CIT. Journal of Computing and Information Technology, Vol. 28, No. 2, June 2020, 73–87
doi: 10.20532/cit.2020.1005188

Network Connectivity Game

Darko Skorin-Kapov[1] and Jadranka Skorin-Kapov[2]

[1]Robert B. Willumstad School of Business, Adelphi University, Garden City, New York, United States
[2]College of Business, Stony Brook University, Stony Brook, New York, United States

We investigate the cost allocation strategy associated with the problem of providing service/communication between all pairs of network nodes. There is a cost associated with each link and the communication between any pair of nodes can be delivered via paths connecting those nodes. The example of a cost efficient solution which could provide service for all node pairs is a (non-rooted) minimum cost spanning tree. The cost of such a solution should be distributed among users who might have conflicting interests. The objective of this paper is to formulate the above cost allocation problem as a cooperative game, to be referred to as a *Network Connectivity (NC)* game, and develop a stable and efficient cost allocation scheme. The *NC* game is related to the *Minimum Cost Spanning Tree* games and to the *Shortest Path* games. The profound difference is that in those games the service is delivered from some common source node to the rest of the network, while in the *NC* game there is no source and the service is established through the two-way interaction among all pairs of participating nodes. We formulate *Network Connectivity (NC)* game and construct an efficient cost allocation algorithm which finds some points in the core of the *NC* game. Finally, we discuss the *Egalitarian Network Cost Allocation (ENCA)* rule and demonstrate that it finds an additional core point.

ACM CCS (2012) Classification: Networks → Network algorithms → Control path algorithms → Network resources allocation

Networks → Network performance evaluation → Network performance analysis

Keywords: networks, cost allocation, cooperative games, mathematical programming

1. Introduction

Consider a complete symmetric network in which all pairs of nodes need to communicate with each other. Examples of such communication networks are: telecommunication networks in which messages are exchanged between node pairs, networks of roads in which communities are connected to each other and are sending traffic to each other, train networks, *etc.* In such networks, the service is established and received by node pairs. For example, if the message is sent between two nodes of the communication network, the service is received by a pair of nodes (one sending a message and one receiving). If connection (friendship) is established between two users of the social network (Facebook/LinkedIn) a pair of nodes is benefiting from this service/connection. If a pair of cities are connected in the road network, the pair of cities receives the network service. Consequently, it seems reasonable to assume that actors/players are pairs of network nodes which need to communicate or deliver some service to each other. Note that in most network cost allocation problems considered in the literature, the service cost is delivered from some source(s) to network users residing at nodes. In those cases, source nodes are typically not paying for the cost and it is natural to identify players with nodes who receive the service. In our case, there is no source and the service is established for and by pairs of nodes. Our model is motivated by the natural context of a number of real networks (transportation networks, peer to peer telecommunication networks) in which service is established between pairs of nodes (cities, airports, computers, individual network users).

The communication in such networks is established through edges/links. It is assumed that each undirected edge enables communication in both directions. The two-way communication between any two nodes in our network can be delivered via any path connecting those two nodes. With each edge (link), we associate the cost of using that edge to establish the communication. We refer to a problem of finding the cheapest network which provides communication for a certain set of node pairs, as to the Network Connectivity (*NC*) problem. Herein, we study the special case of the *NC* problem in which all pairs of nodes require two-way communication. For this special case, the cost efficient solution which provides connection between all node pairs is given by the minimum cost spanning tree. The cost of this solution is shared by users who possibly have conflicting objectives. However, they might cooperate in order to decrease their joint cost. These individuals or organizations are likely to support the network only if their expectations for a "fair share" of the cost are met.

There are multiple proposed solutions, but no unique one-size-fits-all approach to cost allocation in networks. Cooperative game theory has been used to analyze several classes of such problems. Some related examples include: spanning tree games [1–6], Steiner tree games [7, 8], network flow games [9], cost allocation arising from routing in networks [10], capacitated network design games [11], hub network games [12, 13], shortest path games [14], social enterprise tree network games [15] and link games [16]. For a survey and some references on cost allocation models in networks see, for example, [17]. A common approach in the above papers is to formulate the associated cost allocation problem as a cooperative game in characteristic function form, followed by the evaluation of various game theoretic solution concepts in the context of a particular problem. We take a similar approach in this study of the cost allocation problem associated with the *Network Connectivity (NC)* problem.

Cooperative game theory offers the concept of a cost allocation solution known as the *core of a cooperative game*. The core consists of stable cost allocation solutions which provide no incentive for any coalition of users to secede and build their own subnetwork, *i.e.* it avoids crosssubsidies.

Let us first informally describe the *Network Connectivity (NC)* game. Let $G = (N, E)$ be a weighted complete symmetric network and let the set of players P consist of all pairs of network nodes. The characteristic function value for each subset S of players is defined as the cost of the network which would provide service to all node pairs in S.

This game is related to the extensively studied *Minimum Cost Spanning Tree (MCST)* games, *Steiner Tree (ST)* games and *Shortest Path (SP)* games. The profound difference between *NC* game and *MCST*, *ST* and *SP* games is that in the *MCST*, *ST* and *SP* games the service is delivered from a distinguished source node as in electric power networks, streaming TV services and in natural gas networks. In contrast, in the *NC* game there is no source and the communication is delivered between pairs of nodes, like in traffic systems (network of roads, trains or social peer to peer networks).

The non-emptiness of the core of the related *MCST* game was demonstrated and analyzed in the literature, see, *e.g.*, [1–3, 18–20]. However, the entire core of the *MCST* game has not been characterized, see [21]. It turns out that for the *ST* game, stable cost allocations do not necessarily exist, see, *e.g.*, [22]. The sufficient conditions for the existence of core points of the *ST* game, as well as the heuristic algorithm for finding them, were presented in [7, 8]. For approximation results on the *ST* problem and their application to associated cooperative games see, *e.g.*, [23, 24].

In the related study of *Social Enterprise Tree Network (SETN)* games, [15] introduced several games in which the cost of non-rooted directed minimum cost spanning tree is allocated among network nodes. It was demonstrated therein that the core of the *SETN* game(s) is not empty. There are a couple of important differences between the *SETN* game and the *NC* game. In the *SETN* game, players are nodes and the value of the characteristic function for a coalition $C \subseteq N$ is the cost of providing service from all network nodes in N to nodes in C. On the other hand, in the *NC* game the players are pairs of nodes and for any coalition $S \subseteq N * N$ the value of the characteristic function is the cost of the cheapest network which provides service to all node pairs in S. Nevertheless, in this paper we use some ideas about allocating cuts from [15] and apply it to the *NC* game.

In another related study of link games [16], the cost of the minimum cost spanning tree is allocated among the players who are edges of the network. Clearly, if the network is complete, then the players in link games would be all pairs of nodes like in our *NC* game under consideration. The main difference between the link game and the *NC* game is in the definition of the characteristic function. In the link game, the characteristic function value for the coalition *S* of edges is the cost of the network connecting all endpoints of edges in that coalition without use of any edges out of that coalition. In the *NC* game, the characteristic function value for the coalition of pairs *C* is the cost of the cheapest network providing service to all pairs in *C*. Namely, we assume the monotonic version in which the service to the coalition of pairs *C* can be delivered using any network edges, including edges whose endpoints do not appear in *C*. (In Table 1 we summarize how the characteristic function is defined for those related games.)

The main objective of this paper is to develop a cost allocation scheme for the *NC* game which might have practical applications in various communication systems. The input to our cost allocation problem is the complete symmetric network *G* and the minimum cost non-rooted spanning tree in *G*. We then define the *NC* game in characteristic function form. Then we, develop the *Network Connectivity Cost Allocation (NCCA)* algorithm which efficiently finds some cost allocation solutions in the core of the *NC* game. Finally, we introduce the *Egalitarian Network Cost Allocation (ENCA)* rule which potentially generates an additional core point.

Hence, the main contributions of this work are:

1. the formulation of the *NC* game and
2. the development and analysis of a computationally tractable cost allocation scheme(s) for generating some stable cost allocation solutions for the *NC* game.

The remainder of the paper is organized as follows. In Section 2, we review some standard definitions and formulate the *NC* game in the characteristic function form. In Section 3, we present the *Network Connectivity Cost Allocation (NCCA)* algorithm and prove that the *NCCA* algorithm efficiently generates some points in the core of the *NC* game. In Section 4, we discuss the *Egalitarian Network Cost Allocation (ENCA)* rule and prove that it also generates a core allocation. Finally, in Section 5, we summarize our findings and discuss future research.

Table 1. The characteristic functions.

Players / Tree	**Rooted tree**	**Non-rooted tree**
Nodes	*MCST* Min cost network connecting coalition *C* to the root	*SETN* Min cost network connecting coalition *C* to all nodes
Pairs of Nodes		Link (non-monotonic) Min cost network connecting all endpoints of edges in coalition *C* but using only edges in *C*. *NC* (monotonic) Min cost network connecting all node pairs in *C*

2. Definitions and Preliminaries

Let $G^* = (N, E^*)$ be a complete undirected weighted network with an edge weight (cost) function $w^*: E^* \to R^+$. (Remark: we equivalently consider a directed version in which each edge (i, j) is replaced with two directed edges (i, j) and (j, i) in the opposite direction each costing half of the original undirected edge. Hence, we actually consider the complete directed weighted network $G = (N, E)$ with an edge weight (cost) function $w: E \to R^+$ under the assumption that edge weights are symmetric, *i.e.* $w(i, j) = w(j, i)$, for all edges $(i, j) \in E$. Moreover, we assume that when some service is enabled via edge (i, j) so is the service via edge (j, i).)

The service in this network is delivered between pairs of nodes. Consequently, we define players as all subsets of nodes of cardinality two, namely $P = \{\{i, j\}: i, j \in N \text{ and } i \neq j\}$. The goal for each pair set $\{i, j\} \in P$ is to establish the cheapest two way connection via directed paths $p(i, j)$ and $p(j, i)$ and deliver the service from i to j and j to i through those paths, respectively. (Remark: Since we assumed that each link delivers service in both directions, paths $p(i, j)$ and $p(j, i)$ will use the same nodes and links in the opposite direction.)

Consider the following minimization problem. Find the minimum cost network $T = (N, E_T)$ in G such that for each pair of vertices $\{i, j\} \in P$ there exist directed paths $p(i, j)$ and $p(j, i)$ in T. The optimal solution $T = (N, E_T)$ is the directed (non-rooted) *Minimum Cost Spanning Tree* (*MCST*) in G. The objective is to allocate the cost of T among users in P.

In order to analyze this cost allocation problem we define the associated cooperative game in the characteristic function form. Let P be a set of players and let us define a characteristic function c by $c: 2^{|P|} \to R^+$, such that $c(\varnothing) = 0$ and for each $Q \subseteq P$, $c(Q)$ is the characteristic function value. We can interpret the value of the characteristic $c(Q)$ as the cost of providing service to a set of users in Q. The pair (P, c) is a cooperative game.

Specifically, we define characteristic function c as follows. The value of $c(P)$ is equal to the cost of the entire non-rooted minimum cost directed spanning tree T, *i.e.* the cost of providing service to all pairs of nodes. The characteristic function value $c(Q)$ for each subset of players Q, $Q \subseteq P$ is the cost associated with the service received by Q. It represents the cost of delivering two-way service between pairs in Q. Clearly, there is a question how should the above cost $c(Q)$ be determined. For example, it could represent the cost of providing a service to a coalition Q by using only nodes appearing in Q or perhaps allowing the use of nodes outside Q. If we allow the use of nodes out of Q, we might still restrict the coalition Q to use only edges of the optimal solution obtained for the grand coalition P, *i.e.* edges of the non-rooted directed spanning tree $T = (N, E_T)$ or allow Q to also use of edges in $E \setminus E_T$. We choose to define the characteristic function more generally, and seek the cheapest network that would provide service to Q in G. Namely, we consider the monotonic version in which we allow the coalition Q to use nodes out of Q and any edges in E. Consequently, $c(Q)$ represents the cost of a cheapest directed forest F in G such that for each $\{k, l\} \in Q$ there exist directed paths $p(k, l)$ and $p(l, k)$ in F. Then, the pair (P, c) is a cooperative game to be referred to as the *Network Connectivity (NC) game*.

Central to the theory of cooperative games is the solution concept referred to as the core of a game. The core $C(P, c)$ of a game (P, c) consists of all cost allocation vectors $x \in R^{|P|}$ $(x(P) = c(P))$, such that $x(Q) \leq c(Q)$ for all $Q \subseteq P$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede and build their own subnetwork, *i.e.* there is no cross-subsidization.

A cost cooperative game (P, c) is called *concave or submodular* if

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \text{ for all } S, T \subseteq P.$$

It is known that concave games [25] have non-empty cores. Next, we present an example of the *NC* game which is not concave but nevertheless has a non-empty core.

Example 1. Consider the network depicted in Figure 1. This network is the directed equivalent of the network used by [26] to demonstrate that the monotonic version of the minimum cost spanning tree game is not concave. We introduce the *NC* game on this network. Bold links indicate the optimal *NC* network.

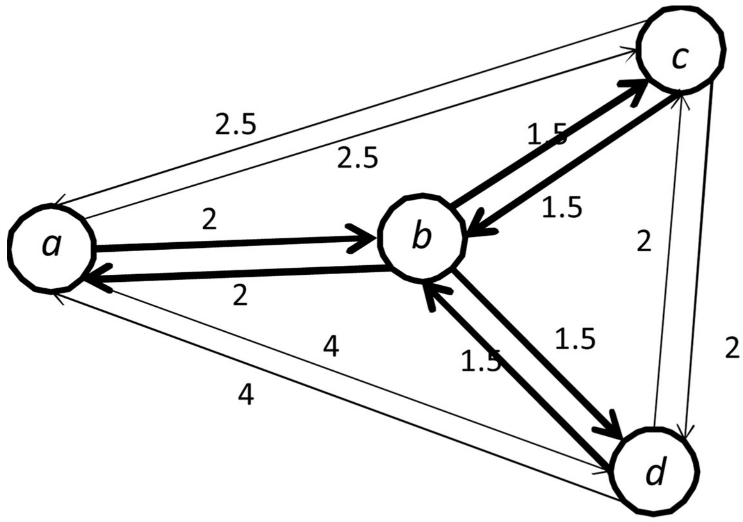

Figure 1. NC game is not concave.

The players are

$P = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

The characteristic function values:

$$
\begin{aligned}
&c(\{a, b\}) = 4,\\
&c(\{a, c\}) = 5,\\
&c(\{a, d\}) = 7,\\
&c(\{b, c\}) = 3,\\
&c(\{b, d\}) = 3,\\
&c(\{c, d\}) = 4,\\
&c(\{a, b\}, \{a, c\}) = 7,\\
&c(\{a, b\}, \{a, d\}) = 7,\\
&c(\{a, b\}, \{b, c\}) = 7,\\
&c(\{a, b\}, \{b, d\}) = 7,\\
&c(\{a, b\}, \{c, d\}) = 8,\\
&c(\{a, c\}, \{a, d\}) = 9,\\
&c(\{a, c\}, \{b, c\}) = 7,\\
&c(\{a, c\}, \{b, d\}) = 8,\\
&c(\{a, c\}, \{c, d\}) = 9,\\
&c(\{a, d\}, \{b, c\}) = 10,\\
&c(\{a, d\}, \{b, d\}) = 7,\\
&c(\{a, d\}, \{c, d\}) = 9,\\
&c(\{b, c\}, \{b, d\}) = 6,\\
&c(\{b, c\}, \{c, d\}) = 6,\\
&c(\{b, d\}, \{c, d\}) = 6,\\
&\ldots,\\
&c(P) = 10.
\end{aligned}
$$

Consider the following coalitions:

$$S = \{\{a, b\}, \{a, d\}\} \text{ and}$$
$$T = \{\{a, c\}, \{a, d\}\}.$$

Then,

$$c(S) + c(T) = 7 + 9 = 16 \text{ and}$$
$$c(S \cup T) + c(S \cap T) = c(\{a, b\}, \{a, d\}, \{a, c\}) + \\ + c(\{a, d\}) = 10 + 7 = 17,$$

which violates the concavity constraint.

It can be verified that the cost allocation

$$x = (x_{ab}, x_{ac}, x_{ad}, x_{bc}, x_{bd}, x_{cd}) = (4, 0, 0, 3, 3, 0)$$

is in the core of the above game (P, c). Although this allocation is in the core, one might argue that it seems unfair that players $\{a, c\}$, $\{a, d\}$ and $\{c, d\}$ do not pay anything for their services. Our *NCCA* algorithm will enable us to generate additional core cost allocations in which more players will participate.

Our network connectivity cost allocation scheme for the *NC* game constructed in the next section will allocate the service cost based on certain cut sets in G. Hence, we need to define the concept of a cut set. For a directed edge $l = (i, j)$ we refer to i as the tail and j as the head of l, and for a subset of vertices $S \subseteq N$, we denote by $\delta(S)$ the set of all directed edges having their heads, but not their tails, in S. A subset

$S \subseteq N$, is said to be a cut set of $G = (N, E)$, if $S \neq \emptyset$ and the undirected subgraph $G(S)$ of G induced by S is connected. We denote by φ the set of all cut sets of G.

The *non-rooted minimum spanning directed tree* problem can be formulated as the following integer programming problem:

$$IP(N)\text{: } \min\left\{wu\text{: } u(\delta(S)) \geq 1, \text{ for all } S \in \varphi, \; u_{(i,j)} - u_{(j,i)} = 0 \text{ for all } (i,j) \in E, u \in \{0,1\}^E\right\},$$

where $u \equiv u(i, j) = 1$ if $(i, j) \in E$ is used in the directed spanning tree T, and 0 otherwise. Similar integer programming formulation for the rooted minimum spanning tree is presented in [27]. Therein, the LP relaxation is used to develop a primal-dual algorithm which generates some core allocations for the minimum cost spanning tree game. It was shown in the study of the social tree network game [15], that due to the integrality gap this primal-dual result does not hold for the minimum cost unrooted spanning tree game. Nevertheless, we will use some ideas from the above primal-dual algorithm for the minimum cost rooted spanning tree game to construct the cost allocation scheme for the *NC* game.

3. The Network Connectivity Cost Allocation (*NCCA*) Algorithm

Based on the results obtained in the *MCST* related literature, we intuitively expect that the core of the *NC* game is not empty. Indeed, in this section, we construct a Network Connectivity Cost Allocation (*NCCA*) algorithm and demonstrate that it generates some core points of the *NC* game (P, c). This algorithm is a modification of the primal-dual algorithm used in the context of the minimum cost spanning tree games [27] and social enterprise network games [15].

The input to our algorithm is the symmetric directed weighted complete network $G = (N, E)$, the minimum cost non-rooted directed spanning tree $T = (N, E_T)$ in G obtained by some known algorithm and a set of players $P = \{\{i,j\}: i,j \in N\}$ and $i \neq j\}$. If we assume that the service for each edge is always established in both directions, then it is easy to verify that T is the optimal network which provides two-way service to all pairs of nodes in P. Note that in the absence of this assumption, for example, the Hamiltonian cycle could be a cheaper solution than the minimum cost directed spanning tree.

In the initial step of the *NCCA* algorithm, we set the cost allocation vector $x \in R^{|P|}$ to zero, and then in each iteration, we will construct some cut set S and allocate some of the cost of edges in $E_T \cap \delta(S)$.

In each subsequent step, we will construct a minimal cut set S, for which the weights of all edges in $\delta(S)$ have positive values. We refer to such cut sets as allocating cuts. We find the smallest weight edge in $\delta(S)$ and denote its weight by w_S. Then for each $(i, j) \in \delta(S) \cap E_T$ we allocate the amount w_S arbitrarily to the users in the allocating set $iS = \{\{i, s\}, s \in S\}$ induced by the allocating cut set S. It will be demonstrated in the proof of Theorem 3 that this allocation does not violate core constraints. We also reduce the weights of all the edges in set $\delta(S)$ by w_S. Once a directed edge $e \in \delta(S)$ has its weight reduced to zero (we refer to it as a directed zero edge), S cannot be used as an allocating cut. A directed path $p_0(k, l)$ is called a directed zero path if all of its edges are directed zero edges. Clearly, the idea of the algorithm is to use minimal cut sets in such a way to allocate as much cost as possible without violating core constraints.

Next, we will show that the *NCCA* algorithm generates points in the core of the *NC* game (P, c). We will demonstrate that the algorithm allocates the entire cost of the optimal non-rooted minimum cost directed spanning tree $T = (N, E_T)$ and that the cost allocation satisfies core constraints. We need a couple of Lemmas. Those two Lemmas were already proven in the context of the social enterprise tree network cost allocation algorithm [15], so we omit their proofs.

Lemma 1. The construction of the *NCCA* algorithm implies that for each allocating cut set S of cardinality $|S| > 1$ considered in the course of the algorithm there exists a directed zero weight path between each pair of nodes contained in S.

Lemma 2. All edges in the network whose weights are reduced to zero during the execution of the *NCCA* algorithm must be the edges of the optimal non-rooted directed minimum cost spanning tree $T = (N, E_T)$.

Algorithm 1. The *NCCA* algorithm.

Input: A symmetric complete directed weighted network $G = (N, E)$ with an edge weight (cost) function $w: E \rightarrow R^+$ with initial weights $w(i, j) = W(i, j)$ and the optimal non-rooted minimum cost directed spanning tree $T = (N, E_T)$ in G obtained using some known algorithm.

Preprocessing: We modify the weights of $G = (N, E)$ by increasing the weights of all edges that do not belong to T, by an arbitrarily small weight $\varepsilon > 0$

Note: By slightly increasing the above edge weights, we make the minimum cost directed spanning tree T in G unique.

Initialization:
Set the cost allocations $x_{\{i,j\}} = x(\{i, j\}) = 0$, for all $\{i, j\} \in P$.
  Main Step:
  "Find a potential allocating cut set S (the smallest in size)".
    Do for $k = 1, \ldots |N - 1|$
      Do for each $n \in N$
        Let $S = \{n\}$.
        Do until $|S| = k$ or no additional nodes can be added to S
          If $\exists\ (i, j) \in \delta(S)$ such that $w(i, j) = 0$, let $S = S \cup \{i\}$
        End Do
        If for all $e \in \delta(S)$, $w(e) > 0$
          Let $w_S = \min\{w(e), e \in \delta(S)\}$.
          For each $(i, j) \in \delta(S) \cap E_T$ allocate the amount w_S arbitrary to users in allocating set $iS = \{\{i,s\}, s \in S\}$.
          Do for all $(i, j) \in \delta(S) \cap E_T$
            Choose $a_{\{is\}}$, $\{is\} \in iS$ such that $\sum_{s \in S} a_{\{is\}} = w_S$.
            Do for all $\{is\} \in iS$
              Let $x_{\{is\}} = x_{\{is\}} + a_{\{is\}}$
            EndDo
          EndDo
          Do for all $e \in \delta(S)$
            Let $w(e) = w(e) - w_S$
            EndDo
        EndIf
      EndDo
    EndDo
    End

Theorem 3. Let $T = (N, E_T)$ be the *non-rooted Minimum Cost Directed Spanning Tree* in a network $G = (N, E)$ and let (P, c) be the associated *NC* game. Then the *NCCA* algorithm generates a cost allocation in the core of this game.

Proof. First, we show that in the course of the *NCCA* algorithm, the entire cost of T gets allocated. By construction, whenever any edge in E_T was reduced by some amount, that amount was allocated. Further, by construction, Lemma 1 and Lemma 2, the weights of all edges in E_T are reduced to zero. Hence, in the course of the *NCCA* algorithm the entire cost of T has been allocated, *i.e.* $c(P) = x(P)$. Next, we will show that the other core constraints are satisfied, namely that for each Q, $Q \subset P$, $x(Q) \le c(Q)$.

For a coalition $Q \subset P$ let $F = (N_Q, E_Q)$ be the minimum cost directed forest providing service to Q and whose overall cost is $c(Q)$. Let F consist of K connected components

$$F_1 = (N_{Q_1}, E_{Q_1}), \ldots, F_K = (N_{Q_K}, E_{Q_K})$$

which provide two way connections for all pairs of nodes in $Q_1, \ldots, Q_K$, respectively, and

$$Q = Q_1 \cup \ldots \cup Q_K.$$

Note that, in order to demonstrate that the core constraints are satisfied, we do not need to find forest F explicitly.

If $F = (N_Q, E_Q)$ is a single component spanning all nodes in N then F is a minimum cost spanning tree and can provide service to all players in P and $c(Q) = c(P) = x(P) \geq x(Q)$.

Otherwise, we start with a forest $F = (N_Q, E_Q)$ and then keep adding to F some directed edges of the minimum cost spanning tree $T = (N, E_T)$ whose cost was paid by players in $P \setminus Q$ in the course of the *NCCA* algorithm until we build a network F' which provides service to all players in P.

It is easy to verify that for each connected component $F_m = (N_{Q_m}, E_{Q_m})$ of F there must exist a two-way connection between every pair of vertices in that component. Namely, F_{Q_m} is a non-rooted directed Steiner tree spanning all nodes contained in the set of pairs Q_m and possibly some other intermediate nodes. Hence, $F_m = (N_{Q_m}, E_{Q_m})$ actually provides service to all pairs of nodes $\{i, j\}$, $i, j \in N_{Q_m}$.

Observe that by construction, for each directed edge $(i, j) \in E_T$ such that $i \notin N_Q$ the cost $w(i, j)$ was allocated to some pairs of nodes which all contain i. Since vertex $i \notin N_Q$, i is not contained in any vertex pair in Q and the entire cost of (i, j) is paid by some users in $P \setminus Q$.

Initially, $F' = F$.

Then, we add to F' all nodes in $N \setminus N_Q$ and all directed edges $(i, j) \in E_T$ such that $i \notin N_Q$ and $j \notin N_Q$. Clearly, if (i, j) was added to F' in this step, so was (j, i) and the costs for both of these edges were paid by players in $P \setminus Q$. At this point F' consists of connected components

$$F_1 = (N_{Q_1}, E_{Q_1}), \ldots, F_K = (N_{Q_K}, E_{Q_K})$$

which comprise F and potentially some connected components

$$F_{K+1} = (N_{K+1}, E_{K+1}), \ldots, F_{K+L} = (N_{K+L}, E_{K+L})$$

(subtrees of T and/or some isolated nodes) added to F. See Figure 2 for the illustration of this step. Note that by this construction

$$N_{Q_1} \cup \ldots \cup N_{Q_K} \cup N_{K+1} \cup \ldots \cup N_{K+L=N}.$$

Since, $T = (N, E_T)$ is a directed spanning tree, there exist directed edges in E_T which connect all components of F' in G.

Next, we perform the main building step. Among pairs of opposite edges $(i, j), (j, i) \in E_T$, such that i and j are not in the same connected component of F', find the first such pair for which $w(i, j)$ and $w(j, i)$ were both reduced to zero during the course of the *NCCA* algorithm. Then i is in some connected component F_m of F' (and j is not in F_m) and every allocating cut set S such that $(i, j) \in \delta(S) \cap E_T$ must have had an empty intersection with a node set of F_m. This is implied by Lemma 1 and Lemma 2. Namely, for every two nodes in an allocating set S there are zero directed paths between them in both directions (Lemma 1) and these zero paths belong to the directed minimum cost spanning tree (Lemma 2). Consequently, if S had a non-empty intersection with the node set of F_m then the pair (i, j), $(j, i) \in E_T$ could not have been a two way connection between two components which was first reduced to zero. This further implies that the cost $w(i, j)$ has been paid by users in $P \setminus Q$. By the same reasoning the edge (j, i) is also paid by users in $P \setminus Q$. We add (i, j) and (j, i) to F'. Now F' has one less connected component with all nodes being two way connected within each component. We repeat this building step until F' becomes a single connected component spanning all nodes in N.

As a result, we constructed a network F' which provides service to all pairs of nodes, *i.e.* to all players in P. Moreover, the cost of $F' \setminus F$ did not exceed the cost of $x(P \setminus Q)$ assigned by the *NCCA*.

Thus, $x(P \setminus Q) + c(Q) \geq c(P) = x(P)$, and then $c(Q) \geq x(P) - x(P \setminus Q) = x(Q)$. □

Corollary 4. Given an instance of the *NC* game on a graph $G = (N, E)$ the *NCCA* algorithm runs $O(|N|^3)$ time.

Proof. For any given k, the algorithm examines at most n links for each of n nodes. Since k takes on the values 1 through $|N - 1|$, the running time of the *NCCA* algorithm is $O(|N|^3)$. □

Remark. The cost allocations obtained by the *NCCA* algorithm are telling us how much each pair of users are charged to establish their service within the network. We might also interpret the *NCCA* in the relation to the rooted

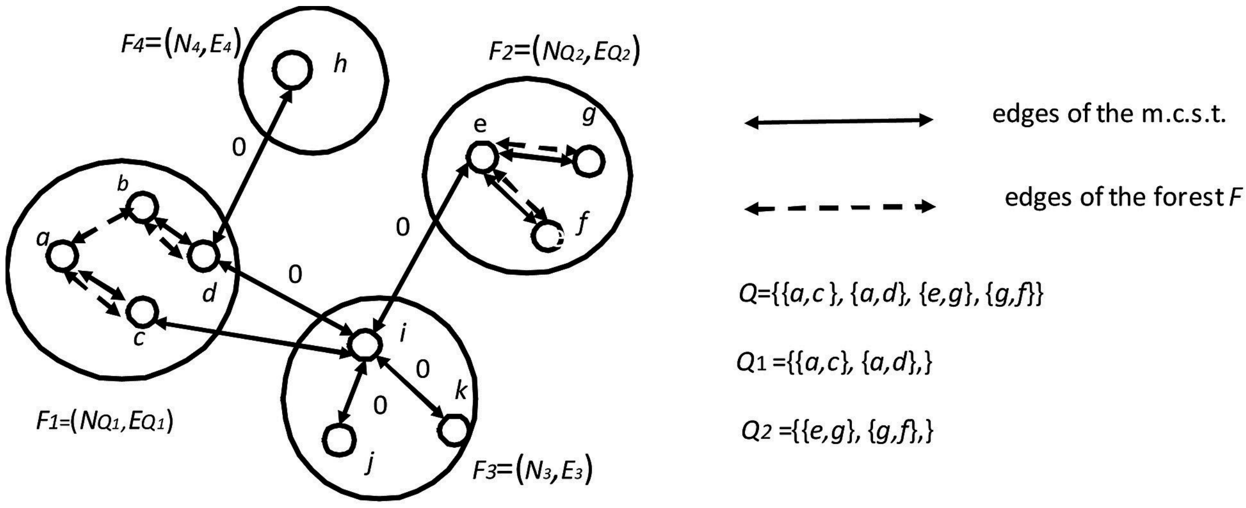

Figure 2. The core constraint for the coalition Q.

minimum cost spanning tree games. In fact, we might modify a set of players P to consist of all ordered pairs of nodes. In this situation, for each specific node i and all $j \in \mathrm{N}$, we can interpret x_{ij} as the cost j pays to get service from i. That can be considered in the context of the rooted minimum cost spanning tree games. Consider for each $i \in N$ the rooted minimum cost spanning tree rooted in i and associated minimum spanning tree game $(N \setminus \{i\}, c)$. Here, for each coalition $S \subseteq N \setminus \{i\}$, the characteristic function value $c(S)$ is assumed to be the cost of minimum cost directed Steiner tree rooted in i which is spanning $S \cup \{i\}$. It can be easily verified that by the construction of the *NCCA* algorithm and dual feasibility in minimum cost spanning tree games (see for example [27] and [15]), for each $i \in N$ the cost allocation $(x_{ij}, j \in N$), satisfies the core constraints for the minimum spanning tree game $(N \setminus \{i\}, c)$. However, note that the cost allocation $(x_{ij}, j \in N)$, is not necessarily a core allocation of the game $(N \setminus \{i\}, c)$. Namely, due to overlapping of the above rooted trees, the *NCCA* algorithm might not allocate the entire cost of the minimum cost spanning tree rooted in i.

In the following example, we illustrate the *NCCA* algorithm.

Example 2. We consider in Figure 3 the same network as the one in Figure 1.

Let us follow the *NCCA* algorithm. Considered allocating cut sets are: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{b, c, d\}$. During the algorithm when the cost is allocated to some pairs in iS (S being an allocating cut set), we allocate the cost equally to all considered node pairs. Consequently, the cost allocation is obtained as follows:

$$x_{ab} = 2 + 1.5 + 0.5 / 3,$$
$$x_{ac} = 0.5 / 3,$$
$$x_{ad} = 0.5 / 3,$$
$$x_{bc} = 1.5 + 1.5,$$
$$x_{bd} = 1.5 + 1.5,$$
$$x_{cd} = 0.$$

Thus, the cost allocation is

$$x = (x_{ab}, x_{ac}, x_{ad}, x_{bc}, x_{bd}, x_{cd}) = (22/6, 1/6, 1/6, 3, 3, 0).$$

In another execution of the above *NCCA* algorithm on this network we could allocate the cost of each tree edge to the pair of end nodes of that edge. In that case, the cost allocation is

$$x' = (4, 0, 0, 3, 3, 0).$$

Both of these cost allocation solutions are in the core. For the solution x' three players ($\{a, c\}$, $\{a, d\}$, $\{c, d\}$) have a free ride. Solution x involves two of them but player $\{c, d\}$ still has a free ride. It is easy to verify that all executions of the *NCCA* algorithm on this example enable a free ride for player $\{c, d\}$.

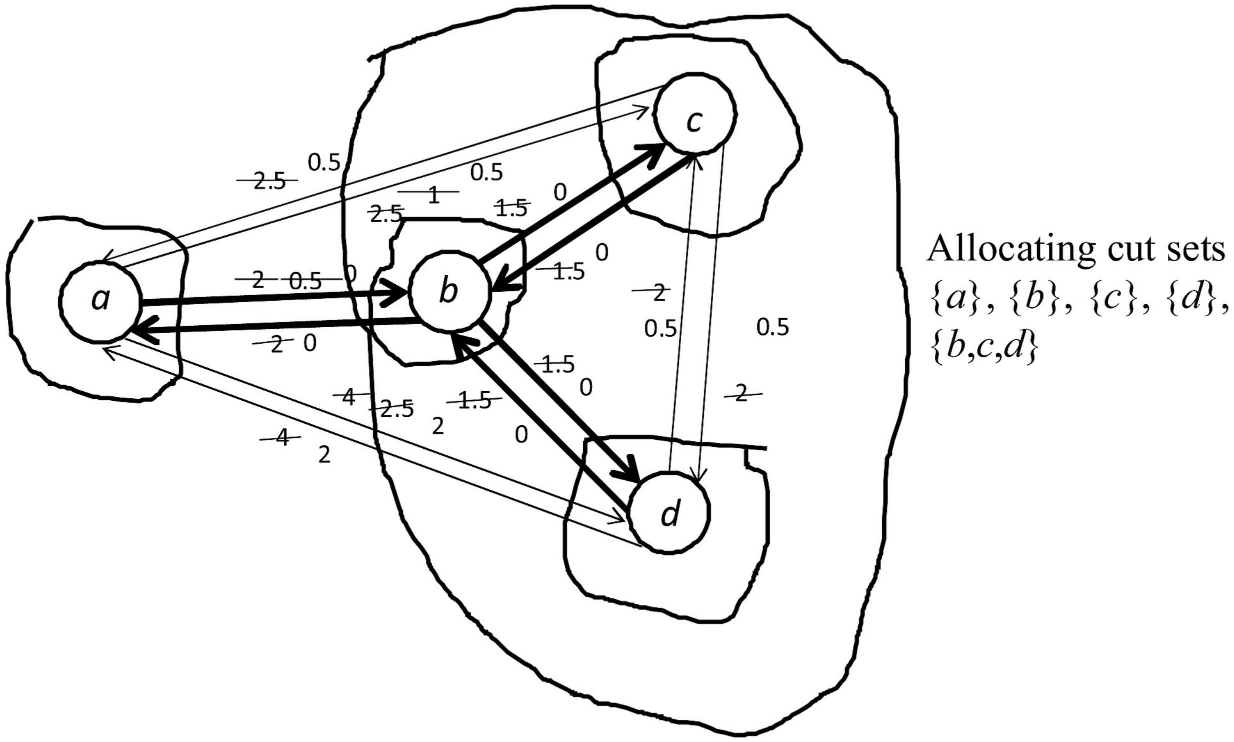

Figure 3. Allocations generated by the NCCA algorithm.

Clearly, the *NCCA* algorithm provides numerous core allocations. Naturally, we would like to determine whether there exist additional core allocations to those generated by the *NCCA* algorithm. Moreover, is there a cost allocation rule which produces a core allocation which does not allow a free ride to any player? The affirmative answer is given in the next section.

4. The Egalitarian Cost Allocation Rule

We now introduce the *Egalitarian Network Cost Allocation (ENCA)* rule. We divide the cost of each link equally between all users/players which use that link. Next, we show that the *ENCA* rule generates a cost allocation in the core of the *NC* game. Since we work under the assumption that each edge consist of two symmetric directed links in the opposite direction, we can equivalently analyze this cost allocation rule on the undirected network.

Theorem 5. Let $T = (N, E_T)$ be the non-rooted Minimum Cost Spanning Tree in an undirected network $G = (N, E)$ with users/players $P = N * N$ and let (P, c) be the corresponding *NC* game. Then the cost allocation x generated by the Egalitarian Network Cost Allocation rule is in the core of the game (P, c).

Proof. Clearly by definition of the *ENCA* rule $x(P) = c(P)$. We will also demonstrate that for every coalition $S \subset P$, the core constraint $c(S) \geq x(S)$ holds. Let F be the minimum cost network providing service to all pairs in S. Note that F is some forest (we do not need to find this forest explicitly). If F has multiple connected components $F_1, ..., F_k$ then there could be no player (pair of nodes) in S which has nodes in two different components. Consequently, we can decompose S in such a way that $S = S_1 \cup ... \cup S_k$, and $c(S_1), ..., c(S_k)$ are the optimal connectivity costs of components $F_1, ..., F_k$, respectively, and $c(S) = c(S_1) + ... + c(S_k)$. This implies that it is enough to consider only those coalitions S for which the optimal solution connecting all players in S is the single connected component *i.e.* the tree $T_S = (N_S, E_S)$. Moreover, without loss of generality, we can assume that S consists of all pairs of nodes in N_S (any proper subset would only pay less).

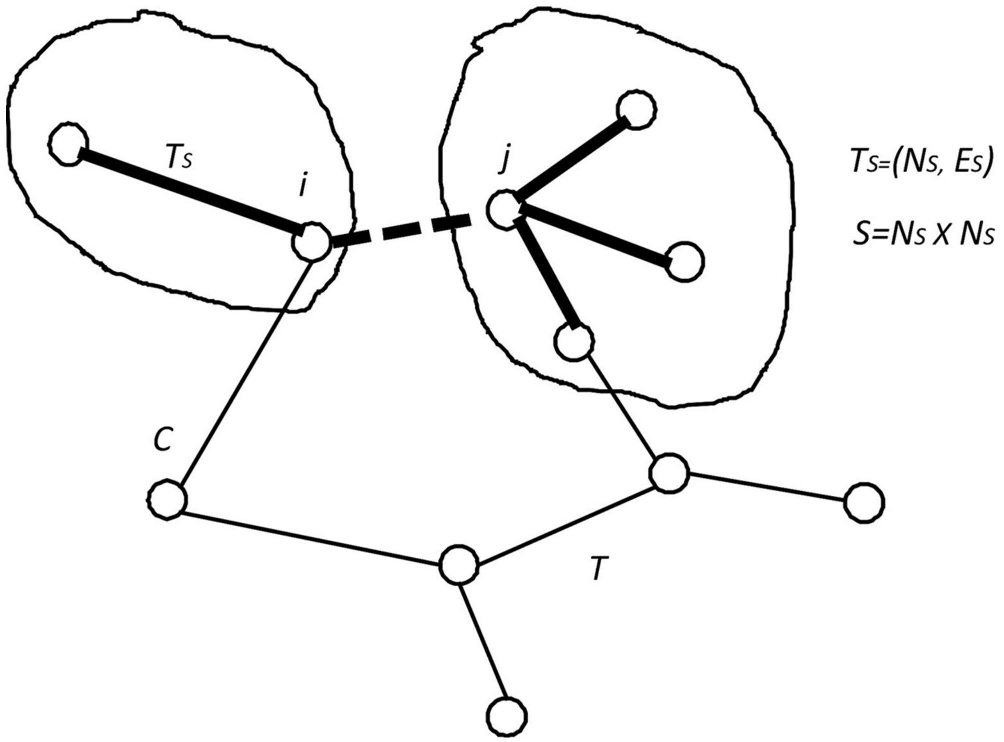

Figure 4. The core constraint for the coalition S.

Let $S = N_S * N_S$ be a coalition of users for which the service cost $c(S)$ is the weight of a minimum cost tree $T_S = (N_S, E_S)$ used to provide service to S.

If T_S is a subtree of T then by the construction players in S participate in payments only for edges of T_S. In such a case obviously $c(S) \geq x(S)$. It remains to analyze the case when T_S contains edge(s) which do not belong to the minimum cost spanning tree T. This situation is illustrated in Figure 4. Solid lines belong to the minimum cost spanning tree T and thick lines belong to T_S. Note that the thick (dotted) link (i, j) belongs to T_S but does not belong to T. If link (i, j) is added to T it creates a cycle C and (i, j) is the link with the largest weight in C. On the other hand removal of link (i, j) from T_S breaks into two connected components like those circled in Figure 4. Clearly, every pair in S which has nodes in two different components of T_S created by the removal of (i, j) from T_S, participated in paying the costs of all links in cycle C except the link (i, j). Also note that C must be at least partially out of T_S since otherwise (i, j) would not be needed in T_S.

Let m be the number of pairs in S which used link (i, j) to deliver service to coalition S via T_S and participated in paying part of the cost of links in the cycle C which are not in T_S. Let l be the number of links which are in C but not in T_S. It is easy to verify that each of those l links has also been partially paid by at least lm players which are not in S (these are players having one node in one of those l links and the other in one of those m pairs which used (i, j) in T_S). Since the cost of each link is divided equally between all players which use it, then it follows that the users in S paid for each link (u, v) in C but not in T_S at most $m * w(u, v) / m * l$ Consequently, total amount that users in S paid for l links in C which are not in T_S is at most

$$\max\{w(u, v), (u, v) \text{ in } C\} = \mathrm{w}(i, j).$$

Hence, if *ENCA* rule is applied, users in S pay at most the total weight of T_S, *i.e.* $c(S) \geq x(S)$. □

Example 3. Consider now the network presented in Figure 5. It is the same network which we analyzed in Example 2 but modified to undirected case. If we apply *ENCA* rule to that network we get the following cost allocation:

$$x_{ab} = 4/3,$$
$$x_{ac} = 4/3 + 3/3,$$
$$x_{ad} = 4/3 + 3/3,$$
$$x_{bc} = 3/3,$$
$$x_{bd} = 3/3,$$
$$x_{cd} = 3/3 + 3/3.$$

Hence, by Theorem 5 cost allocation,

$$\begin{aligned} x &= (x_{ab}, x_{ac}, x_{ad}, x_{bc}, x_{bd}, x_{cd}) \\ &= (4/3, 7/3, 7/3, 1, 1, 2) \end{aligned}$$

is in the core of the *NC* game and it does not allow free ride to any player.

Example 4. For the comparison with Moretti's link games, in Figure 6 we also present the network taken from [16].

Let $N = \{a, b, c, d\}$, $P = N * N$, (N, c) our *NC* game and (N, c') Moretti's link game. Recall that in the link game each coalition can use only edges in the coalition while in NC game each coalition can use any network edge. Which game is more appropriate is application driven.

Clearly, by definition for each coalition $S \subseteq P$, $c(S) \leq c'(S)$. Consequently, the core of the *NC* game is a subset of the core of the link game. Please note that determining the entire core of these games remains an open question.

Moretti's decomposition algorithm (*DA*) found the following core cost allocation of the link game:

$$\begin{aligned} x^{DA} &= \left(x_{ab}^{DA}, x_{ac}^{DA}, x_{ad}^{DA}, x_{bc}^{DA}, x_{bd}^{DA}, x_{cd}^{DA}\right) \\ &= \left(1, 1, 0, 0, 1.5, 0.5\right). \end{aligned}$$

(It turns out that this particular allocation is also in the core of the *NC* game.)

Next, we apply our *NCCA*. Here, we modify the graph into a directed one. We replace each undirected edge with two directed edges in the opposite direction each having half of the original weight. Allocating cuts are {1}, {2}, {3}, {4}, {1, 2, 3} and they lead to the following core allocation of the *NC* game:

$$\begin{aligned} x^{NCCA} &= \left(x_{ab}^{NCCA}, x_{ac}^{NCCA}, x_{ad}^{NCCA}, x_{bc}^{NCCA}, x_{bd}^{NCCA}, x_{cd}^{NCCA}\right) \\ &= \left(1, 1, 1/6, 0, 10/6, 1/6\right). \end{aligned}$$

Finally, if we apply our *ENCA* rule (on the undirected version), we get the core allocation of the *NC* game:

$$\begin{aligned} x^{ENCA} &= \left(x_{ab}^{ENCA}, x_{ac}^{ENCA}, x_{ad}^{ENCA}, x_{bc}^{ENCA}, x_{bd}^{ENCA}, x_{cd}^{ENCA}\right) \\ &= \left(3/12, 4/12, 11/12, 7/12, 8/12, 15/12\right). \end{aligned}$$

Observe that the *DA* algorithm allowed two free riders, *NCCA* had one free rider and *ENCA* did not allow any free riders. Nevertheless, it remains to investigate which of these allocations can be considered more fair.

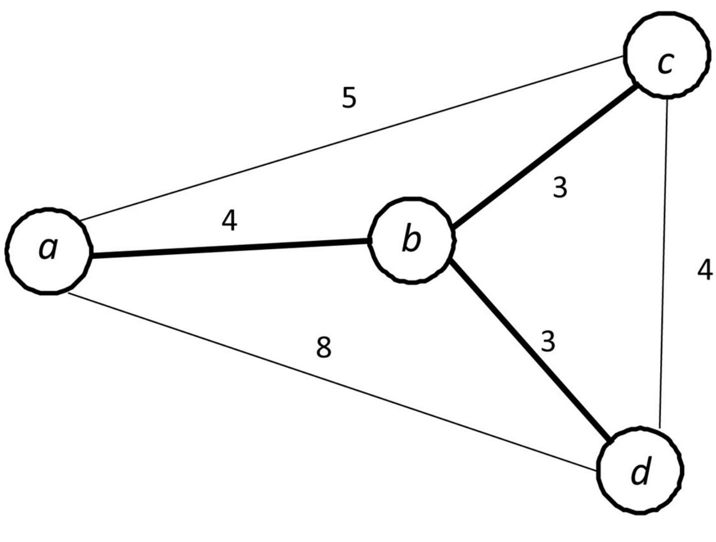

Figure 5. ENCA rule.

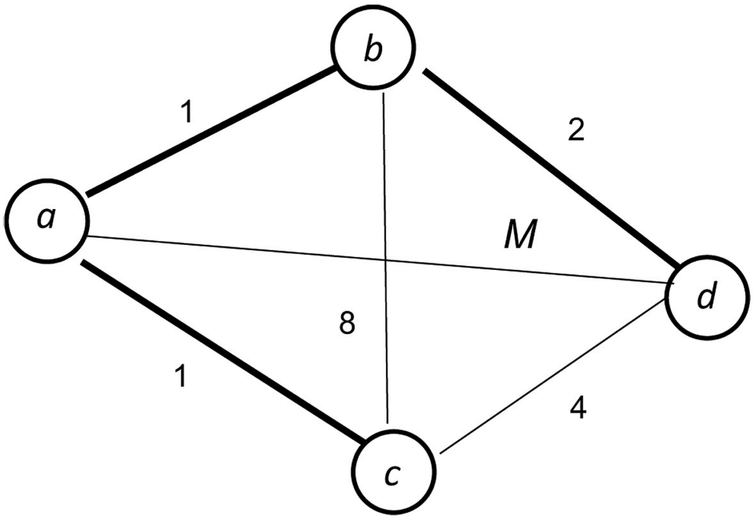

Figure 6. Comparison Link Game and *NC* game.

5. Conclusions and Future Research

We studied the cost allocation problem associated with the network connectivity enterprise in which service is delivered among all network nodes via the non-rooted directed minimum cost spanning tree. We modeled this cost allocation problem as a cooperative game in characteristic function form which we called the *Network Connectivity (NC)* game. Our *NC* game is related to minimum cost spanning tree games and Steiner tree network games previously considered in the literature. The difference is that in those games the network service is delivered from a certain source to other network users, while in the *NC* games there is no central source and the service is delivered from all nodes.

The *NC* game is also related to the *SETN* games and Link games in which there is interaction between nodes but there is no source. However, there are some substantial differences. In the *SETN* games, nodes are players, while in the *NC* game, the set of players consist of pairs of nodes. In the link games, the players are edges which in case of a complete network would be all pairs of nodes. The main difference is in the definition of the characteristic function. In the link games, the coalition of edges is allowed to use only links of that coalition. In our framework for the *NC* game, the best potential solution for a particular coalition could also use nodes and edges out of that coalition.

Our objective was to find cost allocations for which no coalition has incentive to secede and act on their own, or equivalently, to find some points in the core of the *NC* game. We demonstrated that the core of our *NC* game is not empty and we constructed the cost allocation schemes *NCAA* and *ENCA* which efficiently find some core points.

We suggest avenues for future research of the *NC* game. It would be interesting to try to characterize the set of core allocations and investigate if there is a procedure to obtain all core allocations. Furthermore, it would be interesting to try to compute the nucleolus of the *NC* game *i.e.* another specific (unique) core cost allocation solution with some desirable properties. Further, it would be interesting to determine whether there exists a cost monotonic and/or population monotonic cost allocation scheme for this game. Namely, the challenge is to determine if there exists a cost allocation scheme for which the cost reduction of some edge would not increase cost to any coalition and/or the addition of a player would not increase the cost to any original player. The analysis and comparison of desirable properties of such various solutions should follow. Finally, it would be interesting to study the general case of the *NC* game in which we do not require service between all node pairs in the network. In this case, we would need to allocate the cost of the optimal solution for a grand coalition which would be some Steiner forest instead of the minimum cost spanning tree.

References

[1] S. Bird, "On Cost Allocation for a Spanning Tree: A Game Theoretic Approach", *Networks*, vol. 6, pp. 335–350, 1976.

[2] N. Megiddo, "Computational Complexity and the Game Theory Approach to Cost Allocation for a Tree", *Mathematics of Operations Research*, vol. 3, pp. 189–196, 1978.

[3] D. Granot, "A Generalized Linear Production Model: A Unifying Model", *Mathematical Programming*, vol. 34, pp. 212–223, 1986.

[4] B. Dutta and A. Kar, "Cost Monotonicity, Consistency and Minimum Cost Spanning Tree Games", *Games and Economic Behavior*, vol. 48, pp. 223–248, 2004.

[5] L. Lorenzo and S. Lorenzo-Freire, "A Characterization of Kruskal Sharing Rules for Minimum Cost Spanning Tree Problems", *International Journal of Game Theory*, 2009.

[6] G. Bergantinos and J. Vidal-Puga, "Characterization of Monotonic Rules in Minimum Cost Spanning Trees", *International Journal of Game Theory*, pp. 1–34, 2014.

[7] D. Skorin-Kapov, "On the Core of the Minimum Steiner Tree Game in Networks", *Annals of Operations Research*, vol. 57, pp. 233–249, 1995.

[8] D. Skorin-Kapov and J. Skorin-Kapov, "A Note on Steiner Tree Games", *Networks*, vol. 59, 215–225, 2012.

[9] D. Granot and F. Granot, "On Some Network Flow Games", *Mathematics of Operations Research*, vol. 17, no. 4, pp. 792–841, 1992.

[10] J. A. M. Potters *et al.*, "Traveling Salesman Games", *Mathematical Programming*, vol. 53, pp. 199–211, 1987.

[11] D. Skorin-Kapov, "On a Cost Allocation Problem Arising From a Capacitated Concentrator Covering Problem", *Operations Research Letters*, vol. 13, no. 5, pp. 315–323, 1993.

[12] D. Skorin-Kapov, "Hub Network Games," *Networks*, vol. 31, pp. 293–302, 1998.

[13] D. Skorin-Kapov, and J. Skorin-Kapov, "Threshold Based Discounting Network: The Cost Allocation Provided by the Nucleolus", *European Journal of Operations Research*, vol. 166, pp. 154–159, 2005.

[14] E. C. Rosenthal, "Shortest Path Games," *European Journal of Operations Research*, vol. 224, pp. 132–140, 2013.

[15] D. Skorin-Kapov, "Social Enterprise Tree Network Game", *Annals of Operations Research*, vol. 268, pp. 5–20, 2018.

[16] S. Moretti, "On Cooperative Connection Situations Where the Players are Located at the Edges", in Volume 10767 of Lecture Notes in Computer Science Lecture Notes in Artificial Intelligence, Springer, pp. 339–353, 2018.

[17] N. Nisan *et al.*, "Algorithmic Game Theory," Cambridge University Press, 2007.

[18] D. Granot and G. Huberman, "Minimum Cost Spanning Tree Games", *Mathematical Programming*, vol. 21, pp. 1–18, 1981.

[19] D. Granot and G. Huberman, "On the Core and Nucleolus of the M.C.S.T. Games", *Mathematical Programming*, vol. 29, pp. 323–347, 1984.

[20] H. S. Moretti Norde and S. Tijs, "Minimum Cost Spanning Tree Games and Population Monotonic Cost Allocation Schemes", *European Journal of Operational Research*, vol. 154, pp. 84–97, 2004.

[21] U. Faigle *et al.*, "On the Complexity of Testing Membership in the Core of Min-Cost Spanning Tree Games", *International Journal of Game Theory*, vol. 26, pp. 361–366, 1997.

[22] A. Tamir, "On the Core of Network Synthesis Game", *Mathematical Programming*, vol. 50, pp. 123–135, 1991.

[23] V. V. Vazirani, "Recent Results on Approximating the Steiner Tree Problem and its Generalizations", *Theoretical Computer Science*, vol. 235, pp. 205–216, 2000.

[24] K. Jain and V. V. Vazirani, "Applications of Approximation Algorithms to Cooperative Games", in *Proc. of the STOC'01*, 2001, Hersonissos, Crete, Greece.

[25] L. S. Shapley, "Cores and Convex Games", *International Journal of Game Theory*, vol. 1, pp. 1–26, 1971.

[26] H. Aarts, "Minimum Cost Spanning Tree Games and Set Games", PhD Thesis, University of Twente, 1994.

[27] D. Skorin-Kapov, "On Primal-Dual Cost Allocation Schema in Multicast Communication: Combinatorial Game Theory Model", in *Proc. of the 17th International Conference on Industrial Engineering and Engineering Management*, 2010, pp. 433–438.

Received: January 2021
Revised: April 2021
Accepted: May 2021

Contact addresses:
Darko Skorin-Kapov
Robert B. Willumstad School of Business
Adelphi University
Garden City
New York
United States
e-mail: skorin@adelphi.edu

Jadranka Skorin-Kapov
College of Business
Stony Brook University
Stony Brook
New York
United States
e-mail: jadranka.skorin-kapov@stonybrook.edu

Darko Skorin-Kapov received his PhD in Management Science from the University of British Columbia, Vancouver, Canada, MSc in Information Science and BSc in Mathematics from the University of Zagreb, Croatia. He has extensive international academic and industrial experience. He is currently a professor of information technology and operations management at the R.B. Willumstad School of Business, Adelphi University, New York, USA. Previously he worked at Financial Technologies International, New York, Stony Brook University, New York, the University of Zagreb, Croatia, Croatian Hydro-Meteorological Institute and Zagreb Bank, Croatia. His research interests are focusing on optimization and cost allocation in communication networks. He authored over 60 research papers in management science and related areas. He serves as an associate editor of several journals, including Electronic Commerce Research and Telecommunication Systems. Dr. Skorin-Kapov is a recipient of a number of awards and grants, including Bender Award, Fulbright Award, USA National Science Foundation Grants and Professor Recognition Award.

Jadranka Skorin-Kapov's scholarship spans diverse fields of operations research, business ethics, film studies, and philosophy. She holds three PhD degrees: in Operations Research, in Philosophy, and in Art History and Criticism. Dr. Skorin-Kapov's research papers in Operations Research, funded through five National Science Foundation grants, provided novel algorithmic developments for notoriously hard combinatorial problems, especially relevant in the current era of big data and search for nonlinear pattern recognition. In 2016, she received the SUNY Chancellor's Award for Excellence in Scholarship and Creative Activities. For her innovative course on business ethics, in 2017, she received the prestigious Ideas Worth Teaching Award from the Aspen Institute, and her course was named among "the 20 most interesting business school classes in the world" by Quarz at Work. In 2020, dr. Skorin-Kapov was elected as the corresponding member to the Croatian Academy of Sciences and Arts.