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Weak convergence to a class of two-parameter Gaussian processes from a Lévy sheet*

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Abstract. In this paper, we show an approximation in law, in the space of continuous functions on $[0,1]^2$, of two-parameter Gaussian processes that can be represented as a Wiener type integral by processes constructed from processes that converge to the Brownian sheet. As an application, we obtain a sequence of processes constructed from a Lévy sheet that converges in law towards the fractional Brownian sheet.

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1. Introduction

A Brownian sheet $B = \{B_{s,t}\,; (s,t) \in [0,1]^2\}$ is a zero mean real continuous Gaussian process with covariance function $E[B_{s_1,t_1}B_{s_2,t_2}] = (s_1 \wedge s_2)(t_1 \wedge t_2)$ for any $(s_1,t_1),(s_2,t_2) \in [0,1]^2$.

Let us consider a family of random kernels θ_n such that the processes

$$\zeta_n(s,t) = \int_0^t \int_0^s \theta_n(x,y)dxdy, \quad (s,t) \in [0,1] \times [0,1],$$

converge in law in the space of continuous functions $\mathcal{C}([0,1]^2)$, with its usual topology, to the Brownian sheet. Our aim is to give sufficient conditions on the family θ_n and on a couple of deterministic kernels K_1 and K_2 to ensure that the processes

$$X_n(s,t) = \int_0^1 \int_0^1 K_1(s,u)K_2(t,v)\theta_n(u,v)dudv, \tag{1}$$

converge in law in $\mathcal{C}([0,1]^2)$ to the process

$$W_{s,t}^{K_1,K_2} := \int_0^1 \int_0^1 K_1(s,u)K_2(t,v)dB_{u,v}. \tag{2}$$

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As an example, we obtain the convergence to the fractional Brownian sheet of a family defined using random kernels based on a Lévy sheet. In Section 5, the reader can find the definition of the fractional Brownian sheet and the corresponding expressions of the kernels K_1 and K_2.

Compared with the previous literature, our result generalizes widely the family of random kernels used in the construction of the approximating processes. By clearly distinguishing the essential properties needed by the kernels to obtain the convergence, we are able to extend the approximations that have been known so far only by using Poisson process (see [4]) to Lévy processes.

The proces $W^{K_1,K_2} = \left\{ W^{K_1,K_2}_{s,t},\ (s,t) \in [0,1]^2 \right\}$ given by (2) is characterized by the fact that it is centered, Gaussian and its covariance function factorizes in the following way:

$$E\left[W^{K_1,K_2}_{s_1,s_2} W^{K_1,K_2}_{s_1',s_2'} \right] = \prod_{i=1}^{2} \left(\int_0^1 K_i(s_i,u) K_i(s_i',u) du \right).$$

There are several papers in the literature dealing with the weak convergence to the fractional Brownian motion. In [7, 9, 10] the authors built up the approximations using Poisson processes, while in [11], the approximation sequence is based on a Lévy process.

On the other hand, in [3], it is proved that the family of processes

$$n \int_0^t \int_0^s \sqrt{xy}(-1)^{N(\sqrt{n}x,\sqrt{n}y)} dx dy, \quad n \in \mathbb{N},$$

where $\{N(x,y),\ (x,y) \in \mathbb{R}^2_+\}$ is a standard Poisson process in the plane, converges in law in $\mathcal{C}([0,1]^2)$ to an ordinary Brownian sheet. Using this result, in [4], the authors show that the sequence

$$n \int_0^t \int_0^s K_1(s,u) K_2(t,v) \sqrt{uv}(-1)^{N(\sqrt{n}u,\sqrt{n}v)} du dv,$$

converges in law to the process W^{K_1,K_2} defined in (2). Actually, this convergence is a particular case of our Theorem 1 since the kernels $\theta^0_n(x,y) = n\sqrt{xy}(-1)^{N(\sqrt{n}x,\sqrt{n}y)}$ will satisfy our hypothesis (see Section 2).

The result of [3] is generalized in [5]. The authors consider $\{L(x,y);\ x,y \geq 0\}$ a Lévy sheet with Lévy exponent $\Psi(\xi) := a(\xi) + ib(\xi)$, $\xi \in \mathbb{R}$. Given $\theta \in (0,2\pi)$ such that $a(\theta)a(2\theta) \neq 0$, for any $n \in \mathbb{N}$ and $(s,t) \in [0,1]^2$, they define

$$\bar{\zeta}_n(s,t) := nN_\theta \int_0^t \int_0^s \sqrt{xy} \left\{ \cos(\theta L(\sqrt{n}x,\sqrt{n}y)) + i \sin(\theta L(\sqrt{n}x,\sqrt{n}y)) \right\} dx dy,$$

where the constant N_θ is given by

$$N_\theta = \frac{1}{\sqrt{2}} \frac{a(\theta)^2 + b(\theta)^2}{a(\theta)}. \tag{3}$$

Then they prove that, as n tends to infinity, $\bar{\zeta}_n$ converges in law, in the space of complex-valued continuous functions $\mathcal{C}([0,1]^2;\mathbb{C})$, to a complex Brownian sheet.

That is, the real part and the imaginary part converge to two independent Brownian sheets.

In our paper, we show that the random kernels presented in [5], that is,

$$\theta_n^1(x,y) = nN_\theta\sqrt{xy}\cos(\theta L(\sqrt{n}x,\sqrt{n}y)$$

and

$$\theta_n^2(x,y) = nN_\theta\sqrt{xy}\sin(\theta L(\sqrt{n}x,\sqrt{n}y)$$

satisfy the set of conditions in Section 2. Thus, they can also be used to construct approximations to the fractional Brownian sheet.

Actually, we will present two sets of conditions (H1) and (H1') on the deterministic kernels K_1 and K_2. (H1) is satisfied for the kernels that can be used to define the fractional Brownian sheet with a parameter greater than $\frac{1}{2}$ while the kernels used to define the fractional Brownian sheet with a parameter less than or equal to $\frac{1}{2}$ satisfy only hypothesis (H1') that is weaker than (H1). Moreover, deterministic kernels that satisfy only (H1') need random kernels θ_n that satisfy an extra hypothesis (H4), that also depends on the properties of deterministic kernels.

We have organized the paper as follows: Section 2 is devoted to the sets of hypotheses for the deterministic kernels K_1 and K_2 and for random kernels θ_n. In Section 3, we prove our main result that under the hypothesis presented in the previous section we can obtain weak convergence. In Section 4, we prove that the kernels θ_n^1 and θ_n^2 satisfy the set of hypotheses, and, finally, in Section 5, we give some examples to which our result applies, pointing out the case of the fractional Brownian sheet.

Along the paper we will consider two probability spaces. On the one hand, we will consider a probability space $(\Omega, \mathcal{F}, P)$, where we have defined the approximating processes, and another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, where we have defined the limit processes. The mathematical expectation of these probability spaces will be denoted by E and $\tilde{E}$, respectively.

The multiplicative constants that appear along the paper are denoted with capital letters. They may vary from one expression to another. We only specify the parameters on which they depend when this dependence is important for the bounds.

2. Hypothesis

Since our aim is to study convergence in law to a Gaussian process

$$W_{s,t}^{K_1,K_2} = \int_0^1\int_0^1 K_1(s,u)K_2(t,v)dB_{u,v},$$

where B is a standard Brownian sheet and K_1 and K_2 are deterministic kernels, we need to fix the conditions that will satisfy K_1 and K_2 and allow us to get the convergence to a fractional Brownian sheet. We consider two sets of hypotheses on K_1 and K_2 that we recall from [3]:

(H1)

(i) For $i = 1,2$, K_i is measurable and $K_i(0,r) = 0$ for all $r \in [0,1]$ almost everywhere.

(ii) For $i = 1,2$, there exists an increasing continuous function $G_i : [0,1] \longrightarrow \mathbb{R}$ and $\alpha_i > 1$ such that for all $0 \le s < s' \le 1$,

$$\int_0^1 \left(K_i(s',r) - K_i(s,r)\right)^2 dr \le \left(G_i(s') - G_i(s)\right)^{\alpha_i}.$$

(H1')

(i) For $i = 1,2$, K_i is measurable and $K_i(0,r) = 0$ for all $r \in [0,1]$ almost everywhere.

(ii') For $i = 1,2$, there exists an increasing continuous function $G_i : [0,1] \longrightarrow \mathbb{R}$ and $0 < \rho_i \le 1$ such that for all $0 \le s < s' \le 1$,

$$\int_0^1 \left(K_i(s',r) - K_i(s,r)\right)^2 dr \le \left(G_i(s') - G_i(s)\right)^{\rho_i}.$$

(iii) For $i = 1,2$, there exist constants $M_i > 0$ and $\beta_i > 0$ such that for all $0 \le s < s' \le 1$ and $0 \le s_0 < s_0' \le 1$,

$$\int_{s_0}^{s_0'} \left(K_i(s',r) - K_i(s,r)\right)^2 dr \le M_i (s_0' - s_0)^{\beta_i}.$$

Remark 1. *Clearly, condition* (ii) *implies condition* (ii'). *Condition* (iii) *is added in order to obtain tightness under the weak condition* (ii') *(see Theorem 1).*

For instance, the deterministic kernels associated to a fractional Brownian sheet with Hurst parameter $H > \frac{1}{2}$ satisfy (ii)*, while when $H \le \frac{1}{2}$, the deterministic kernels satisfy* (ii') *and* (iii).

On the other hand, to build the approximation sequence X_n given by (1), we deal with a set of kernels $\theta_n \in L^\infty([0,1]^2)$. We consider the following hypotheses on these kernels:

(H2) The processes

$$\zeta_n(s,t) = \int_0^t \int_0^s \theta_n(x,y)dxdy, \quad (s,t) \in [0,1] \times [0,1],$$

converge in law in $\mathcal{C}([0,1]^2)$, as n tends to infinity, to the Brownian sheet.

(H3) For any $f,\ g \in L^2([0,1])$

$$E\left(\int_{[0,1]^2} f(x)g(y)\theta_n(x,y)dxdy\right)^2 \le C\int_{[0,1]^2} f(x)^2 g(y)^2 dxdy.$$

Finally, under (H1') we need another hypothesis on the kernels θ_n. Notice that this hypothesis (H4) will also depend on the properties of K_1 and K_2. We need first to introduce some notation. Given a real function X defined on $\mathbb{R}_+^2$, and $(s,t),(s',t') \in \mathbb{R}_+^2$ such that $s \le s'$ and $t \le t'$, we denote by $\Delta_{s,t}X(s',t')$ the increment of X over the rectangle $((s,t),(s',t')]$, that is

$$\Delta_{s,t}X(s',t') = X(s',t') - X(s',t) - X(s,t') + X(s,t).$$

When we consider an increment of the processes defined in (1), we have that

$$\begin{aligned}\Delta_{s,t}X_n(s',t') &= \int_{[0,1]^2} (K_1(s',x) - K_1(s,x))(K_2(t',y) - K_2(t,y))\theta_n(x,y)dxdy \\ &= \Delta_{0,0}Y_n(1,1),\end{aligned}$$

where the process Y_n, that depends on s,t,s',t',K_1 and K_2, is defined by

$$Y_n(s_0,t_0) := \int_{[0,s_0]\times[0,t_0]} (K_1(s',x) - K_1(s,x))(K_2(t',y) - K_2(t,y))\theta_n(x,y)dxdy. \quad (4)$$

Now, we can state (H4) when (H1') holds:

> **(H4)** Consider the processes Y_n defined in (4). For any $0 < s_0 < s_0' < 2s_0$, $0 < t_0 < t_0' < 2t_0$ there exists an even number $m > \frac{4}{\min\{\rho_1,\rho_2\}}$ such that
>
> $$\begin{aligned}E\left[\Delta_{s_0,t_0}Y_n(s_0',t_0')\right]^m \le & C_{m,M}\left(G_1(s') - G_1(s)\right)^{\frac{m\rho_1}{4}}\left(G_2(t') - G_2(t)\right)^{\frac{m\rho_2}{4}} \\ & \times \left[(s_0' - s_0)(t_0' - t_0)\right]^{m\gamma}, \qquad (5)\end{aligned}$$
>
> where γ is a parameter belonging to the interval $(0,1)$ (that will only depend on β_1 and β_2) and the constant $C_{m,M}$ depends only on m, M_1 and M_2.

Remark 2. *Notice that under* (H1) *or* (H1') *the processes X_n are continuous. Indeed, for all $0 < s \le s' < 1$, $0 < t \le t' < 1$ using condition (ii') on hypothesis* (H1') *we have that*

$$|\Delta_{s,t}X_n(s',t')| \le \|\theta_n\|_\infty \left(G_1(s') - G_1(s)\right)^{\frac{\rho_1}{2}}\left(G_2(t') - G_2(t)\right)^{\frac{\rho_2}{2}},$$

where G_1 and G_2 are continuous functions.

3. Convergence in law to two-parameter Gaussian processes

In this section, we present our main result. It states as follows:

Theorem 1. *Assume one of the following sets of hypotheses:*

(J1) *K_1 and K_2 satisfy* (H1) *and the kernels θ_n satisfy* (H2) *and* (H3)

(J2) *K_1 and K_2 satisfy* (H1')*, the kernels θ_n satisfy* (H2) *and* (H3) *and, all together, they satisfy* (H4).

Then, the laws of the processes $\{X_n(s,t),\ (s,t) \in [0,1]^2\}$*, given by (1), converge weakly to the law of* $\{W^{K_1,K_2}_{s,t},\ (s,t) \in [0,1]^2\}$*, given by (2), in* $\mathcal{C}([0,1]^2)$*, when* n *goes to infinity.*

Before the proof, we need to recall a technical lemma from [4] (see Lemma 3.2 therein), that will be useful for our computations.

Lemma 1. *Let* $Z = \{Z_{u,v};\ (u,v) \in [0,1]^2\}$ *be a continuous process. Assume that for a fixed even* $m \in \mathbb{N}$ *and some* $\delta_1, \delta_2 \in (0,1)$ *there exists a constant* $Q > 0$ *such that*

$$E\left(\Delta_{u,v} Z(u',v')\right)^m \le Q\,(u'-u)^{m\delta_1}(v'-v)^{m\delta_2}$$

for any $0 < u < u' < 2u$, $0 < v < v' < 2v$*. Then, there exists a constant* C *that depends only on* m, δ_1 *and* δ_2 *such that*

$$E\left(\Delta_{u,v} Z(u',v')\right)^m \le CQ\,(u'-u)^{m\delta_1}(v'-v)^{m\delta_2}$$

for any $0 \le u < u' \le 1$, $0 \le v < v' \le 1$.

Proof of Theorem 1. We will prove the convergence in law checking the tightness of the family of laws of the family $\{X_n\}$ and identifying the limit using the convergence of finite dimensional distributions.

This structure of the proof is similar to those in [3] and [4] to obtain the convergence. Nevertheless, we present a general version. [3] and [4] deal with specific kernels θ_n constructed from a Poisson process, and therefore some computations could be made directly. In our case, we can only use the bounds given by hypotheses (H3) and (H4). As we will see in Section 4, these hypotheses are satisfied not only by the kernels constructed from a Poisson process but also by those constructed from Lévy processes.

We first prove the tightness. Using the criterion given by Bickel and Wichura in [6] and that our processes are null on the axes, it suffices to show that for some $m \ge 2$ there exist two constants, $C > 0$ and $\eta > 1$, and two increasing continuous functions, G_1 and G_2, such that

$$\sup_n E\left[\Delta_{s,t} X_n(s',t')\right]^m \le C\left[\left(G_1(s') - G_1(s)\right)\left(G_2(t') - G_2(t)\right)\right]^\eta, \qquad (6)$$

for any $0 \le s \le s' \le 1$, $0 \le t \le t' \le 1$.

Under the set of conditions (J1), using condition (ii) of (H1) and (H3) we have that

$$\begin{aligned} E\left[\Delta_{s,t} X_n(s',t')\right]^2 &= E\left(\int_{[0,1]^2} (K_1(s',x) - K_1(s,x))(K_2(t',y) - K_2(t,y))\theta_n(x,y)\,dxdy\right)^2 \\ &\le C\int_{[0,1]^2} (K_1(s',x) - K_1(s,x))^2 (K_2(t',y) - K_2(t,y))^2\,dxdy \\ &\le C\left(G_1(s') - G_1(s)\right)^{\alpha_1}\left(G_2(t') - G_2(t)\right)^{\alpha_2}, \end{aligned}$$

where α_1 and α_2 are greater than 1. So, choosing $\eta = \min\{\alpha_1, \alpha_2\}$, (6) holds.

Under the set of conditions (J2) we will see that for $m > \frac{4}{\min\{\rho_1,\rho_2\}}$ given in (H4) and for all $0 \le s < s' \le 1$, $0 \le t < t' \le 1$,

$$\sup_n E\left[\Delta_{s,t} X_n(s',t')\right]^m \le C\left(G_1(s') - G_1(s)\right)^{\frac{m\rho_1}{4}}\left(G_2(t') - G_2(t)\right)^{\frac{m\rho_2}{4}},$$

where C is a constant that does not depend on n. Indeed,

$$\begin{aligned} E\left[\Delta_{s,t} X_n(s',t')\right]^m &= E\left[\int_{[0,1]^2} (K_1(s',x) - K_1(s,x))(K_2(t',y) - K_2(t,y))\theta_n(x,y)dxdy\right]^m \\ &= E\left[\Delta_{0,0} Y_n(1,1)\right]^m, \end{aligned}$$

where the process Y_n was defined in (4). By Lemma 1 it suffices to check that for any $0 < s_0 < s_0' < 2s_0$, $0 < t_0 < t_0' < 2t_0$

$$E\left[\Delta_{s_0,t_0} Y_n(s_0',t_0')\right]^m \le L\left[(s_0' - s_0)(t_0' - t_0)\right]^{m\gamma},$$

where $L = C\left(G_1(s') - G_1(s)\right)^{\frac{m\rho_1}{4}}\left(G_2(t') - G_2(t)\right)^{\frac{m\rho_2}{4}}$ and it is true by hypothesis (H4). So (6) holds easily.

We proceed now with the identification of the limit law. We will prove the convergence of finite dimensional distributions of the processes X_n to those of W^{K_1,K_2}. For fixed $k \in \mathbb{N}$, consider $a_1, \ldots, a_k \in \mathbb{R}$ and $(s_1,t_1), \ldots, (s_k,t_k) \in [0,1]^2$. We must see that the random variables

$$\sum_{j=1}^{k} a_j X_n(s_j,t_j) \tag{7}$$

converge in law, as n tends to infinity, to

$$\sum_{j=1}^{k} a_j W^{K_1,K_2}(s_j,t_j). \tag{8}$$

Actually, we will prove the convergence of the characteristic functions.

For any j, consider a sequence $\{\gamma^{j,\ell}\}$ of elementary functions converging in $L^2([0,1])$, as ℓ tends to infinity, to $K_1(s_j,\cdot)$. In the same way, take a sequence $\{\rho^{j,\ell}\}$ of elementary functions tending in $L^2([0,1])$ to $K_2(t_j,\cdot)$. Then, we can introduce random variables

$$X_n^{j,\ell} = \int_{[0,1]^2} \gamma^{j,\ell}(x)\rho^{j,\ell}(y)\theta_n(x,y)dxdy,$$

and

$$X^{j,\ell} = \int_{[0,1]^2} \gamma^{j,\ell}(x)\rho^{j,\ell}(y)dB_{x,y}.$$

Then, for any $\lambda \in \mathbb{R}$ we can bound the difference between characteristic functions of (7) and (8) by

$$
\begin{aligned}
&\Big|E\left[e^{i\lambda\sum_j a_j X_n(s_j,t_j)}\right]-\tilde{E}\left[e^{i\lambda\sum_j a_j W^{K_1,K_2}(s_j,t_j)}\right]\Big|\\
&\qquad\le \big|E\left[e^{i\lambda\sum_j a_j X_n(s_j,t_j)}-e^{i\lambda\sum_j a_j X_n^{j,\ell}}\right]\big|\\
&\qquad\quad+\Big|E\left[e^{i\lambda\sum_j a_j X_n^{j,\ell}}\right]-\tilde{E}\left[e^{i\lambda\sum_j a_j X^{j,\ell}}\right]\Big|\\
&\qquad\quad+\Big|\tilde{E}\left[e^{i\lambda\sum_j a_j X^{j,\ell}}-e^{i\lambda\sum_i a_j W^{K_1,K_2}(s_j,t_j)}\right]\Big|\\
&\qquad:= S_{1,n,\ell}+S_{2,n,\ell}+S_{3,\ell}.
\end{aligned}
\tag{9}
$$

We study first $S_{1,n,\ell}$. By the mean value theorem

$$
S_{1,n,\ell}\le C\max_j\{E|X_n(s_j,t_j)-X_n^{j,\ell}|\}.
$$

Each one of the expectations appearing in the last maximum can be bounded as follows:

$$
\begin{aligned}
&E|X_n(s_j,t_j)-X_n^{j,\ell}|\\
&\quad E\Big|\int_{[0,1]^2}K_1(s_j,x)K_2(t_j,y)\theta_n(x,y)dxdy-\int_{[0,1]^2}\gamma^{j,\ell}(x)\rho^{j,\ell}(y)\theta_n(x,y)dxdy\Big|\\
&\quad\le E\Big|\int_{[0,1]^2}K_1(s_j,x)(\rho^{j,\ell}(y)-K_2(t_j,y))\theta_n(x,y)dxdy\Big|\\
&\quad\quad+E\Big|\int_{[0,1]^2}\rho^{j,\ell}(y)(K_1(s_j,x)-\gamma^{j,\ell}(x))\theta_n(x,y)dxdy\Big|\\
&\quad\le C\left(\int_{[0,1]}K_1(s_j,x)^2dx\right)\left(\int_{[0,1]}(\rho^{j,\ell}(y)-K_2(t_j,y))^2dy\right)\\
&\quad\quad+C\left(\int_{[0,1]}(\rho^{j,\ell}(y))^2dy\right)\left(\int_{[0,1]}(K_1(s_j,x)-\gamma^{j,\ell}(x))^2dx\right),
\end{aligned}
$$

where in the last inequality we have used hypothesis (H3). Since $\{\gamma^{j,\ell}\}$ and $\{\rho^{j,\ell}\}$ converge in $L^2([0,1])$, as ℓ tends to infinity, to $K_1(s_j,\cdot)$ and $K_2(t_j,\cdot)$, respectively, if ℓ is big enough, this last expression can be made arbitrarily small. That is, for any $\varepsilon>0$, there exists ℓ_0 big enough such that for any $\ell>\ell_0$

$$
\sup_n|S_{1,n,\ell}|<\varepsilon. \tag{10}
$$

We deal now with $S_{2,n,\ell}$. Since the functions $\gamma^{j,\ell}$ and $\rho^{j,\ell}$ are elementary functions, the random variables $X_n^{j,\ell}$ are linear combinations of increments of the process $\zeta_n(s,t)$ defined in (H2). Due to (H2), the laws of these last processes converge weakly, in $\mathcal{C}([0,1]^2)$, to the law of the Brownian sheet. Then, the linear combinations of the increments of ζ_n will converge in law to the same linear combinations of the increments of the Brownian sheet, that is, to $X^{j,\ell}$. So, for fixed $\ell\in\mathbb{N}$,

$$
\lim_{n\to\infty}S_{2,n,\ell}=0. \tag{11}
$$

Finally, we consider $S_{3,\ell}$. Applying the mean value theorem as in the study of $S_{1,n,\ell}$ and using the isometry of the stochastic integral, we can write

$$\begin{aligned} S_{3,\ell} &\leq C \max_j \{\tilde{E}|X^{j,\ell} - W^{K_1,K_2}(s_j,t_j)|\} \\ &= C \max_j \{\tilde{E}| \int_{[0,1]^2} [\gamma^{j,\ell}(x)\rho^{j,\ell}(y) - K_1(s_j,x)K_2(t_j,y)] dB_{x,y}|\} \\ &\leq C \max_j \left(\int_{[0,1]^2} \left(\gamma^{j,\ell}(x)\rho^{j,\ell}(y) - K_1(s_j,x)K_2(t_j,y)\right)^2 dxdy \right)^{\frac{1}{2}}. \end{aligned}$$

This last norm in $L^2([0,1]^2)$ tends to zero as ℓ tends to infinity. That is

$$\lim_{\ell \to \infty} S_{3,\ell} = 0. \tag{12}$$

Putting together (9), (10), (11) and (12) we finish the proof. $\square$

4. Kernels defined from a Lévy sheet

In this section, we will prove that the kernels defined from a Lévy sheet introduced in [5] satisfy our hypothesis.

Remark 3. *In [3], it is proved that if we consider the kernels*

$$\theta_n^0(x,y) = n\sqrt{xy}(-1)^{N(\sqrt{n}x,\sqrt{n}y)},$$

where $\{N(x,y),\, (x,y) \in \mathbb{R}_+^2\}$ is a standard Poisson process, then the corresponding processes $\{\zeta_n(s,t),\ (s,t) \in [0,1]\times[0,1]\}$ converge in law in $\mathcal{C}([0,1]^2)$ to the Brownian sheet. So, hypothesis (H2) *is verified. Moreover, hypothesis* (H3) *corresponds to Lemma 3.1 in [4], and* (H4) *is checked for deterministic kernels satisfying* (H1') *in the proof of Lemma 3.3 in [4]. Thus, the kernels θ_n^0 satisfy hypotheses* (H2), (H3) *and* (H4).

Let us recall some notation and definitions of Lévy sheets. If Q is a rectangle in $\mathbb{R}_+^2$ and Z a random field in $\mathbb{R}_+^2$, we denote by $\Delta_Q Z$ the increment of Z on Q. It is well-known that for any negative definite function Ψ in $\mathbb{R}$, there exists a real-valued random field $L = \{L(s,t);\, s,t \geq 0\}$ such that

- For any family of disjoint rectangles $Q_1, \ldots, Q_n$ in $\mathbb{R}_+^2$, the increments $\Delta_{Q_1} L, \ldots, \Delta_{Q_n} L$ are independent random variables.
- For any rectangle Q in $\mathbb{R}_+^2$, the characteristic function of the increment $\Delta_Q L$ is given by
$$\mathbb{E}\left[e^{i\xi\Delta_Q L}\right] = e^{-\lambda(Q)\Psi(\xi)}, \quad \xi \in \mathbb{R}, \tag{13}$$
where λ denotes the Lebesgue measure on $\mathbb{R}_+^2$.

Definition 1. *A random field $L = \{L(s,t);\, s,t \geq 0\}$ taking values in $\mathbb{R}$ that is continuous in probability and satisfies the above two conditions is called a Lévy sheet with exponent Ψ.*

By the Lévy-Khintchine formula, we have $\Psi(\xi) = a(\xi) + ib(\xi)$, where

$$a(\xi) := \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} [1 - \cos(\xi x)]\eta(dx), \tag{14}$$

and

$$b(\xi) := a\xi + \int_{\mathbb{R}} \left[\frac{x\xi}{1 + |x|^2} - \sin(\xi x) \right] \eta(dx), \tag{15}$$

with $a \in \mathbb{R}$, $\sigma \geq 0$ and η the corresponding Lévy measure, that is a Borel measure on $\mathbb{R} \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}} \frac{|x|^2}{1 + |x|^2} \eta(dx) < \infty.$$

Notice that $a(\xi) \geq 0$ and, if $\xi \neq 0$, $a(\xi) > 0$ whenever $\sigma > 0$ or η is nontrivial.

We are able now to recall the kernels defined from a Lévy sheet introduced in [5]. Consider $\{L(x, y);\, x, y \geq 0\}$ a Lévy sheet and $\Psi(\xi) := a(\xi) + ib(\xi)$, $\xi \in \mathbb{R}$, its Lévy exponent. Let $\theta \in (0, 2\pi)$ and define

$$\theta_n^1(x, y) = nN_\theta \sqrt{xy} \cos(\theta L(\sqrt{n}x, \sqrt{n}y)) \tag{16}$$

and

$$\theta_n^2(x, y) = nN_\theta \sqrt{xy} \sin(\theta L(\sqrt{n}x, \sqrt{n}y)), \tag{17}$$

where we assume that $a(\theta)a(2\theta) \neq 0$ and where the constant N_θ is given by (3).

Our aim is to check that these kernels satisfy hypotheses (H2), (H3) and (H4). In [5], it is proved that the corresponding processes $\{\zeta_n(s, t),\, (s, t) \in [0, 1] \times [0, 1]\}$ converge in law in $\mathcal{C}[0, 1]^2$ to a Brownian sheet. So, θ_n^1 and θ_n^2 verify hypothesis (H2). We will prove that they also satisfy (H3) and (H4) in lemmas 2 and 4, respectively. Notice that to check (H4) we need the additional condition that $a(\theta)a(2\theta) \cdots a(m\theta) \neq 0$, where m is the even integer appearing in hypothesis (H4). We also present an intermediate technical result in Lemma 3.

Lemma 2. *For any $f,\ g \in L^2([0, 1])$ and $k \in \{1, 2\}$,*

$$E\left(\int_{[0,1]^2} f(x)g(y)\theta_n^k(x, y)dxdy \right)^2 \leq \frac{136}{a^2(\theta)} N_\theta^2 \int_{[0,1]^2} f(x)^2 g(y)^2 dxdy.$$

Proof. We will prove the lemma in the case $k = 1$ since the case $k = 2$ can be done using similar computations. We have that

$$\begin{aligned}
&E\left(\int_{[0,1]^2} f(x)g(y)\theta_n^1(x, y)dxdy \right)^2 \\
&\quad= \int_{[0,1]^4} E\left[\prod_{j=1}^{2} \left(f(x_j)g(y_j)\theta_n^1(x_j, y_j) \right) \right] dx_1 \cdots dy_2 \\
&\quad= 2n^2 N_\theta^2 \int_{[0,1]^4} \prod_{j=1}^{2} \left(f(x_j)g(y_j)\sqrt{x_j y_j} \right) \\
&\qquad\times E[\cos(\theta L(\sqrt{n}x_1, \sqrt{n}y_1)) \cos(\theta L(\sqrt{n}x_2, \sqrt{n}y_2))] 1_{\{x_1 \leq x_2\}} dx_1 \cdots dy_2,
\end{aligned} \tag{18}$$

where in the last expression we have used the symmetry between x_1 and x_2 to get that the integral over $[0,1]^2$ is two times the integral over the set $\{x_1 \le x_2\}$.

Notice that using complex notation, we have that

$$\begin{aligned}
&\cos(\theta L(\sqrt{n}x_1,\sqrt{n}y_1))\cos(\theta L(\sqrt{n}x_2,\sqrt{n}y_2))\\
&\quad=\left(\frac{e^{i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)}+e^{-i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)}}{2}\times\frac{e^{i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}+e^{-i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}}{2}\right)\\
&\quad=\frac{1}{4}\Big(e^{i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)+i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}+e^{i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)-i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}\\
&\qquad+e^{-i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)+i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}+e^{-i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)-i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}\Big)\\
&\quad:=A_1+A_2+A_3+A_4.
\end{aligned}$$

Putting this expression in (18), we obtain that the term (18) is equal to the sum of the corresponding four terms obtained from A_1, A_2, A_3 and A_4 that will be denoted by I_1, I_2, I_3 and I_4.

We will deal first with I_1 and so we will begin with the study of A_1. Notice that using that L is null on the axes, when $\{x_1 \le x_2\}$ and $\{y_2 \le y_1\}$, we have that

$$\begin{aligned}
&L(\sqrt{n}x_1,\sqrt{n}y_1)+L(\sqrt{n}x_2,\sqrt{n}y_2)\\
&\quad=\Delta_{0,\sqrt{n}y_2}L(\sqrt{n}x_1,\sqrt{n}y_1)+\Delta_{\sqrt{n}x_1,0}L(\sqrt{n}x_2,\sqrt{n}y_2)+2\Delta_{0,0}L(\sqrt{n}x_1,\sqrt{n}y_2)
\end{aligned}$$

and when $\{x_1 \le x_2\}$ and $\{y_1 \le y_2\}$, we have that

$$\begin{aligned}
&L(\sqrt{n}x_1,\sqrt{n}y_1)+L(\sqrt{n}x_2,\sqrt{n}y_2)\\
&\quad=\Delta_{0,\sqrt{n}y_1}L(\sqrt{n}x_1,\sqrt{n}y_2)+\Delta_{\sqrt{n}x_1,0}L(\sqrt{n}x_2,\sqrt{n}y_1)+\Delta_{\sqrt{n}x_1,\sqrt{n}y_1}L(\sqrt{n}x_2,\sqrt{n}y_2)\\
&\qquad+2\Delta_{0,0}L(\sqrt{n}x_1,\sqrt{n}y_1).
\end{aligned}$$

Then, from (13) we get that

$$\begin{aligned}
&E\left[e^{i\theta L(\sqrt{n}x_1,\sqrt{n}y_1)+i\theta L(\sqrt{n}x_2,\sqrt{n}y_2)}I_{\{x_1\le x_2\}}\right]\\
&\quad=e^{-n[(x_2-x_1)y_2+(y_1-y_2)x_1]\Psi(\theta)-n(x_1y_2\Psi(2\theta))}I_{\{x_1\le x_2\}}I_{\{y_2\le y_1\}}\\
&\qquad+e^{-n(x_2y_2-x_1y_1)\Psi(\theta)-n(x_1y_1\Psi(2\theta))}I_{\{x_1\le x_2\}}I_{\{y_1<y_2\}}\\
&\quad\le e^{-n[(x_2-x_1)y_2+(y_1-y_2)x_1]a(\theta)}I_{\{x_1\le x_2\}}I_{\{y_2\le y_1\}}\\
&\qquad+e^{-n[(x_2-x_1)y_1+(y_2-y_1)x_1]a(\theta)}I_{\{x_1\le x_2\}}I_{\{y_1\le y_2\}},
\end{aligned}$$

where in the last inequality we have bounded by 1 the modulus of all the terms with the factor $ib(\theta)$ in the exponential and we have used that $a(\theta)\ge 0$ to bound also by 1 some exponentials with negative real exponent.

Since both summands in the last expression are equal interchanging the roles of y_1 and y_2, we obtain that

$$\begin{aligned}
|I_1|\le n^2N_\theta^2&\int_{[0,1]^4}\prod_{j=1}^{2}\left|f(x_j)g(y_j)\sqrt{x_jy_j}\right|e^{-n[(x_2-x_1)y_1+(y_2-y_1)x_1]a(\theta)}\\
&\times I_{\{x_1\le x_2\}}I_{\{y_1\le y_2\}}dx_1\cdots dy_2.
\end{aligned}$$

On the other hand, using that $a(-\theta)=a(\theta)$, we can bound the modulus of the integrals I_2, I_3 and I_4 by the same bound. Then

$$
\begin{aligned}
&E\left(\int_{[0,1]^2} f(x)g(y)\theta_n^1(x,y)dxdy\right)^2 \\
&\quad \leq 4n^2N_\theta^2\int_{[0,1]^4}\prod_{j=1}^{2}\left|f(x_j)g(y_j)\sqrt{x_jy_j}\right| e^{-n[(x_2-x_1)y_1+(y_2-y_1)x_1]a(\theta)} \\
&\qquad \times I_{\{x_1\leq x_2\}}I_{\{y_1\leq y_2\}}dx_1\cdots dy_2.
\end{aligned}
$$

From here we can follow closely the proof of Lemma 3.1 in [4]. We have to divide the region of integration into two parts: $A:=\{x_1\leq x_2\leq 2x_1, y_1\leq y_2\leq 2y_1\}$ and A^C. Over the region of integration A we can obtain the bound

$$2\times\frac{4}{a^2(\theta)}N_\theta^2\int_{[0,1]^2}\left(f(x)g(y)\right)^2dxdy,$$

while over the region A^C, we get the bound

$$2\times\frac{64}{a^2(\theta)}N_\theta^2\int_{[0,1]^2}\left(f(x)g(y)\right)^2dxdy,$$

and the proof finishes easily.

□

Let us present an intermediate technical result.

Lemma 3. *Consider $L=\{L(s,t);\, s,t\geq 0\}$ a Lévy sheet with exponent $\Psi(\xi)=a(\xi)+ib(\xi)$, $0<s_0<s_0'<2s_0$, $0<t_0<t_0'<2t_0$, m an even number and $\theta\in(0,2\pi)$ such that $a(\theta)a(2\theta)\cdots a(m\theta)\neq 0$. Then, for any $(x_1,y_1),\ldots,(x_m,y_m)$ such that $s_0<x_j<s_0'$ and $t_0<y_j<t_0'$ for all $j\in\{1,\ldots,m\}$, it holds that*

$$
\begin{aligned}
&E\left[\prod_{j=1}^{m}\cos(\theta L(\sqrt{n}x_j,\sqrt{n}y_j))\right] \\
&\quad\leq \exp\left[-\frac{n}{2}a^*(\theta)\left[x_{(m-1)}(y_{(m)}-y_{(m-1)})+x_{(m-3)}(y_{(m-2)}-y_{(m-3)})\right.\right. \\
&\qquad \left.\left.+\cdots+x_{(1)}(y_{(2)}-y_{(1)})\right]\right]\exp\left[-\frac{n}{2}a^*(\theta)\left[y_{(m-1)}(x_{(m)}-x_{(m-1)})\right.\right. \\
&\qquad \left.\left.+y_{(m-3)}(x_{(m-2)}-x_{(m-3)})+\cdots+y_{(1)}(x_{(2)}-x_{(1)})\right]\right],
\end{aligned}
$$

where $a^*(\theta)=\min\{a(\theta),a(2\theta),\ldots,a(m\theta)\}$ and $x_{(1)},\ldots,x_{(m)}$ and $y_{(1)},\ldots,y_{(m)}$ are $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$ after being ordered.

Proof. Notice that

$$\begin{aligned}\prod_{j=1}^{m}\cos(\theta L(\sqrt{n}x_j,\sqrt{n}y_j)) &= \prod_{j=1}^{m}\left(\frac{e^{i\theta L(\sqrt{n}x_j,\sqrt{n}y_j)}+e^{-i\theta L(\sqrt{n}x_j,\sqrt{n}y_j)}}{2}\right)\\ &= \frac{1}{2^m}\sum_{(\delta_1,\ldots,\delta_m)\in\{1,-1\}^m}\prod_{j=1}^{m}e^{i\delta_j\theta L(\sqrt{n}x_j,\sqrt{n}y_j)}\\ &= \frac{1}{2^m}\sum_{(\delta_1,\ldots,\delta_m)\in\{1,-1\}^m}e^{i\theta\sum_{j=1}^{m}\delta_j L(\sqrt{n}x_j,\sqrt{n}y_j)}\\ &= \frac{1}{2^m}\sum_{(\delta_1,\ldots,\delta_m)\in\{1,-1\}^m}e^{i\theta\sum_{j=1}^{m}\delta_j \Delta_{0,0}L(\sqrt{n}x_j,\sqrt{n}y_j)}. \quad (19)\end{aligned}$$

For fixed $(\delta_1,\ldots,\delta_m)$, we can write

$$\begin{aligned}\sum_{j=1}^{m}\delta_j\Delta_{0,0}L(\sqrt{n}x_j,\sqrt{n}y_j) &= \sum_{j=1}^{m}\delta_j\Delta_{s_0,t_0}L(\sqrt{n}x_j,\sqrt{n}y_j)+\sum_{j=1}^{m}\delta_j\Delta_{s_0,0}L(\sqrt{n}x_j,\sqrt{n}t_0)\\ &\quad+\sum_{j=1}^{m}\delta_j\Delta_{0,t_0}L(\sqrt{n}s_0,\sqrt{n}y_j)+L(\sqrt{n}s_0,\sqrt{n}t_0)\sum_{j=1}^{m}\delta_j,\end{aligned}$$

and so

$$\begin{aligned}e^{\sum_{j=1}^{m}\delta_j\Delta_{0,0}L(\sqrt{n}x_j,\sqrt{n}y_j)} &= e^{\sum_{j=1}^{m}\delta_j\Delta_{s_0,t_0}L(\sqrt{n}x_j,\sqrt{n}y_j)}e^{\sum_{j=1}^{m}\delta_j\Delta_{s_0,0}L(\sqrt{n}x_j,\sqrt{n}t_0)}\\ &\quad\times e^{\sum_{j=1}^{m}\delta_j\Delta_{0,t_0}L(\sqrt{n}s_0,\sqrt{n}y_j)}e^{L(\sqrt{n}s_0,\sqrt{n}t_0)\sum_{j=1}^{m}\delta_j}. \quad (20)\end{aligned}$$

Since the interval $((0,0),(s_0,t_0)]$ and the families of intervals $\{((s_0,t_0),(x_j,y_j)], 1\le j\le m\}$, $\{((s_0,0),(x_j,t_0)], 1\le j\le m\}$, $\{((0,t_0),(s_0,y_l)], 1\le j\le m\}$ have support on disjoint sets, the four factors of (20) are independent random variables. Moreover,

$$\begin{aligned}&|E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{0,0}L(\sqrt{n}x_j,\sqrt{n}y_j)}\right)|\\ &\quad= |E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{s_0,t_0}L(\sqrt{n}x_j,\sqrt{n}y_j)}\right)||E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{s_0,0}L(\sqrt{n}x_j,\sqrt{n}t_0)}\right)|\\ &\quad\quad\times|E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{0,t_0}L(\sqrt{n}s_0,\sqrt{n}y_j)}\right)||E\left(e^{i\theta L(\sqrt{n}s_0,\sqrt{n}t_0)\sum_{j=1}^{m}\delta_j}\right)|\\ &\quad\le |E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{s_0,0}L(\sqrt{n}x_j,\sqrt{n}t_0)}\right)||E\left(e^{i\theta\sum_{j=1}^{m}\delta_j\Delta_{0,t_0}L(\sqrt{n}s_0,\sqrt{n}y_j)}\right)|, \quad (21)\end{aligned}$$

where in the last step we have bounded two factors by 1.

Let us study first the second term in (21). We need to introduce the notation $(y_{(1)},\delta_{(1)}),\ldots,(y_{(n)},\delta_{(n)})$ for the variables $(y_1,\delta_1),\ldots,(y_n,\delta_n)$ sorted in increasing order by the variables y_i. Using this notation, we can write

$$\begin{aligned}\sum_{j=1}^m \delta_j \Delta_{0,t_0} L(\sqrt{n}s_0,\sqrt{n}y_j) &= \sum_{j=1}^m \delta_{(j)} \Delta_{0,t_0} L(\sqrt{n}s_0,\sqrt{n}y_{(j)})\\ &= \delta_{(m)}\Delta_{0,y_{(m-1)}} L(\sqrt{n}s_0,\sqrt{n}y_{(m)}) + (\delta_{(m)}+\delta_{(m-1)})\Delta_{0,y_{(m-2)}}\\ &\quad \times L(\sqrt{n}s_0,\sqrt{n}y_{(m-1)}) + \cdots + \left(\sum_{j=1}^m \delta_{(j)}\right) \Delta_{0,t_0} L(\sqrt{n}s_0,\sqrt{n}y_{(1)}).\end{aligned}$$

Since in the last expression all the rectangles where we consider the increments of the Lévy sheet are disjoint, all the terms in the last expression are independent random variables. Then, if we bound by 1 all the factors with an even number of summands in $\delta_{(m)}+\delta_{(m-1)}+\cdots+\delta_{(j)}$, we obtain that

$$\begin{aligned}&|E\left(e^{i\theta\sum_{j=1}^m \delta_j \Delta_{0,t_0} L(\sqrt{n}s_0,\sqrt{n}y_j)}\right)|\\ &\quad \le \exp\big[-ns_0(y_{(m)}-y_{(m-1)})\Psi(\delta_{(m)}\theta)\\ &\qquad -ns_0(y_{(m-2)}-y_{(m-3)})\Psi((\delta_{(m)}+\delta_{(m-1)}+\delta_{(m-2)})\theta)\\ &\qquad -\cdots - ns_0(y_{(2)}-y_{(1)})\Psi((\sum_{j=2}^m \delta_{(j)})\theta)\big].\end{aligned}$$

Let us recall that $\Psi(h\theta)=a(h\theta)+ib(h\theta)$ for all $h\in\mathbb{R}$. Bounding again by 1 the modulus of all the terms with the factor $ib(h\theta)$ in the exponential we obtain that

$$\begin{aligned}&|E\left(e^{i\theta\sum_{j=1}^m \delta_j \Delta_{0,t_0} L(\sqrt{n}s_0,\sqrt{n}y_j)}\right)|\\ &\quad \le \exp\big[-ns_0(y_{(m)}-y_{(m-1)})a(\delta_{(m)}\theta)\\ &\qquad -ns_0(y_{(m-2)}-y_{(m-3)})a((\delta_{(m)}+\delta_{(m-1)}+\delta_{(m-2)})\theta)\\ &\qquad -\cdots - ns_0(y_{(2)}-y_{(1)})a((\sum_{j=2}^m \delta_{(j)})\theta)\big]\\ &\quad \le \exp\big[-ns_0(y_{(m)}-y_{(m-1)})a^*(\theta)\\ &\qquad -ns_0(y_{(m-2)}-y_{(m-3)})a^*(\theta)-\cdots-ns_0(y_{(2)}-y_{(1)})a^*(\theta)\big], \qquad (22)\end{aligned}$$

where $a^*(\theta)=\min\{a(\theta),a(2\theta),\ldots,a(m\theta)\}$.

Using the same arguments, we can also bound the first term in (21) and we get that

$$\begin{aligned}&|E\left(e^{i\theta\sum_{j=1}^m \delta_j \Delta_{s_0,0} L(\sqrt{n}x_j,\sqrt{n}t_0)}\right)|\\ &\quad \le \exp\big[-nt_0(x_{(m)}-x_{(m-1)})a^*(\theta)-nt_0(x_{(m-2)}-x_{(m-3)})a^*(\theta)\\ &\qquad -\cdots-nt_0(x_{(2)}-x_{(1)})a^*(\theta)\big]. \qquad (23)\end{aligned}$$

Then, putting together (21), (23) and (22)

$$
\begin{aligned}
&|E\left(e^{i\theta\sum_{j=1}^m \delta_j \Delta_{0,0} L(\sqrt{n}x_j,\sqrt{n}y_j)}\right)| \\
&\quad \le \exp\left[-na^*(\theta)s_0\left[(y_{(m)}-y_{(m-1)})+(y_{(m-2)}-y_{(m-3)})+\cdots+(y_{(2)}-y_{(1)})\right]\right] \\
&\qquad \times \exp\left[-na^*(\theta)t_0\left[(x_{(m)}-x_{(m-1)})+(x_{(m-2)}-x_{(m-3)})+\cdots+(x_{(2)}-x_{(1)})\right]\right].
\end{aligned}
$$

Finally, using that $2t_0 > t_0'$ and $2s_0 > s_0'$, the last expression can be bounded by

$$
\begin{aligned}
&\exp\left[-na^*(\theta)\frac{s_0'}{2}\left[(y_{(m)}-y_{(m-1)})+(y_{(m-2)}-y_{(m-3)})+\cdots+(y_{(2)}-y_{(1)})\right]\right] \\
&\times \exp\left[-na^*(\theta)\frac{t_0'}{2}\left[(x_{(m)}-x_{(m-1)})+(x_{(m-2)}-x_{(m-3)})+\cdots+(x_{(2)}-x_{(1)})\right]\right] \\
&\quad \le \exp\left[-\frac{n}{2}a^*(\theta)\left[x_{(m-1)}(y_{(m)}-y_{(m-1)})+x_{(m-3)}(y_{(m-2)}-y_{(m-3)})+\cdots\right.\right. \\
&\qquad \left.\left.+x_{(1)}(y_{(2)}-y_{(1)})\right]\right] \exp\left[-\frac{n}{2}a^*(\theta)\left[y_{(m-1)}(x_{(m)}-x_{(m-1)})\right.\right. \\
&\qquad \left.\left.+y_{(m-3)}(x_{(m-2)}-x_{(m-3)})+\cdots+y_{(1)}(x_{(2)}-x_{(1)})\right]\right]. \qquad (24)
\end{aligned}
$$

Since this last bound does not depend on $(\delta_1,\ldots,\delta_m)$, putting together bound (24) with (19), we finish the proof easily. □

We are now able to prove (H4) under (H1').

Lemma 4. *Assume* (H1'). *Let us consider processes (4) defined using the kernels θ_n^1 or θ_n^2, with $\theta\in(0,2\pi)$ such that $a(\theta)a(2\theta)\cdots a(m\theta)\neq 0$ for an even integer number $m>\frac{4}{\min\{\rho_1,\rho_2\}}$. Then, for any $0<s_0<s_0'<2s_0$, $0<t_0<t_0'<2t_0$,*

$$
\begin{aligned}
E\left[\Delta_{s_0,t_0}Y_n(s_0',t_0')\right]^m \le\ & C_{m,M}\left(G_1(s')-G_1(s)\right)^{\frac{m\rho_1}{4}}\left(G_2(t')-G_2(t)\right)^{\frac{m\rho_2}{4}} \\
&\times\left[(s_0'-s_0)(t_0'-t_0)\right]^{m\gamma}, \qquad (25)
\end{aligned}
$$

where γ is a parameter belonging to the interval $(0,1)$ (that will depend only on β_1 and β_2) and the constant $C_{m,M}$ depends only on m, M_1 and M_2.

Proof. As in Lemma 2, we will prove the result only for the kernels θ_n^1 since the case θ_n^2 is very similar. From (4), the definition of Y_n, we can write that

$$
\begin{aligned}
&E\left[\Delta_{s_0,t_0}Y_n(s_0',t_0')\right]^m \\
&\quad = n^m N_\theta^m E\Big[\int_{[0,1]^{2m}}\prod_{j=1}^m\Big(I_{[s_0',s_0]}(x_j)I_{[t_0',t_0]}(y_j)(K_1(s',x_j)-K_1(s,x_j))(K_2(t',y_j) \\
&\qquad -K_2(t,y_j))\sqrt{x_jy_j}\cos(\theta L(\sqrt{n}x_j,\sqrt{n}y_j))\Big)dx_1\cdots dy_m\Big].
\end{aligned}
$$

Using Lemma 3, we obtain easily that

$$
\begin{aligned}
&E\left[\Delta_{s_0,t_0} Y_n(s'_0,t'_0)\right]^m \\
&\quad\le (m!)^2 n^m N_\theta^m \int_{[0,1]^{2m}} \prod_{j=1}^m \Bigg(I_{[s'_0,s_0]}(x_j) I_{[t'_0,t_0]}(y_j) \\
&\quad\quad \times (K_1(s',x_j)-K_1(s,x_j))(K_2(t',y_j)-K_2(t,y_j))\sqrt{x_j y_j}\Bigg) \\
&\quad\quad \times \exp\left[-\frac{n}{2}a^*(\theta)\left[x_{m-1}(y_m-y_{m-1})+\cdots+x_1(y_2-y_1)\right]\right] \\
&\quad\quad \times \exp\left[-\frac{n}{2}a^*(\theta) y_{m-1}(x_m-x_{m-1})+\cdots+y_1(x_2-x_1)\right] I_{\{x_1\le\cdots\le x_m\}} \\
&\quad\quad \times I_{\{y_1\le\cdots\le y_m\}} dx_1\cdots dy_m \\
&\quad\le C_m\Bigg(n^2 N_\theta^2 \int_{[0,1]^4} \prod_{j=1}^2 \Bigg(I_{[s'_0,s_0]}(x_j) I_{[t'_0,t_0]}(y_j) \\
&\quad\quad \times (K_1(s',x_j)-K_1(s,x_j))(K_2(t',y_j)-K_2(t,y_j))\sqrt{x_j y_j}\Bigg) \\
&\quad\quad \times \exp\left[-\frac{n}{2}a^*(\theta)\left[x_1(y_2-y_1)+y_1(x_2-x_1)\right]\right] I_{\{x_1\le x_2\}} I_{\{y_1\le y_2\}} dx_1\cdots dy_2 \Bigg)^{\frac{m}{2}},
\end{aligned}
$$

where C_m is a constant depending only on m.

By the computations of Lemma 2, this last expression is bounded by

$$
\begin{aligned}
&C_m \frac{N_\theta^2}{(a^*(\theta))^2}\Big(\int_{[0,1]^2} I_{[s'_0,s_0]}(x) I_{[t'_0,t_0]}(y)(K_1(s',x)-K_1(s,x))^2 \\
&\quad\quad \times (K_2(t',y)-K_2(t,y))^2 dxdy\Big)^{\frac{m}{2}} \\
&\quad \le C_m \frac{N_\theta^2}{(a^*(\theta))^2}\Big(\int_{[0,1]^2} I_{[s'_0,s_0]}(x) I_{[t'_0,t_0]}(y)(K_1(s',x)-K_1(s,x))^2 \\
&\quad\quad \times (K_2(t',y)-K_2(t,y))^2 dxdy\Big)^{\frac{m}{4}} \\
&\quad\quad \times \Big(\int_{[0,1]^2} (K_1(s',x)-K_1(s,x))^2 (K_2(t',y)-K_2(t,y))^2 dxdy\Big)^{\frac{m}{4}}.
\end{aligned}
$$

Using conditions (iii) and (ii') on the kernels K_1 and K_2, the last expression is bounded by

$$
\begin{aligned}
&C_m M_1^{\frac{m}{4}} M_2^{\frac{m}{4}} \frac{N_\theta^2}{(a^*(\theta))^2}(s'_0-s_0)^{\frac{\beta_1 m}{4}}(t'_0-t_0)^{\frac{\beta_2 m}{4}}(G_1(s')-G_1(s))^{\frac{m\rho_1}{4}} \\
&\quad (G_2(t')-G_2(t))^{\frac{m\rho_2}{4}} \\
&\quad\le C_{m,M}\frac{N_\theta^2}{(a^*(\theta))^2}\left[(s'_0-s_0)(t'_0-t_0)\right]^{\gamma m}(G_1(s')-G_1(s))^{\frac{m\rho_1}{4}}(G_2(t')-G_2(t))^{\frac{m\rho_2}{4}},
\end{aligned}
$$

where $\gamma = \frac{1}{4}\inf\{\beta_1, \beta_2\}$. So, we have proved inequality (25) and the proof is now complete.

□

Putting together all the lemmas of this section, we get the following result:

Corollary 1. *Let $L = \{L(s,t);\, s,t \geq 0\}$ be a Lévy sheet with exponent $\Psi(\xi) = a(\xi) + ib(\xi)$, where $a(\xi)$ and $b(\xi)$ are defined in (14) and (15), respectively. Consider $\theta \in (0, 2\pi)$ with $a(\theta)a(2\theta) \neq 0$. Then, the kernels θ_n^1 and θ_n^2 defined in (16) and (17) satisfy hypotheses* (H2) *and* (H3). *Furthermore, given deterministic kernels K_1 and K_2 satisfying* (H1'), *if for an even integer number $m > \frac{4}{\min\{\rho_1,\rho_2\}}$ $a(\theta)a(2\theta)\cdots a(m\theta) \neq 0$, then* (H4) *is also satisfied.*

5. The fractional Brownian sheet and other examples

In this section, we apply Theorem 1 to obtain the convergence to the fractional Brownian sheet. More precisely, we use the anisotropic fractional Wiener random field introduced by [8] and [2]. This is a centered Gaussian process, defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, denoted by $W^{\alpha,\beta} = \{W^{\alpha,\beta}_{s,t},\ (s,t) \in \mathbb{R}^2_+\}$, with the covariance function given by

$$\tilde{E}\left(W^{\alpha,\beta}_{s,t} W^{\alpha,\beta}_{s',t'}\right) = \frac{1}{2}\left(s'^{2\alpha} + s^{2\alpha} - |s'-s|^{2\alpha}\right)\frac{1}{2}\left(t'^{2\beta} + t^{2\beta} - |t'-t|^{2\beta}\right), \tag{26}$$

where α and β are two parameters belonging to the interval $(0,1)$. By definition, this process is null almost surely on the axes, and it is proved in [8] and [2] that it possesses a continuous version. Observe that if $\alpha = \beta = \frac{1}{2}$, then we obtain the standard Brownian sheet.

Recall that a fractional Brownian motion of Hurst parameter $\alpha \in (0,1)$, $W^\alpha = \{W^\alpha_t,\ t \in \mathbb{R}_+\}$ is a centered Gaussian process with covariance function

$$R(t,s) = E\left(W^\alpha_t W^\alpha_s\right) = \frac{1}{2}\left(t^{2\alpha} + s^{2\alpha} - |t-s|^{2\alpha}\right).$$

This process admits an integral representation of the form (see for instance [1])

$$W^\alpha_t = \int_0^t K_\alpha(t,s)\, dB_s, \tag{27}$$

where B is a standard Brownian motion and the kernel K_α is given by

$$K_\alpha(t,s) = \left[d_\alpha (t-s)^{\alpha-\frac{1}{2}} + d_\alpha\left(\frac{1}{2}-\alpha\right)\int_s^t (u-s)^{\alpha-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-\alpha}\right) du\right] I_{(0,t)}(s), \tag{28}$$

with the following normalizing constant

$$d_\alpha = \left(\frac{2\alpha\Gamma(\frac{3}{2}-\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2-2\alpha)}\right)^{\frac{1}{2}}.$$

Taking into account expression (27) for the fractional Brownian motion, one can consider the following representation in law of the fractional Brownian sheet:

$$\int_0^t \int_0^s K_\alpha(s,u) K_\beta(t,v) dB_{u,v}.$$

To see the equality in law of these processes and the fractional Brownian sheet it suffices to realize that this is a centered Gaussian process with the same covariance function as $W^{\alpha,\beta}$.

Our result states as follows.

Theorem 2. *Let $L = \{L(s,t);\, s,t \geq 0\}$ be a Lévy sheet with exponent $\Psi(\xi) = a(\xi) + ib(\xi)$, where $a(\xi)$ and $b(\xi)$ are defined in (14) and (15), respectively. Consider the random kernels θ_n^1 or θ_n^2 defined in (16) and (17), with $\theta \in (0, 2\pi)$, and the deterministic kernels K_α and K_β defined in (28). Consider the conditions:*

(A) $\min(\alpha,\beta) > \frac{1}{2}$ *and* $a(\theta)a(2\theta) \neq 0$.

(B) $\min\{\alpha,\beta\} \leq \frac{1}{2}$ *and there exists an even integer number* $m > \frac{4}{\min(2\alpha,2\beta)}$ *such that*

$$a(\theta)a(2\theta)\cdots a(m\theta) \neq 0.$$

Then, if (A) or (B) is satisfied, the laws of the processes $\{X_n(s,t),\ (s,t) \in [0,1]^2\}$ given by (1) (with $K_1 = K_\alpha$ and $K_2 = K_\beta$) converge weakly to the law of a fractional Brownian sheet, $\{W_{s,t}^{K_\alpha,K_\beta},\ (s,t) \in [0,1]^2\}$, given by (2), in $\mathcal{C}([0,1]^2)$ when n goes to infinity.

Proof. If we consider the kernels K_α (or K_β), we have that

$$\int_0^1 \left(K_\alpha(s',r) - K_\alpha(s,r)\right)^2 dr = E\left(W_{s'}^\alpha - W_s^\alpha\right)^2 = (s'-s)^{2\alpha}.$$

And then the set of conditions (H1) is satisfied if $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$. In [4], it is proved that if α or β belongs to $(0, \frac{1}{2}]$, then the kernels satisfy the set of conditions (H1').

In Corollary 1, we have checked that θ_n^1 and θ_n^2 satisfy (H2) and (H3). We have also checked that under (B) (H4) is also true. We finish the proof by applying Theorem 1.

□

Remark 4. *If we consider the kernel processes θ_n^1 and θ_n^2 with the same Lévy process, it can be proved that we will obtain two families of approximation processes that converge to two independent fractional Brownian sheets. The proof follows the ideas in [5] for the case of the Brownian sheet.*

5.1. Other examples

In [4], we can find other examples of kernels that satisfy the set of conditions (H1'). On the other hand, in Lemma 4, we have seen that using random kernels θ_n^1 or θ_n^2 defined by (16) and (17) condition (H4) will also be satisfied. So Theorem 1 can be applied to processes with representation (2), where K_1 and K_2 are some of these kernels. For the sake of completeness, let us recall these examples.

5.1.1. Goursat kernels

The kernel

$$K(t,r) = \sum_{i=1}^{I} g_i(t) h_i(r) I_{[0,t]}(r)$$

for some $I \in \mathbb{N}$, with $g_i \in \mathrm{Lip}_{\gamma_1}(0 < \gamma_1 \leq 1)$ and $h_i \in L^2([0,1])$. We also impose that F, defined by $F(t) = \int_0^t h_i^2(r)dr$, belongs to $\mathrm{Lip}_{\gamma_2}(0 < \gamma_2 \leq 1)$.

5.1.2. The Holmgren-Riemann-Liouville fractional integral

The kernel

$$K(t,r) = \sqrt{2\pi}(t-r)^{H-\frac{1}{2}} I_{[0,t]}(r),$$

with $0 < H < 1$. This kernel satisfies the set of conditions (H1') for all $0 < H < 1$.

5.1.3. A Lipschitz function

The kernel

$$K(t,r) = h(t-r) I_{[0,t]}(r),$$

with h a Lipschitz function of parameter $\gamma \in (0,1]$.

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