chi_3^{-1} \times \chi_2 \rtimes \mathrm{St}_\zeta)$, *or* $\chi_2 \times \chi_3 \rtimes \mathrm{St}_\zeta$, *if* $a_1 = \frac{1}{2}$ *and* $a_2 = a_3 = 0$.

(vi) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \rtimes \mathrm{St}_\zeta)$, *if* $a_1 = \frac{1}{2}$ *and* $a_3 = 1 - a_2$ *with* $a_3 > \frac{1}{2}$, *also* $\chi_2 \cong \chi_3^{-1}$, *or* $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \zeta \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_3 = 1 - a_2$ *with* $a_3 > \frac{1}{2}$, *also* $\chi_2 \cong \chi_3^{-1}$.

(vii) $L(\nu^{-a_1}\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times})$, *if* $a_1 > 0$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i, j\} = \{2, 3\}$, *or* $\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i, j\} = \{2, 3\}$.

(viii) $\delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes \mathrm{St}_\zeta$, *if* $a_l = \frac{1}{2}$, *for all* $l \in \{1, 2, 3\}$, *and if* $\chi_i \cong \chi_j^{-1}$, *for* $\{i, j\} = \{2, 3\}$.

(b) *If* $\nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3$ *reduces, then* $a_3 - a_2 = 1$, *also* $\chi_2 \cong \chi_3$ *and all irreducible subquotients of the* $\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_1}\zeta \times \nu^{-a_3}\chi_2^{-1} \times \nu^{-a_2}\chi_2^{-1} \rtimes 1_{F^\times})$, *if* $a_1 \geq a_3$ *and* $a_2 > 0$, *or* $L(\nu^{-a_3}\chi_2^{-1} \times \nu^{-a_1}\zeta \times \nu^{-a_2}\chi_2^{-1} \rtimes 1_{F^\times})$, *if* $0 < a_2 \leq a_1 < a_3$, *or* $L(\nu^{-a_3}\chi_2^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, *if* $0 < a_1 < a_2$.

(ii) $L(\nu^{-a_3}\chi_2^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \zeta \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 > 0$, *or* $L(\nu^{-1}\chi_2^{-1} \times \zeta \times \chi_2 \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = 0$.

(iii) $L(\nu^{-a_1}\zeta \times \nu^{-1}\chi_2^{-1} \times \chi_2 \rtimes 1_{F^\times})$, *if* $a_1 \geq 1$ *and* $a_2 = 0$, *or* $L(\nu^{-1}\chi_2^{-1} \times \nu^{-a_1}\zeta \times \chi_2 \rtimes 1_{F^\times})$, *if* $0 < a_1 < 1$ *and* $a_2 = 0$.

(iv) $L(\nu^{-a_1}\zeta \times \delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \rtimes 1_{F^\times})$, *if* $a_1 \geq a_3 - \frac{1}{2}$, *or* $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, *if* $0 < a_1 < a_3 - \frac{1}{2}$.

(v) $L(\nu^{-a_3}\chi_2^{-1} \times \nu^{-a_2}\chi_2^{-1} \rtimes \mathrm{St}_\zeta)$, *if* $a_1 = \frac{1}{2}$ *and* $a_2 > 0$, *or if* $a_1 = \frac{1}{2}$ *and* $a_2 = 0$, *then* $L(\nu^{-1}\chi_2^{-1} \times \chi_2 \rtimes \mathrm{St}_\zeta)$.

(vi) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \rtimes \mathrm{St}_\zeta)$, *if* $a_1 = \frac{1}{2}$, *or if* $a_1 = 0$, *then* $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \zeta \rtimes 1_{F^\times})$.

PROOF. If $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau) \leq \nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$, where $\zeta, \chi_1, \chi_2 \in \widehat{F^\times}$ are such that $\zeta^2 \cong 1_{F^\times}$, $\chi_2^2 \not\cong 1_{F^\times}$ and $\chi_3^2 \not\cong 1_{F^\times}$, also $a_i \geq 0$, for all $i \in \{1,2,3\}$, where $0 \leq a_2 \leq a_3$, then $k \leq 3$. For $k = 3$, all irreducible subquotients of the given induced representation we get using the Lemma 3.2. And the rest of the irreducible subquotients of the induced representation $\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ one gets using the approach used in the proof of the Lemma 4.1. □

In continuation we consider two cases. The case when characters that are not quadratic are mutually isomorphic, which is given with the next theorem, and the case when characters that are not quadratic are not mutually isomorphic.

THEOREM 6.2. *Let* $\zeta, \chi_2, \chi_3 \in \widehat{F^\times}$ *such that* $\zeta^2 \cong 1_{F^\times}$, $\chi_2^2 \not\cong 1_{F^\times}$ *and* $\chi_3^2 \not\cong 1_{F^\times}$, *also let* $\chi_2 \cong \chi_3$. *Further, let* $a_i \geq 0$, *for all* $i \in \{1,2,3\}$, *where* $0 \leq a_2 \leq a_3$. *Then all irreducible unitarizable subquotients of the induced representation* $\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_1}\zeta \times \chi_2 \times \chi_2 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq \frac{1}{2}$ *and* $a_2 = a_3 = 0$.
(ii) $\zeta \times \chi_2 \times \chi_2 \rtimes 1_{F^\times}$, *if* $a_i = 0$, *for all* $i \in \{1,2,3\}$.
(iii) $\chi_2 \times \chi_2 \rtimes \mathrm{St}_\zeta$, *if* $a_1 = \frac{1}{2}$ *and* $a_2 = a_3 = 0$.

PROOF. All irreducible subquotients of the induced representation

$$\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_2 \rtimes 1_{F^\times} \tag{6.1}$$

are given with the Lemma 6.1, and we need to determine which ones among them are the unitarizable ones.

Since we have a quadratic character and two characters that are not quadratic, using [20, Theorem 4.2], we conclude that the unitarity problem reduces to the unitarity problem of representations of the general linear group $GL(2,F)$ and representations of the special orthogonal group $SO(3,F)$. Therefore, we have $\pi \cong \theta \rtimes \pi'$, where θ is an irreducible subquotient of the $\nu^{a_2}\chi_2 \times \nu^{a_3}\chi_2$, and π' is an irreducible subquotient of the $\nu^{a_1}\zeta \rtimes 1_{F^\times}$.

Following the notations from [22, Section 2.16.], since two characters that are not quadratic are mutually isomorphic we see that $m = 1$ and $\delta \cong \chi_2$. Now we can conclude, using the unitary dual of the general linear group from [18], that $\theta \cong \chi_2 \times \chi_2$, for $a_2 = a_3 = 0$, is the only irreducible unitarizable representation of the group $GL(2,F)$ we have.

Further, using [22, Section 7.1.], we conclude that $a_1 \leq \frac{1}{2}$. Hence, it is either $\pi' \cong L(\nu^{-a_1}\zeta \rtimes 1_{F^\times})$, for $0 < a_1 \leq \frac{1}{2}$, or it is $\pi' \cong \mathrm{St}_\zeta$, if $a_1 = \frac{1}{2}$, and $\pi' \cong \zeta \rtimes 1$, for $a_1 = 0$.

One now easily gets all irreducible unitarizable subquotients of the induced representation (6.1). □

It is left to consider the case when characters that are not quadratic are not mutually isomorphic.

THEOREM 6.3. *Let $\zeta, \chi_2, \chi_3 \in \widehat{F^\times}$ such that $\zeta^2 \cong 1_{F^\times}$, $\chi_2^2 \ncong 1_{F^\times}$ and $\chi_3^2 \ncong 1_{F^\times}$, also let $\chi_2 \ncong \chi_3$. Further, let $a_i \geq 0$, for all $i \in \{1,2,3\}$, where $0 \leq a_2 \leq a_3$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times})$, *if* $0 < a_2 < \frac{1}{2}$ *and* $a_2 \leq a_1 \leq \frac{1}{2}$, *also* $a_2 = a_3$ *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or if* $0 < a_1 < a_2 < \frac{1}{2}$ *and* $a_2 = a_3$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *then it is* $L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = a_3 = \frac{1}{2}$ *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or* $L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, *if* $0 < a_1 < \frac{1}{2}$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$.

(iii) $L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or* $L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times})$, *if* $a_1 = 0$ *and* $0 < a_2 < \frac{1}{2}$, *also* $a_2 = a_3$ *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$.

(iv) $L(\nu^{-a_1}\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq \frac{1}{2}$ *and* $a_2 = a_3 = 0$, *or* $\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, *if* $a_i = 0$, *for all* $i \in \{1,2,3\}$.

(v) $L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes \mathrm{St}_\zeta)$, *if* $a_l = \frac{1}{2}$, *for all* $l \in \{1,2,3\}$, *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or* $L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes \mathrm{St}_\zeta)$, *if* $a_1 = \frac{1}{2}$ *and* $0 < a_2 < \frac{1}{2}$, *also* $a_2 = a_3$ *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or* $\chi_2 \times \chi_3 \rtimes \mathrm{St}_\zeta$, *if* $a_1 = \frac{1}{2}$ *and* $a_2 = a_3 = 0$.

(vi) $L(\nu^{-a_1}\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times})$, *if* $0 < a_1 \leq \frac{1}{2}$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$, *or* $\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$.

(vii) $\delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes \mathrm{St}_\zeta$, *if* $a_l = \frac{1}{2}$, *for all* $l \in \{1,2,3\}$, *and* $\chi_i \cong \chi_j^{-1}$, *for* $\{i,j\} = \{2,3\}$.

PROOF. All irreducible subquotients of the induced representation

$$\nu^{a_1}\zeta \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times} \tag{6.2}$$

are given with the Lemma 6.1. Thus, it is necessary to determine which ones among them are unitarizable.

Using [20, Theorem 4.2], because we have one quadratic character and two characters that are not quadratic, we conclude that the unitarity problem now reduces to the unitarity problem of representations of the general linear group and representations of the special orthogonal group $SO(3,F)$. Thus, we have $\pi \cong \theta \rtimes \pi'$, where θ is an irreducible subquotient of the $\nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3$, and π' is an irreducible subquotient of the $\nu^{a_1}\zeta \rtimes 1_{F^\times}$.

Further, using [22, Section 7.1.], we conclude that $a_1 \leq \frac{1}{2}$. Thus, we can see that it is either $\pi' \cong L(\nu^{-a_1}\zeta \rtimes 1_{F^\times})$, for $0 < a_1 \leq \frac{1}{2}$, or it is $\pi' \cong \mathrm{St}_\zeta$, for $a_1 = \frac{1}{2}$, and $\pi' \cong \zeta \rtimes 1$, for $a_1 = 0$.

Foremost, if we take a look at the set

$$\left\{\nu^\alpha\sigma \times \nu^{-\alpha}\sigma \colon \sigma \in B_{\mathrm{rigid}},\ 0 < \alpha < \frac{1}{2}\right\},$$

where B_{rigid} denotes the set of all Speh representations, we can conclude that $\theta \cong \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i$ is an irreducible unitarizable representation of the group $GL(2,F)$, if $0 < a_2 < \frac{1}{2}$ and $a_2 = a_3$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.

Now we can find all irreducible subquotients of the induced representation (6.2) which are unitarizable. Using the irreducibility criterion from [20] and [22, Corollary 9.4.], we can now easily conclude that the representation $\theta \rtimes \pi' \cong \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes L(\nu^{-a_1}\zeta \rtimes 1_{F^\times})$ is irreducible and unitarizable. Suppose that $a_1 \geq a_2$. Then we have

$$\begin{aligned}\nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes L(\nu^{-a_1}\zeta \rtimes 1_{F^\times}) &\hookrightarrow \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \times \nu^{-a_1}\zeta \rtimes 1_{F^\times}\\ &\cong \nu^{-a_1}\zeta \times \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes 1_{F^\times}\\ &\cong \nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{a_2}\chi_i \rtimes 1_{F^\times}\\ &\cong \nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times},\end{aligned}$$

and the representation $\nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times}$ has the unique irreducible subrepresentation, so we can now conclude that the subquotient $\theta \rtimes \pi' \cong L(\nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times})$, if $0 < a_2 < \frac{1}{2}$ and $a_3 \leq a_1 \leq \frac{1}{2}$, also $a_2 = a_3$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. Now analogous in the case when $a_1 < a_2$, since

$$\nu^{-a_1}\zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times} \cong \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times},$$

we can conclude that the $\theta \rtimes \pi' \cong L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, if $0 < a_1 < a_2 < \frac{1}{2}$ and $a_2 = a_3$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. Further, using the irreducibility criterion from [20] and [22, Corollary 9.4.], we can conclude that the representation $\theta \rtimes \pi' \cong \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes \mathrm{St}_\zeta$ is irreducible and unitarizable. Then

$$\begin{aligned}\nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes \mathrm{St}_\zeta &\cong \nu^{-a_2}\chi_i \times \nu^{a_2}\chi_i \rtimes \mathrm{St}_\zeta\\ &\cong \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes \mathrm{St}_\zeta,\end{aligned}$$

and the representation $\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes \mathrm{St}_\zeta$ has the unique irreducible subrepresentation, so we conclude that $\theta \rtimes \pi' \cong L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes \mathrm{St}_\zeta)$, if $a_1 = \frac{1}{2}$ and $0 < a_2 < \frac{1}{2}$, also $a_2 = a_3$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. Furthermore, using the irreducibility criterion from [20] and [22, Corollary 9.4.], we conclude that the $\theta \rtimes \pi' \cong \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \times \zeta \rtimes 1_{F^\times}$

is irreducible and unitarizable. Then

$$\begin{aligned}\nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \times \zeta \rtimes 1_{F^\times} &\cong \zeta \times \nu^{a_2}\chi_i \times \nu^{-a_2}\chi_i \rtimes 1_{F^\times} \\ &\cong \zeta \times \nu^{-a_2}\chi_i \times \nu^{a_2}\chi_i \rtimes 1_{F^\times} \\ &\cong \zeta \times \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \rtimes 1_{F^\times} \\ &\cong \nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times},\end{aligned}$$

and the representation $\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times}$ has the unique irreducible subrepresentation, so we conclude that $\theta \rtimes \pi' \cong L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times})$, if $a_1 = 0$ and $0 < a_2 < \frac{1}{2}$, also $a_2 = a_3$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable.

If we now take a look at the set B_{rigid}, since we have two characters that are not quadratic and one quadratic character, we conclude that $1 \leq m \leq 2$, where m is as in [22, Section 2.16.].

In case that $m = 1$, then $\delta \cong \chi_i$, for $i \in \{2,3\}$, or $\delta \cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i])$, if $a_2 = a_3 = \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$. Using the unitary dual of general linear group, we now conclude that for $m = 1$ it is either $\theta \cong \chi_2 \times \chi_3$, if $a_2 = a_3 = 0$, or it is $\theta \cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i])$, if $a_2 = a_3 = \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$. In case that $m = 2$, then the representation θ is isomorphic to the unique irreducible subrepresentation of the $\nu^{-\frac{1}{2}}\chi_i \times \nu^{\frac{1}{2}}\chi_i$, if $a_2 = a_3 = \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.

And now, if we take a look at the irreducible subquotients of the induced representation (6.2), we can easily, using [20, Theorem 4.2], determine the remaining unitarizable subquotients.

Let us now find all unitarizable subquotients of the given induced representation for $m = 1$, when $\theta \cong \chi_2 \times \chi_3$, sequentially for all possible π'. Namely, since

$$\begin{aligned}L(\nu^{-a_1}\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}) &\hookrightarrow \nu^{-a_1}\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times} \\ &\cong \chi_2 \times \chi_3 \times \nu^{-a_1}\zeta \rtimes 1_{F^\times} \\ &\cong \chi_2 \times \chi_3 \rtimes L(\nu^{-a_1}\zeta \rtimes 1_{F^\times}),\end{aligned}$$

we conclude that the $\theta \rtimes \pi' \cong L(\nu^{-a_1}\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times})$, if $0 < a_1 \leq \frac{1}{2}$ and $a_2 = a_3 = 0$, is unitarizable. Further, we can at once conclude that the $\theta \rtimes \pi' \cong \chi_2 \times \chi_3 \rtimes \mathrm{St}_\zeta$, if $a_1 = \frac{1}{2}$ and $a_2 = a_3 = 0$, is unitarizable. Finally, since

$$\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times} \cong \chi_2 \times \chi_3 \times \zeta \rtimes 1_{F^\times} \cong \theta \rtimes \pi',$$

we conclude that $\zeta \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, if $a_i = 0$, for all $i \in \{1,2,3\}$, is unitarizable.

Further, all unitarizable subquotients of the given induced representation for $m = 1$, in the case when $\theta \cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i])$, sequentially for all possible

π', we find in the continuation. Namely, since

$$\begin{aligned} L(\nu^{-a_1}\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}) &\hookrightarrow \nu^{-a_1}\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times} \\ &\cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \times \nu^{-a_1}\zeta \rtimes 1_{F^\times} \\ &\cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes L(\nu^{-a_1}\zeta \rtimes 1_{F^\times}), \end{aligned}$$

it follows that the $\theta \rtimes \pi' \cong L(\nu^{-a_1}\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times})$, if $0 < a_1 \leq \frac{1}{2}$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. Especially, observe that the representation $\delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$ is irreducible and tempered. Also, we conclude at once that the $\theta \rtimes \pi' \cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes \mathrm{St}_\zeta$, if $a_l = \frac{1}{2}$, for all $l \in \{1,2,3\}$, and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. At the end, since

$$\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times} \cong \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \times \zeta \rtimes 1_{F^\times} \cong \theta \rtimes \pi',$$

we conclude that the $\zeta \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable.

Finally, it remains to determine all unitarizable subquotients of the given induced representation when θ is isomorphic to the unique irreducible subrepresentation of the representation $\nu^{-\frac{1}{2}}\chi_i \times \nu^{\frac{1}{2}}\chi_i$. Using the irreducibility criterion from [20] and [22, Corollary 9.4.], we can conclude that the representation $\theta \rtimes \pi' \cong \theta \rtimes L(\nu^{-a_1}\zeta \rtimes 1_{F^\times})$ is irreducible and unitarizable. Let $a_1 < \frac{1}{2}$. Since

$$\begin{aligned} \theta \rtimes \pi' &\hookrightarrow \nu^{-\frac{1}{2}}\chi_i \times \nu^{\frac{1}{2}}\chi_i \times \nu^{-a_1}\zeta \rtimes 1_{F^\times} \\ &\cong \nu^{-\frac{1}{2}}\chi_i \times \nu^{-a_1}\zeta \times \nu^{\frac{1}{2}}\chi_i \rtimes 1_{F^\times} \\ &\cong \nu^{-\frac{1}{2}}\chi_i \times \nu^{-a_1}\zeta \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes 1_{F^\times} \\ &\cong \nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times}, \end{aligned}$$

and the $\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times}$ has the unique irreducible subrepresentation, it follows that the $\theta \rtimes \pi' \cong L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \nu^{-a_1}\zeta \rtimes 1_{F^\times})$, if $0 < a_1 < \frac{1}{2}$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. Also, by analogy we conclude that if $a_1 = a_2 = a_3 = \frac{1}{2}$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, the unitarizable subquotient is equal to the $\theta \rtimes \pi' \cong L(\nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes 1_{F^\times})$. Further, using the irreducibility criterion from [20] and [22, Corollary 9.4.], we conclude that the representation $\theta \rtimes \pi' \cong \theta \rtimes \mathrm{St}_\zeta$ is irreducible and unitarizable. Now, since

$$\begin{aligned} \theta \rtimes \mathrm{St}_\zeta &\hookrightarrow \nu^{-\frac{1}{2}}\chi_i \times \nu^{\frac{1}{2}}\chi_i \rtimes \mathrm{St}_\zeta \\ &\cong \nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes \mathrm{St}_\zeta, \end{aligned}$$

and the representation $\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes \mathrm{St}_\zeta$ has the unique irreducible subrepresentation, it follows that $\theta \rtimes \pi' \cong L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \rtimes \mathrm{St}_\zeta)$, if $a_l = \frac{1}{2}$, for all $l \in \{1,2,3\}$, and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable. At the end, we can conclude, using the irreducibility criterion from [20] and [22, Corollary 9.4.], that the representation $\theta \rtimes \pi' \cong \theta \times \zeta \rtimes 1_{F^\times}$ is irreducible and unitarizable. Since

$$\begin{aligned}\theta \times \zeta \rtimes 1_{F^\times} &\hookrightarrow \nu^{-\frac{1}{2}}\chi_i \times \nu^{\frac{1}{2}}\chi_i \times \zeta \rtimes 1_{F^\times}\\ &\cong \nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times},\end{aligned}$$

and the representation $\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times}$ has the unique irreducible subrepresentation, it follows that $\theta \rtimes \pi' \cong L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \zeta \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, is unitarizable.

As we have found all unitarizable subquotients of the given induced representation, the proof of the theorem is finished. ∎

7. Irreducible unitarizable subquotients in the case when none of the characters are quadratic

The remaining irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ are determined in this section, and they are the ones we get in the case when none of the characters that appear in the induced representation are quadratic.

Since for finding irreducible unitarizable subquotients of the induced representation in this case we first have to find irreducible ones, let us begin with the following lemma that gives their description.

Lemma 7.1. *Let $\chi_i \in \widehat{F^\times}$ such that $\chi_i \ncong \chi_i^{-1}$, for all $i \in \{1,2,3\}$, and let $0 \le a_1 \le a_2 \le a_3$. Then:*

(a) If $(a_s, \chi_s) \neq (a_t - 1, \chi_t)$, for all $s, t \in \{1,2,3\}$, then all irreducible subquotients of the induced representation $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ are as follows:

(i) $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \nu^{-a_1}\chi_1^{-1} \rtimes 1_{F^\times})$, if $a_1 > 0$, or if $a_1 = 0$ and $a_2 > 0$, then $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \chi_1 \rtimes 1_{F^\times})$, or $L(\nu^{-a_3}\chi_3^{-1} \times \chi_1 \times \chi_2 \rtimes 1_{F^\times})$, if $a_1 = a_2 = 0$ and $a_3 > 0$, or $\chi_1 \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, if $a_1 = a_2 = a_3 = 0$.

(ii) $L(\nu^{-a_i}\chi_i^{-1} \times \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \rtimes 1_{F^\times})$, if $a_i \ge a_j - \frac{1}{2}$ and $a_j = 1 - a_l$ with $a_j > \frac{1}{2}$, also $\chi_l \cong \chi_j^{-1}$, for $\{i,j,l\} = \{1,2,3\}$, or $L(\delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \nu^{-a_i}\chi_i^{-1} \rtimes 1_{F^\times})$, if $0 < a_i < a_j - \frac{1}{2}$ and $a_j = 1 - a_l$ with $a_j > \frac{1}{2}$, also $\chi_l \cong \chi_j^{-1}$, for $\{i,j,l\} = \{1,2,3\}$.

(iii) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \chi_l \rtimes 1_{F^\times})$, if $a_l = 0$ and $a_3 = 1 - a_j$ with $a_3 > \frac{1}{2}$, also $\chi_j \cong \chi_3^{-1}$, for $\{j,l\} = \{1,2\}$.

(iv) $L(\nu^{-a_l}\chi_l^{-1} \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times})$, *if* $a_l \neq 0$ *and* $a_i = a_j = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i, j, l\} = \{1, 2, 3\}$.

(v) $\chi_1 \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = \frac{1}{2}$, *also* $\chi_i \cong \chi_j^{-1}$, *for* $\{i, j\} = \{2, 3\}$.

(b) *If* $(a_s, \chi_s) = (a_t - 1, \chi_t)$, *for some* $s, t \in \{1, 2, 3\}$, *but no such* s, t, u *exist that* $\{s, t, u\} = \{1, 2, 3\}$, $(a_s, \chi_s) = (a_t - 1, \chi_t)$ *and* $(a_t, \chi_t) = (a_u - 1, \chi_u)$, *then all irreducible subquotients of the induced representation* $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \nu^{-a_1}\chi_1^{-1} \rtimes 1_{F^\times})$, *if* $a_1 > 0$, *or if* $a_1 = 0$ *and* $a_2 > 0$, *then* $L(\nu^{-a_3}\chi_3^{-1} \times \nu^{-a_2}\chi_2^{-1} \times \chi_1 \rtimes 1_{F^\times})$, *or if* $a_1 = a_2 = 0$, *then* $L(\nu^{-1}\chi_3^{-1} \times \chi_1 \times \chi_2 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_i}\chi_i^{-1} \times \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \rtimes 1_{F^\times})$, *if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$ *or if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = 1 - a_l$ *with* $a_j > \frac{1}{2}$, *also* $\chi_l \cong \chi_j^{-1}$, *for* $\{i, j, l\} = \{1, 2, 3\}$.

(iii) $L(\delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \nu^{-a_i}\chi_i^{-1} \rtimes 1_{F^\times})$, *if* $0 < a_i < a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i, j, l\} = \{1, 2, 3\}$.

(iv) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+2}\chi_3^{-1}]) \rtimes 1_{F^\times})$, *if* $a_3 = 2 - a_1$ *and* $a_3 = a_2 + 1$ *with* $a_3 > 1$, *also* $\chi_3^{-1} \cong \chi_1$ *or if* $a_3 = 2 - a_2$ *and* $a_3 = a_1 + 1$ *with* $a_3 > 1$, *also* $\chi_3^{-1} \cong \chi_2$.

(v) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+1}\chi_3^{-1}]) \times \chi_l \rtimes 1_{F^\times})$, *if* $a_l = 0$ *and* $a_3 = a_j + 1$, *for* $\{j, l\} = \{1, 2\}$, *or* $L(\delta([\nu^{-1}\chi_3^{-1}, \chi_3^{-1}]) \times \chi_l \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = 0$ *and* $\chi_1 \times \chi_2 \cong \chi_l \times \chi_3^{-1}$, *for* $l \in \{1, 2\}$.

(vi) $L(\nu^{-\frac{3}{2}}\chi_3^{-1} \times \delta([\nu^{-\frac{1}{2}}\chi_3, \nu^{\frac{1}{2}}\chi_3]) \rtimes 1_{F^\times})$, *if* $a_1 = a_2 = \frac{1}{2}$ *and* $\{\chi_1, \chi_2\} = \{\chi_3, \chi_3^{-1}\}$.

(vii) $\delta([\nu^{-1}\chi_1, \nu\chi_1]) \rtimes 1_{F^\times}$, *if* $a_1 = 0$ *and* $a_2 = a_3 = 1$, *also* $\{\chi_2, \chi_3\} = \{\chi_1, \chi_1^{-1}\}$.

(c) *If* $(a_1, \chi_1) = (a_2 - 1, \chi_2)$ *and* $(a_2, \chi_2) = (a_3 - 1, \chi_3)$, *then all irreducible subquotients of the induced representation* $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ *are as follows:*

(i) $L(\nu^{-a_3}\chi_1^{-1} \times \nu^{-a_2}\chi_1^{-1} \times \nu^{-a_1}\chi_1^{-1} \rtimes 1_{F^\times})$, *if* $a_1 > 0$, *or if* $a_1 = 0$, *then* $L(\nu^{-2}\chi_1^{-1} \times \nu^{-1}\chi_1^{-1} \times \chi_1 \rtimes 1_{F^\times})$.

(ii) $L(\nu^{-a_i}\chi_j^{-1} \times \delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \rtimes 1_{F^\times})$, *if* $a_i \geq a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i, j, l\} = \{1, 2, 3\}$, *or if* $0 < a_i < a_j - \frac{1}{2}$ *and* $a_j = a_l + 1$, *for* $\{i, j, l\} = \{1, 2, 3\}$, *then* $L(\delta([\nu^{-a_j}\chi_j^{-1}, \nu^{-a_j+1}\chi_j^{-1}]) \times \nu^{-a_i}\chi_j^{-1} \rtimes 1_{F^\times})$.

(iii) $L(\delta([\nu^{-a_3}\chi_3^{-1}, \nu^{-a_3+2}\chi_3^{-1}]) \rtimes 1_{F^\times})$.

(iv) $L(\delta([\nu^{-2}\chi_3^{-1}, \nu^{-1}\chi_3^{-1}]) \times \chi_3 \rtimes 1_{F^\times})$, *if* $a_1 = 0$.

First, we consider the case when all three characters that are not quadratic are mutually isomorphic.

Theorem 7.2. *Let $\chi_i \in \widehat{F^\times}$ such that $\chi_i \ncong \chi_i^{-1}$, for all $i \in \{1,2,3\}$, and let $\chi_1 \cong \chi_2 \cong \chi_3$. Further, let $0 \leq a_1 \leq a_2 \leq a_3$. Then $\chi_1 \times \chi_1 \times \chi_1 \rtimes 1_{F^\times}$, if $a_1 = a_2 = a_3 = 0$, is the unique irreducible unitarizable subquotient of the induced representation $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$.*

Proof. Since none of the characters are quadratic, using [20, Theorem 4.2], we conclude that the unitarity problem reduces to the unitarity problem of representations of the general linear group $GL(3,F)$ and we see at once, using [18, Theorem 7.5.], what the unitarizable subquotient is. ∎

The description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ when exactly two among the characters that are not quadratic are mutually isomorphic is given with the following theorem.

Theorem 7.3. *Let $\chi_i \in \widehat{F^\times}$ such that $\chi_i \ncong \chi_i^{-1}$, for $i \in \{1,2,3\}$, and let $\chi_j \cong \chi_k \ncong \chi_l$, for $\{j,k,l\} = \{1,2,3\}$. Further, let $0 \leq a_1 \leq a_2 \leq a_3$. Then all irreducible unitarizable subquotients of the induced representation $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \chi_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, or $L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \chi_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $0 < a_2 < \frac{1}{2}$, also $a_2 = a_3$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.
(ii) $\chi_1 \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, if $a_1 = a_2 = a_3 = 0$.
(iii) $\chi_1 \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.
(iv) $\delta([\nu^{-1}\chi_1, \nu\chi_1]) \rtimes 1_{F^\times}$, if $a_1 = 0$ and $a_2 = a_3 = 1$, also $\{\chi_2, \chi_3\} = \{\chi_1, \chi_1^{-1}\}$.
(v) $L(\nu^{-1}\chi_1 \times \nu^{-1}\chi_1^{-1} \times \chi_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = a_3 = 1$, also $\{\chi_2, \chi_3\} = \{\chi_1, \chi_1^{-1}\}$.

At the end, the description of all irreducible unitarizable subquotients of the induced representation of the group $SO(7,F)$ in which all three characters that are not quadratic are mutually nonisomorphic is given.

Theorem 7.4. *Let $\chi_i \in \widehat{F^\times}$ such that $\chi_i \ncong \chi_i^{-1}$, for $i \in \{1,2,3\}$, and let $\chi_i \ncong \chi_j$ for $i \neq j$ and $i,j \in \{1,2,3\}$. Further, let $0 \leq a_1 \leq a_2 \leq a_3$. Then all irreducible unitarizable subquotients of the $\nu^{a_1}\chi_1 \times \nu^{a_2}\chi_2 \times \nu^{a_3}\chi_3 \rtimes 1_{F^\times}$ are as follows:*

(i) $L(\nu^{-\frac{1}{2}}\chi_i \times \nu^{-\frac{1}{2}}\chi_i^{-1} \times \chi_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$, or $L(\nu^{-a_2}\chi_i \times \nu^{-a_2}\chi_i^{-1} \times \chi_1 \rtimes 1_{F^\times})$, if $a_1 = 0$ and $0 < a_2 < \frac{1}{2}$, also $a_2 = a_3$ and $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.
(ii) $\chi_1 \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$, if $a_1 = a_2 = a_3 = 0$.
(iii) $\chi_1 \times \delta([\nu^{-\frac{1}{2}}\chi_i, \nu^{\frac{1}{2}}\chi_i]) \rtimes 1_{F^\times}$, if $a_1 = 0$ and $a_2 = a_3 = \frac{1}{2}$, also $\chi_i \cong \chi_j^{-1}$, for $\{i,j\} = \{2,3\}$.

The proofs of the previous two theorems and lemma are left to the reader. One should use the ideas from the previous sections.

That completes the unitary dual of p-adic group $SO(7)$ with support on minimal parabolic subgroup.

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