Math. Commun. **26**(2021), 159–175

Designs and binary codes from maximal subgroups and conjugacy classes of the Mathieu group M_{11}*

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Received December 2, 2020; accepted February 1, 2021

Abstract. By using a method of constructing block-primitive and point-transitive 1-designs, in this paper we determine all block-primitive and point-transitive 1-(v, k, λ)-designs from the maximal subgroups and the conjugacy classes of elements of the small Mathieu group M_{11}. We examine the properties of 1-(v, k, λ)-designs and construct the codes defined by the binary row span of the incidence matrices of the designs. Furthermore, we present a number of interesting Δ-divisible binary codes invariant under M_{11}.

AMS subject classifications: 05E20, 05E30, 94B05

Key words: primitive designs, linear code, Mathieu group M_{11}

1. Introduction

One of the main problems in design theory is the classification of structures with given parameters and/or with a prescribed automorphism group. The construction of primitive designs from finite simple groups gives additional information on the group acting on a design, which is interesting from both the group theoretical and the combinatorial points of view. There are a number of known constructions of 1-designs invariant under finite groups. However, two methods proposed by Key and Moori in [25] and [26], respectively, stand out in that they provide the construction of point- and block-primitive 1-designs invariant under finite simple groups.

The technique presented in [26] outlines the construction of block-primitive and point-transitive 1-designs from the maximal subgroups and conjugacy classes of elements of a finite non-abelian simple group. Note that the said 1-designs are not necessarily symmetric.

Examples of the application of this method to some families of finite simple groups of Lie type are given in [26, 27, 28, 30, 31, 32]. In this paper, taking the simple group M_{11} of Mathieu as an example of application, and using the method

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described in [26], we determine all block-primitive and point-transitive 1-(v, k, λ)-designs invariant under M_{11}. To our knowledge, this is the first instance of an application of the said method to obtaining enumeration results which provide a structure description of automorphism groups of the designs and of the corresponding binary codes, on which a simple group of sporadic type acts as a permutation group of automorphisms. With this study we attempt to gain further understanding of the structure description of the automorphism group of 1-(v, k, λ) designs for $\lambda \geq 2$.

For each design we construct the associated binary linear code and the automorphism group. A number of interesting Δ-divisible binary codes are constructed from the incidence matrices of these designs or of their reduced designs. In particular, we construct some irreducible 10-dimensional Δ-divisible codes invariant under M_{11}.

A variety of structures preserved by the Mathieu group M_{11} have been studied. These include combinatorial designs and strongly regular graphs [13], arc-transitive graphs [15], residually-primitive geometries [7], and linear codes [12, 14], to name but a few.

The paper is organized as follows: in Section 2, we give the terminology and in Section 3, we give a brief but complete overview of the construction method used to obtain the designs and codes examined in the paper. In Section 4, we give some background to the Mathieu group M_{11} and some results of the permutation characters defined by the action of M_{11} on the sets of conjugates of its distinct maximal subgroups, and in Sections 4.1, 4.2, 4.3 and 4.4, we present our results on the designs and codes. The latter are obtained as binary row spans of incidence matrices of the designs.

2. Terminology and notation

Our notation is standard, and it is as in [1, 2] for designs, and in [9, 23] for group theory and character theory. For the structure of finite simple groups and their maximal subgroups, as well as for the structure of conjugacy classes, we follow the $\mathbb{ATLAS}$ notation [9]. The notations $G.H$, $G:H$, and $G^{\cdot}H$ are used to denote a general extension, a split extension and a non-split extension, respectively. For a prime p, the symbol p^n denotes an elementary abelian group of that order. If p is an odd prime, p_+^{1+2n} and p_-^{1+2n} denote the extraspecial groups of order p^{1+2n} and exponent p or p^2 respectively. Let Sym_Ω denote the full symmetric permutation group of the set Ω and we write Sym_n for the symmetric group acting on $\{1, \ldots, n\}$ and A_n for the alternating group on $\{1, \ldots, n\}$.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. We say that $\mathcal{D}$ is symmetric if it has the same number of points and blocks.

The code C_F of the design $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. We take F to be a prime field F_p, in which case we also write C_p for C_F, and refer to the dimension of C_p as the *p-rank* of $\mathcal{D}$. In the general case of a 2-design, the prime must divide the order of the design, i.e. $r - \lambda$, where r is the replication number for the design, that is, the number of blocks through a point. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by

$\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus, $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$ and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. For any code C, the *dual* or *orthogonal* code $C^{\perp}$ is the orthogonal under the standard inner product. The *hull* of a design's code over some field is the intersection $C \cap C^{\perp}$. If a linear code over a field of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to represent this information. A linear code C over any field is a *linear code with a complementary dual (LCD)* code if $C \cap C^{\perp} = \{0\}$.

The all-one vector will be denoted by $\jmath$, and it is the constant vector of weight the length of the code, whose coordinate entries consist entirely of 1's. A linear code C over $\mathbb{F}_q$ is said to be Δ-divisible if the Hamming weight $\mathrm{w}(c)$ of every codeword $c \in C$ is divisible by Δ. Binary *doubly-even* and *triply-even* codes are special cases of Δ-divisible codes, where $\Delta = 2^r$ and $r = 2, 3$. The weight distribution of a code C is the sequence $\{A_i | i = 0, 1, \ldots, n\}$, where A_i is the number of codewords of weight i. The polynomial $W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$ is called the weight enumerator of C. The weight enumerator of a code C and its dual $C^{\perp}$ are related via MacWilliams' identity.

Two linear codes of the same length and over the same field are *equivalent* if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of C.

The *graphs*, $\Gamma = (V, E)$ with vertex set V and edge set E, discussed here are undirected with no loops, apart from the case where *all* loops are included, in which case the graph is called *reflexive*. A graph is *regular* if all the vertices have the same valency. An *adjacency matrix* A of a graph of order n is an $n \times n$ matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. An *incidence matrix* of Γ is an $n \times |E|$ matrix B with $b_{ij} = 1$ if the vertex labelled by i is on the edge labelled by j, and $b_{ij} = 0$ otherwise. If Γ is regular with valency k, then the 1-$(|E|, k, 2)$ design with incidence matrix B is called the *incidence design* of Γ. The *neighbourhood design* of a regular graph is the 1-design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix as an incidence matrix for the design.

3. Construction of the designs

In this section, we outline a method of constructing block-primitive and point-transitive 1-designs (see Result 1) given by Key and Moori in [26].

We start by recalling the notion of permutation character, for it is useful in determining one of the parameters of these designs.

Let G be a finite group and Ω a finite G-set. The map $\pi_\Omega : G \longrightarrow \mathbb{N}$ given by $\pi_\Omega(g) = |\mathrm{Fix}_\Omega(g)| = |\{\omega \in \Omega : g \cdot \omega = \omega\}|$ is called the permutation character of the action of G on Ω. A transitive permutation character is the character of a transitive action. If the subgroup M of G is the stabilizer of a point in a transitive G-set, we have $\pi_\Omega = 1_M^G$. We state the following result whose proof we give for the sake of completeness.

Lemma 1. *Let M be a subgroup of G, $x \in G$ and x^G, the conjugacy class of x, and let 1_M^G be the permutation character of the action of G on the right cosets of M. Then*

$$\frac{1_M^G(x) \times |x^G|}{1_M^G(\mathbf{1})} = |x^G \cap M|.$$

Moreover, if M is self-normalizing, then the number of conjugates of M containing x is $1_M^G(x)$.

Proof. Recall that the permutation character of G on the right cosets of M is the induced character obtained from the principal character 1_M^G of M. Applying the formula for induced characters yields:

$$\begin{aligned} 1_M^G(x) &= \frac{1}{|M|}\sum_{x\in G} 1_M^G(g^{-1}xg) \\ &= \frac{|x^G \cap M|}{|M|} \cdot |C_G(x)| \\ &= \frac{|x^G \cap M|}{|M|} \cdot \frac{|G|}{|x^G|} \\ &= |x^G \cap M| \frac{[G{:}M]}{|x^G|} \\ &= |x^G \cap M| \frac{1_M^G(\mathbf{1})}{|x^G|}. \end{aligned}$$

From this we obtain the result. If M is self-normalizing, then the actions of G on the right cosets of M by right multiplication and on the set of conjugates of M by conjugation are equivalent and thus have the same character. □

Remark 1. *(i) Transitive permutation characters are a tool to deduce information about the subgroups of a group G from the character table of G. From the permutation character 1_M^G one can obtain information about the conjugacy classes of M; for example, one can compute the number of elements of a certain order in M.*

(ii) Lemma 1 has numerous applications amongst which are determining the covering number of some finite simple groups [17]; finding possible permutation characters of a finite group [4]; determining multiplicity-free permutation characters [5]; and the construction of combinatorial designs [26]. Restricted to finite simple groups, the results of Lemma 1 are used in [26], where Key and Moori proposed a method of constructing point-transitive and block-primitive 1-(v, k, λ) designs from the maximal subgroups and conjugacy classes of elements of the group. This construction is outlined in Result 1 below and will constitute the main tool used for our investigations.

Result 1. *Let S be a finite simple group, T a maximal subgroup of S and x^S a conjugacy class of elements of order n in S such that $T \cap x^S \neq \varnothing$. Also, let χ_T be the permutation character afforded by the action of S on the set of conjugates of T in S.*

Let $B = \{(T \cap x^S)^y | y \in S\}$. Then we have a 1-$(|x^S|, |T \cap x^S|, \chi_T(x))$ design $\mathcal{D}$. The group S acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive (but not necessarily primitive) on points of $\mathcal{D}$.

Proof. See [26, Theorem 4]. □

Remark 2. *(i) Observe that in Result 1 and elsewhere 1_M^G is denoted by χ_T.*

(ii) The 1-design $\mathcal{D}$ of Result 1 is an $S_r(1, k, v)$ 1-design called a tactical configuration (see [2, Definition 3.5]).

The following lemma shows that if we obtain two of the three parameters of the design, the third may be directly computed.

Lemma 2. *Let $\mathcal{D} = (v, k, \lambda)$ be a design obtained by Result 1. Then*

$$1_M^G(\mathbf{1}) = \frac{\lambda v}{k} \quad \text{and} \quad r = \lambda.$$

Proof. This follows by Lemma 1 by noticing that $1_M^G(\mathbf{1}) = b = [S{:}T]$. The remainder follows since $\mathcal{D}$ is a tactical configuration, in which case we obtain by [2, Proposition 1.1, Chapter II] that $b = \frac{\lambda v}{k}$. Now, $r = \lambda$ is a direct application of [2, Corollary 1.4, Chapter II].

□

In the sequel we consider S to be the simple group M_{11} of Mathieu and apply Result 1 to construct all point- and block-primitive 1-(v, k, λ) designs from its maximal subgroups, and conjugacy classes of elements. We give a brief description of M_{11}, and encourage the reader to consult some standard group theory textbooks such as, for example, [8] for further details.

4. Some results of the Mathieu group M_{11}

The small Mathieu groups $\mathrm{M}_{11}, \mathrm{M}_{12}$ were discovered by the French mathematician Émile Mathieu (1835–1890), who also discovered the large Mathieu groups $\mathrm{M}_{22}, \mathrm{M}_{23}$ and M_{24}. They are remarkable groups; for example, apart from the symmetric and alternating groups, M_{12} and M_{24} are the only 5-transitive permutation groups.

Here, we consider the simple group M_{11} of Mathieu, see $\mathbb{ATLAS}$ (see also [33, Section 5.3.8]). The Mathieu group M_{11} is defined as a stabilizer of a point in M_{12} and this definition shows that it has order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. M_{11} has five classes of maximal subgroups up to conjugation, namely of degrees 11, 12, 55, 66 and 165. In Table 1 below, the first column depicts the ordering of the primitive representations as given by the $\mathbb{ATLAS}$ [9] and as used in our computations; the second gives the maximal subgroups; and the third gives the degrees (the number of cosets of the point stabilizer).

In the next five lemmas, we determine the values of $\lambda = \chi_{T_i}(x)$ for $x \in G$ and using Lemma 2, we obtain the remaining parameters of the designs as given in Table 3.

Line	Max. sub. structure	degree
T_1	$A_6{}^{\cdot}2_3$	11
T_2	$L_2(11)$	12
T_3	$3^2{:}Q_8 \cdot 2$	55
T_4	Sym_5	66
T_5	$2{}^{\cdot}\mathrm{Sym}_4$	165

Table 1: *Maximal subgroups of* M_{11}

Lemma 3. *Let T_1 be a maximal subgroup of M_{11} isomorphic to $A_6{}^{\cdot}2_3$ and suppose that χ_{T_1} is the permutation character afforded by the action of M_{11} on the set of conjugates of T_1. Then the following holds:*

$$\chi_{T_1}(x) = \begin{cases} 3, & \text{if } o(x) = 2 \ \text{ or } \ o(x) = 4; \\ 2, & \text{if } o(x) = 3; \\ 1, & \text{if } o(x) = 5 \ \text{ or } o(x) = 8; \\ 0, & \text{if } o(x) = 6 \ \text{ or } o(x) = 11. \end{cases}$$

Proof. The action of M_{11} on the set of conjugates of T_1 is doubly transitive [6, Lemma 2.1.1(i)] of degree 11. So, the decomposition of the permutation character of M_{11} with respect to this action is $\pi_\Omega = 1 + \varphi$, where φ is an irreducible character of M_{11} of degree 10. Hence, $\varphi(\mathbf{1}) = 11 - 1 = 10$. An inspection of the character table of M_{11} [9, p. 18] shows that M_{11} has three irreducible characters of degree 10, namely χ_2, χ_3 and χ_4. In particular, notice in this case that $\varphi = \chi_2$, since the character values are real, (see also [6, Lemma 2.1.1(i)]). Thus, the values of φ on conjugacy classes of M_{11} are $-1, 0, 1$ and 2. In fact, if $x^{M_{11}}$ is a conjugacy class of M_{11} we have

$$\varphi(x^{M_{11}}) = \begin{cases} 2, & \text{if } o(x) = 2 \ \text{ or } \ o(x) = 4; \\ 1, & \text{if } o(x) = 3; \\ 0, & \text{if } o(x) = 5 \ \text{ or } o(x) = 8; \\ -1, & \text{if } o(x) = 6 \ \text{ or } o(x) = 11. \end{cases}$$

Notice that the value $\varphi(x^{M_{11}}) = -1$ occurs if $x^{M_{11}} \cap T_1 = \varnothing$. This implies that there are no elements of orders 6 and 11, respectively, in $A_6{}^{\cdot}2_3$. Direct application of Lemma 1 shows this. $\square$

Lemma 4. *Let T_2 be a maximal subgroup of M_{11} isomorphic to $L_2(11)$ and suppose that χ_{T_2} is the permutation character afforded by the action of M_{11} on the set of conjugates of T_2. Then the following holds:*

$$\chi_{T_2}(x) = \begin{cases} 4, & \text{if } o(x) = 2; \\ 3, & \text{if } o(x) = 3; \\ 2, & \text{if } o(x) = 5; \\ 1, & \text{if } o(x) = 6 \ \text{ or } o(x) = 11; \\ 0, & \text{if } o(x) = 4 \ \text{ or } o(x) = 8. \end{cases}$$

Proof. The proof follows by using arguments similar to those used in the proof of Lemma 3 bearing in mind that the action of M_{11} on the set of conjugates of T_2

is doubly transitive of degree 12 [6, Lemma 2.1.1(ii)], and that there is a unique irreducible character of degree 11, namely $\varphi = \chi_5$. □

Lemma 5. *Let T_3 be a maximal subgroup of M_{11} isomorphic to $3^2{:}\mathrm{Q}_8{\cdot}2$ and suppose that χ_{T_3} is the permutation character afforded by the action of M_{11} on the set of conjugates of T_3. Then the following holds:*

$$\chi_{T_3}(x) = \begin{cases} 7, & \textit{if } o(x) = 2; \\ 3, & \textit{if } o(x) = 4; \\ 1, & \textit{if } o(x) = 3,\ o(x) = 6 \ \text{ or } o(x) = 8; \\ 0, & \textit{if } o(x) = 5 \ \text{ or } o(x) = 11. \end{cases}$$

Proof. Notice first that the action of M_{11} on the set of conjugates of T_3 is rank 3 of degree 55 with subdegrees 1, 18, and 36. In particular, it follows from [6, Lemma 2.1.1(iii)] that the permutation character is $\varphi = 1 + 10a + 44a$. We obtain that the values of φ on conjugacy classes of M_{11} are $-1, 0, 1, 2$ and 4, and the result follows. □

Lemma 6. *Let T_4 be a maximal subgroup of M_{11} isomorphic to Sym_5 and suppose that χ_{T_4} is the permutation character afforded by the action of M_{11} on the set of conjugates of T_4. Then the following holds:*

$$\chi_{T_4}(x) = \begin{cases} 10, & \textit{if } o(x) = 2; \\ 3, & \textit{if } o(x) = 3; \\ 2, & \textit{if } o(x) = 4; \\ 1, & \textit{if } o(x) = 5 \ \text{ or } o(x) = 6; \\ 0, & \textit{if } o(x) = 8 \ \text{ or } o(x) = 11. \end{cases}$$

Proof. Omitted. □

Lemma 7. *Let T_5 be a maximal subgroup of M_{11} isomorphic to $2^{\cdot}\mathrm{Sym}_4$ and suppose that χ_{T_5} is the permutation character afforded by the action of M_{11} on the set of conjugates of T_5. Then the following holds:*

$$\chi_{T_5}(x) = \begin{cases} 13, & \textit{if } o(x) = 2; \\ 3, & \textit{if } o(x) = 3; \\ 1, & \textit{if } o(x) = 4,\ o(x) = 6 \ \text{ or } o(x) = 8; \\ 0, & \textit{if } o(x) = 5 \ \text{ or } o(x) = 11. \end{cases}$$

Proof. Omitted. □

Let H be a subgroup of G. We say that H controls G-fusion in itself if each pair of elements in H which are conjugates in G are also conjugates in H, or equivalently, if for $x \in H$ we have $x^G \cap H = x^H$.

We adopt the following definition from [30]:

Definition 1. *Let $H \le G$ and k be a positive integer. We define*

$$\mathrm{cn}_H^G(k) := |\{x^G | x \in H, o(x) = k\}|.$$

Also, we write $\mathrm{cn}_H(k) := \mathrm{cn}_H^H(k)$. It is easy to see that $\mathrm{cn}_H^G(k) \le \mathrm{cn}_H(k)$ and, if the equality holds, then, for every $x \in H$ with $o(x) = k$, we have $x^G \cap H = x^H$.

We need the following result on the maximal subgroups of M_{11} that satisfies the control fusion property.

Lemma 8. *Let T_i be a maximal subgroup of M_{11}. Then T_i controls M_{11}-fusion in itself in the following cases:*

(*a*) T_i *is any maximal subgroup of* M_{11} *and* $o(x) = 3$;

(*b*) $T_i \cong \mathrm{A}_6{}^{\cdot}2_3$ *for all* $x \in T_i$;

(*c*) $T_i \cong \mathrm{L}_2(11)$ *and* $o(x) = 2$ *or* $o(x) = 6$;

(*d*) $T_i \cong 3^2{:}\mathrm{Q}_8 \cdot 2$ *and* $o(x) = 6, 8$ *or* $o(x) = 11$;

(*e*) $T_i \cong \mathrm{Sym}_5$ *and* $o(x) = 4, 5$ *or* $o(x) = 6$;

(*f*) $T_i \cong 2{}^{\cdot}\mathrm{Sym}_4$ *and* $o(x) = 4$ *or* $o(x) = 8$.

Proof. We will illustrate case (a). The remaining cases follow by inspecting the $\mathbb{ATLAS}$ [9] or by direct computations with Magma [3] or GAP [19].
(a) Let $x \in T_i$ be a non-trivial element and suppose that $o(x) = 3$. Then we obtain $\mathrm{cn}_{T_i}(3) = 1$, and so we have $x^{\mathrm{M}_{11}} \cap T_i = x^{T_i}$. In particular, if $T_i \cong \mathrm{A}_6{}^{\cdot}2_3$ or $T_i \cong \mathrm{L}_2(11)$, this follows by a direct inspection of [9, p. 4] and [9, p. 7], where the reader will notice that each such maximal subgroup of M_{11} has exactly one conjugacy class of elements of order 3. □

We end this section with the following useful result which deals with designs constructed using Result 1 when the subgroup T_i controls M_{11}-fusion in itself.

Lemma 9. *Let S be a simple group with a maximal subgroup T and assume that T controls S-fusion in itself. Then the designs constructed by Result* 1 *are* 1-$(|x^S|, |x^T|, [C_S(x) : C_T(x)])$ *designs, where x is an element of T.*

Proof. See [32, Proposition 3.4]. □

4.1. The designs

Using results of Section 4 and Section 3 we are able to state and prove our main result for designs in Proposition 1. In Table 2, we reproduce from the $\mathbb{ATLAS}$ the information on the element orders, conjugacy class sizes and centralizer orders for the elements of M_{11}.

Cycle type	1^{11}	$1^3 2^4$	$1^2 3^3$	$1^3 4^2$	$1^1 5^2$	$2^1 3^1 6^1$	$1^1 2^1 8^1_A$	$1^1 2^1 8^1_B$	11^1_A	11^1_B
$\lvert x^S \rvert$	1	165	440	990	1584	1320	990	990	720	720
Centralizer order	7920	48	18	8	5	6	8	8	11	11

Table 2: *Conjugacy classes of elements of* M_{11}

By $\mathcal{D}(x, T_i)$ we denote the design obtained from each maximal subgroup T_i of M_{11} and a non-trivial element x of order n in $x^{\mathrm{M}_{11}}$.

Proposition 1. *Let T_i be a maximal subgroup of M_{11} as given in lines 1 to 5 in Table 1 and let $x \in \mathrm{M}_{11}$ be a non-trivial element given in Table 2. Then the parameters of all non-trivial 1-designs $\mathcal{D}(x, T_i) = (v, k, \lambda)$ are as given in Table 3.*

Proof. By Result 1, $\mathcal{D}(x, T_i) = (|x^{\mathrm{M}_{11}}|, |T_i \cap x^{\mathrm{M}_{11}}|, \chi_{T_i}(x))$. The first parameter $|x^{\mathrm{M}_{11}}|$ is given in Table 2. By Lemma 2 we can deduce the value of $|T_i \cap x^{\mathrm{M}_{11}}|$ and by Lemmas 3 - 7 we obtain the value of $\chi_{T_i}(x)$. Alternatively, once two of the parameters are known, the third can be calculated directly using Lemma 2. The result now follows. $\square$

In Table 3 below, we give the parameters of the designs constructed from all maximal subgroups and conjugacy classes of elements of M_{11}.

Remark 3. *The designs $\mathcal{D}$ constructed by using Result 1 are not symmetric in general. In fact, $\mathcal{D}$ is symmetric if and only if*

$$\begin{aligned} b \,=\, |\mathcal{B}| = |\mathcal{P}| = v &\Leftrightarrow [\mathrm{M}_{11}{:}T_i] = |x^{\mathrm{M}_{11}}| \\ &\Leftrightarrow [\mathrm{M}_{11}{:}T_i] = [\mathrm{M}_{11}{:}C_{\mathrm{M}_{11}}(x)] \Leftrightarrow |T_i| = |C_{\mathrm{M}_{11}}(x)|. \end{aligned}$$

Observe that $\mathcal{D}(x, T_5) = (165, 13, 13)$ *(in Line 21) is the only symmetric design in Table 3.*

Line	Max	$t = o(x)$	$v = \lvert x^{\mathrm{M}_{11}} \rvert$	$k = \lvert T_i \cap x^{\mathrm{M}_{11}} \rvert$	$\lambda = \chi_{T_i}(x)$	$b = [\mathrm{M}_{11}{:}T_i]$	$\mathrm{Aut}(\mathcal{D}(x, T_i))$
1	T_1	$t = 2$	165	45	3	11	Sym_{11}
2	T_1	$t = 3$	440	80	2	11	$(\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{11}$
3	T_1	$t = 4$	990	270	3	11	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{11}$
4	T_1	$t = 5$	1584	144	1	11	$(\mathrm{Sym}_{144})^{11}{:}\mathrm{Sym}_{11}$
5	T_1	$t = 8$	990	90	1	11	$(\mathrm{Sym}_{90})^{11}{:}\mathrm{Sym}_{11}$
6	T_2	$t = 2$	165	55	4	12	M_{11}
7	T_2	$t = 3$	440	110	3	12	$2^{220}{:}\mathrm{Sym}_{12}$
8	T_2	$t = 5$	1584	264	2	12	$(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{12}$
9	T_2	$t = 6$	1320	110	1	12	$(\mathrm{Sym}_{110})^{12}{:}\mathrm{Sym}_{12}$
10	T_2	$t = 11$	720	60	1	12	$(\mathrm{Sym}_{60})^{12}{:}\mathrm{Sym}_{12}$
11	T_3	$t = 2$	165	21	7	55	M_{11}
12	T_3	$t = 3$	440	8	1	55	$(\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{55}$
13	T_3	$t = 4$	990	54	3	55	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{11}$
14	T_3	$t = 6$	1320	24	1	55	$(\mathrm{Sym}_{24})^{55}{:}\mathrm{Sym}_{55}$
15	T_3	$t = 8$	990	18	1	55	$(\mathrm{Sym}_{18})^{55}{:}\mathrm{Sym}_{55}$
16	T_4	$t = 2$	165	25	10	66	M_{11}
17	T_4	$t = 3$	440	20	3	66	$2^{220}{:}\mathrm{Sym}_{12}$
18	T_4	$t = 4$	990	30	2	66	$2^{495}{:}\mathrm{M}_{11}$
19	T_4	$t = 5$	1584	24	1	66	$(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{66}$
20	T_4	$t = 6$	1320	20	1	66	$(\mathrm{Sym}_{20})^{66}{:}\mathrm{Sym}_{66}$
21	T_5	$t = 2$	165	13	13	165	M_{11}
22	T_5	$t = 3$	440	8	3	165	$2^{220}{:}\mathrm{M}_{11}$
23	T_5	$t = 4$	990	6	1	165	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{165}$
24	T_5	$t = 6$	1320	8	1	165	$(\mathrm{Sym}_8)^{165}{:}\mathrm{Sym}_{165}$
25	T_5	$t = 8$	990	6	1	165	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{165}$

Table 3: *Non-trivial* 1-(v, k, λ)*-designs from* M_{11} *constructed using Result 1*

4.2. Automorphism groups of the designs

As stated in Section 1, a study of the structure of the automorphism group $\mathrm{Aut}(\mathcal{D})$ of a 1-(v,k,λ)-design $\mathcal{D}$ constructed using Result 1 was carried out in [29]. In that paper, the notion of a reduced design was introduced with a view to providing a description of the structure of $\mathrm{Aut}(\mathcal{D})$ when $\lambda \geq 2$.

For the sake of completeness, in what follows we restate the definition of reduced designs and some subsequent results.

Definition 2 ([29], Definition 2.10)**.** *For a point $x \in \mathcal{P}$ of the 1-(v,k,λ) design $\mathcal{D}(x,T_i)$, let $B_1,\dots,B_\lambda$ be all distinct blocks containing x. Define*

$$I_x = \bigcap_{s=1}^{\lambda} B_s.$$

Lemma 10. *Let $\mathcal{D} = (\mathcal{P},\mathcal{B})$ be a 1-(v,k,λ) design. Then we can construct a 1-$(v/|I_x|, k/|I_x|, \lambda)$ design $\mathcal{D}_I$ called the reduced design of $\mathcal{D}$.*

In this section, we examine the structure of $\mathrm{Aut}(\mathcal{D})$ of the designs $\mathcal{D}$ given in Table 3, placing emphasis on those cases where $\lambda \geq 2$. To this end we state the following known observations:

Remark 4. *If $\lambda = 1$, then $\mathcal{D}$ is a 1-$(|x^S|,k,1)$ design with b pairwise disjoint blocks. It follows from [26, Remark 3], (see also [27, Remark 4.3]) that* $\mathrm{Aut}(\mathcal{D}) \cong {\mathrm{Sym}_k}^b \wr \mathrm{Sym}_b = (\mathrm{Sym}_k)^b{:}\mathrm{Sym}_b$.

If $r = \lambda = 2$, then $\mathcal{D}$ is a 1-$(|x^S|,k,2)$ design with b blocks. The study of t-$(v,k,2)$-designs for $t = 2$ is of current interest and many papers have devoted attention thereto. We refer the reader to [16] (see also some of the references therein), which examines the classification of flag-transitive 2-$(v,k,2)$ designs.

In the case of a 1-$(|x^S|,k,2)$ design, a regular graph can be defined, where the vertices are the blocks of the design, and two vertices labelled by the blocks b_i and b_j are adjacent if b_i and b_j meet. Moreover, the incidence matrix for the design is an incidence matrix for the graph. In particular, when the graph is an undirected graph without multiple edges, the following result can be used.

Result 2. ([18, Lemma 1]). *Let $\Gamma = (V,E)$ be a regular graph with $|V| = N$, $|E| = e$ and valency v. Let $\mathcal{G}$ be the 1-$(e,v,2)$ incidence design from an incidence matrix A for Γ. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{G})$.

Remark 5. *If the graph Γ is also connected, then using an inductive argument it can be shown that $\mathrm{rank}_p(A) \geq |V| - 1$ for all p. Equality holds when $p = 2$. If, in addition, the minimum weight is the valency of the graph, and the words of this weight are the scalar multiples of the rows of the incidence matrix, then it follows that* $\mathrm{Aut}(C_p(\mathcal{G})) = \mathrm{Aut}(\mathcal{G})$.

As an application of the preceding results to the study of $\mathrm{PSL}(2,q)$-invariant 1-$(|x^S|,k,2)$-designs in [27, Section 5.2] (see also [26, Section 6]), the authors showed that $\mathrm{Aut}(\mathcal{D})$ is isomorphic to $2^{\frac{1}{2}q(q+1)}{:}\mathrm{Sym}_{q+1}$ or to Sym_{q+1}, depending on whether or not the non-trivial element $x \in x^S$ is an involution. However, this does not seem to provide a general structure description for the automorphism groups of the 1-$(|x^S|, k, 2)$-designs constructed using Result 1.

The reader will notice in Proposition 2 below that the automorphism groups of the 1-$(|x^S|, k, 2)$-designs constructed from the maximal subgroups and conjugacy classes of elements of M_{11} possess a different structure description.

Proposition 2. *Let $\mathcal{D}$ be a block-primitive and point-transitive* 1-(v, k, λ) *design constructed from a maximal subgroup T_i and a conjugacy class $x^{\mathrm{M}_{11}}$ of elements of order n in M_{11}. Then $\mathcal{D}$ is a* 1-$(v, k, 2)$ *design if and only if $\mathcal{D}$ is constructed from $T_i \cong \mathrm{A}_6 \cdot 2_3, \mathrm{S}_5$ and $\mathrm{L}_2(11)$, and $x^S = 3A, 4A$ and $5A$, respectively. Further,* $\mathrm{Aut}(\mathcal{D})$ *is isomorphic to* $(\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{11}$, $2^{495}{:}\mathrm{M}_{11}$, *and* $(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{12}$, *respectively.*

Proof. Suppose that $T_i \cong \mathrm{A}_6 \cdot 2_3$. By Proposition 1, it follows that $\mathcal{D}$ is a 1-$(440, 80, 2)$ design with $b = 11$ blocks. By results of [29, Sections 2, 4], we observe that for every point $x \in 3A$, the intersection of the two blocks of $\mathcal{D}$ containing x has size 8, i.e., $|I_x| = 8$. Thus, the reduced design $\mathcal{D}_I$ is a 1-$(55, 10, 2)$ design with 11 blocks and the automorphism group isomorphic to Sym_{11} and $S(I) = (\mathrm{Sym}_8)^{55} \trianglelefteq \mathrm{Aut}(\mathcal{D})$ and so $\mathrm{Aut}(\mathcal{D}) \cong (\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{11}$. From this, it follows that $\mathrm{Aut}(\mathcal{D}_I) \cong \mathrm{Aut}(\mathcal{D})/S(I) \cong \mathrm{Sym}_{11}$. Arguing similarly, it can be shown that the automorphism groups of the 1-$(990, 30, 2)$-design constructed from a maximal subgroup isomorphic to Sym_5 and an element of order 4, and a 1-$(1584, 264, 2)$-design obtained from a maximal subgroup isomorphic to $\mathrm{L}_2(11)$ and an element of order 5, are isomorphic to $2^{495}{:}\mathrm{M}_{11}$, and $(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{12}$, respectively. □

Next, we examine the structure of the automorphism group of the 1-(v, k, λ) designs $\mathcal{D}$ for which $r = \lambda \geq 3$. Table 3 shows that these designs are constructed from all maximal subgroups T_i of M_{11} and every conjugacy class $x^{\mathrm{M}_{11}}$ of non-trivial elements.

Proposition 3. *Let T_i be a maximal subgroup of M_{11}. Let $x \in T_i$ be a non-trivial element with $o(x) \neq 2$. Then $\mathcal{D}(x, T_i)$ is a block-primitive and point-transitive* 1-$(v, k, 3)$ *design and*

(*a*) $v = 990,\ k = 270$ *for* $T_i \cong \mathrm{A}_6{}^{\cdot}2$;

(*b*) $v = 440,\ k = 110$ *for* $T_i \cong \mathrm{L}_2(11)$;

(*c*) $v = 990,\ k = 54$ *for* $T_i \cong 3^{\cdot}\mathrm{Q}_8 \cdot 2$;

(*d*) $v = 440,\ k = 20$ *for* $T_i \cong \mathrm{Sym}_5$;

(*e*) $v = 440,\ k = 8$ *for* $T_i \cong 2^{\cdot}\mathrm{Sym}_4$.

Moreover, $|I_x| \geq 2$, and $\mathrm{Aut}(\mathcal{D})$ *is as in lines* 3, 7, 13, 17 *and* 22 *in Table* 3.

In what follows, we consider those cases of a 1-(v, k, λ) design for $\lambda \geq 3$ and $|I_x| = 1$.

Lemma 11. *Let T_i be a maximal subgroup of M_{11}. Let $x \in T_i$ be an element of order 2. Then $\mathcal{D}(x, T_i)$ is a point- and block-primitive* 1-(v, k, λ) *design with parameters* (*a*) $v = 165,\ k = 45$ *and* $\lambda = 3$ *for* $T_i \cong \mathrm{A}_6{}^{\cdot}2$;

(*b*) $v = 165,\ k = 55$ *and* $\lambda = 4$ *for* $T_i \cong \mathrm{L}_2(11)$;

(*c*) $v = 165,\ k = 21$ *and* $\lambda = 7$ *for* $T_i \cong 3^{\cdot}\mathrm{Q}_8 \cdot 2$;

(*d*) $v = 165,\ k = 25$ *and* $\lambda = 10$ *for* $T_i \cong \mathrm{Sym}_5$;

(*e*) $v = 165,\ k = 13$ *and* $\lambda = 13$ *for* $T_i \cong 2^{\cdot}\mathrm{Sym}_4$.

Moreover, $|I_x| = 1$, *and* $\mathrm{Aut}(\mathcal{D})$ *is as in lines* $1, 6, 11, 16$ *and* 21 *in Table* 3.

Proof. (a) Suppose that $T_i \cong \mathrm{A}_6{}^{\cdot}2_3$. Since by Lemma 8(b), T_i controls M_{11}-fusion in itself, we obtain $x^{\mathrm{M}_{11}} \cap T_i = x^{T_i}$. Furthermore, $|x^{\mathrm{M}_{11}} \cap T_i| = 45 = |x^{T_i}|$. Now let χ_{T_i} be the permutation character of M_{11} with respect to the action of M_{11} on the conjugates of T_i. Then, by [27, Corollary 2.4], we have $\chi_{T_i}(x) = \sum_{i=1}^{k} \frac{|C_{\mathrm{M}_{11}}(x)|}{|C_{T_i}(x_i)|}$, where $x_1, x_2, \ldots, x_k$ are representatives of the conjugacy classes of T_i that fuse to the class $x^{\mathrm{M}_{11}}$ in M_{11}. Since T_i controls M_{11}-fusion in itself, we get $k = 1$ and $\chi_{T_i}(x) = |C_{\mathrm{M}_{11}}(x) : C_{T_i}(x)|$.

Observe from Table 2 that $|C_{\mathrm{M}_{11}}(x)| = 48$, and from Table 1 that $C_{\mathrm{M}_{11}}(x) \cong 2^{\cdot}S_4$. Moreover, from [9, p. 4] we deduce that $C_{T_i}(x)$ is a maximal subgroup of T_i isomorphic to 8:2 and $|C_{T_i}(x)| = 16$. Thus, $\chi_{T_i}(x) = |C_{\mathrm{M}_{11}}(x) : C_{T_i}(x)| = 3$. A direct application of Lemma 9 yields a 1-$(165, 45, 3)$ design $\mathcal{D}$ with 11 blocks. As for the automorphism group of $\mathcal{D}$, notice that $C_{\mathrm{M}_{11}}(x)$ is maximal in M_{11} and so $|I_x| = 1$, i.e., for every point $x \in x^{\mathrm{M}_{11}}$ the intersection of the three blocks containing x is of size one. Thus, $\mathcal{D}$ is itself a reduced design. Now, the structure of $\mathrm{Aut}(\mathcal{D})$ can be determined directly as an application of [29, Remark 4.3]. From the abovementioned, we deduce that $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{Sym}_{11}$.

(b) Let $T_i \cong L_2(11)$. According to Lemma 8(c), T_i controls M_{11}-fusion. So we have $x^{\mathrm{M}_{11}} \cap T_i = x^{T_i}$. The remainder follows arguing similarly to (a). The structure of $\mathrm{Aut}(\mathcal{D})$ can be determined by recalling that $\mathrm{Aut}(\mathrm{M}_{11}) = \mathrm{M}_{11}$. As an application of [29, Remark 4.3], it can be shown that $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{M}_{11}$.

(c) Consider $T_i \cong 3^2{:}Q_8 \cdot 2$. The group T_i has two conjugacy classes of elements of order 2, say $2A$ and $2B$, with $|2A| = 9$ and $|2B| = 12$, and thus $|x^{\mathrm{M}_{11}} \cap T_i| = 21 = |x^{T_i}|$. Observe that $|C_{T_i}(2A)| = 16$ and $|C_{T_i}(2B)| = 12$, and $C_{T_i}(2A) \cong 8{:}2$, while $C_{T_i}(2B) \cong \mathrm{A}_4$, see [10, 11]. Now, $\chi_{T_i}(x) = \sum_{i=1}^{2} \frac{|C_{\mathrm{M}_{11}}(x)|}{|C_{T_i}(x_i)|} = \frac{|C_{\mathrm{M}_{11}}(x)|}{|C_{T_i}(2A)|} + \frac{|C_{\mathrm{M}_{11}}(x)|}{|C_{T_i}(2B)|} = \frac{48}{16} + \frac{48}{12} = 7$. Thus, $\mathcal{D}$ is a 1-$(165, 21, 7)$ design with 55 blocks. Again notice that $C_{\mathrm{M}_{11}}(x)$ is maximal in M_{11}, and moreover $|I_x| = 1$, i.e., for every point $x \in x^{\mathrm{M}_{11}}$, the intersection of the seven blocks containing x has size one. Direct calculations using [29, Theorem 4.2] show that $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{M}_{11}$.
Parts (d) and (e) follow arguing similarly to (c).

□

4.3. Binary codes of the designs

In Table 4, we list all codes $\mathrm{C}_2(\mathcal{D}(x, T_i))$ defined by the binary row span of the incidence matrices of the designs $\mathcal{D}(x, T_i)$ given in Table 3. The second column lists the parameters of the design, the third column gives the parameters of the corresponding binary codes, followed by the automorphism groups and the number of codewords of minimum weight.

Line	$\mathcal{D}(x, T_i)$	$C_2(\mathcal{D}(x, T_i))$	$\mathrm{Aut}(C_2(\mathcal{D}(x, T_i)))$	#minwords
1	1-(165, 45, 3)	$[165, 11, 45]_2$	Sym_{11}	11
2	1-(440, 80, 2)	$[440, 10, 80]_2$	$(\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{11}$	11
3	1-(990, 270, 3)	$[990, 11, 270]_2$	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{11}$	11
4	1-(1584, 144, 1)	$[1584, 11, 144]_2$	$(\mathrm{Sym}_{144})^{11}{:}\mathrm{Sym}_{11}$	11
5	1-(990, 90, 1)	$[990, 11, 90]_2$	$(\mathrm{Sym}_{90})^{11}{:}\mathrm{Sym}_{11}$	11
6	1-(165, 55, 4)	$[165, 11, 55]_2$	M_{11}	12
7	1-(440, 110, 3)	$[440, 12, 110]_2$	$2^{220}{:}\mathrm{Sym}_{12}$	12
8	1-(1584, 264, 2)	$[1584, 11, 264]_2$	$(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{12}$	12
9	1-(1320, 110, 1)	$[1320, 12, 110]_2$	$(\mathrm{Sym}_{110})^{12}{:}\mathrm{Sym}_{12}$	12
10	1-(720, 60, 1)	$[720, 12, 60]_2$	$(\mathrm{Sym}_{60})^{12}{:}\mathrm{Sym}_{12}$	12
11	1-(165, 21, 7)	$[165, 55, 20]_2$	M_{11}	55
12	1-(440, 8, 1)	$[440, 55, 8]_2$	$(\mathrm{Sym}_8)^{55}{:}\mathrm{Sym}_{55}$	55
13	1-(990, 54, 3)	$[990, 45, 54]_2$	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{11}$	55
14	1-(1320, 24, 1)	$[1320, 55, 24]_2$	$(\mathrm{Sym}_{24})^{55}{:}\mathrm{Sym}_{55}$	55
15	1-(990, 18, 1)	$[990, 55, 18]_2$	$(\mathrm{Sym}_{18})^{55}{:}\mathrm{Sym}_{55}$	55
16	1-(165, 25, 10)	$[165, 45, 25]_2$	M_{11}	66
17	1-(440, 20, 3)	$[440, 55, 20]_2$	$2^{220}{:}\mathrm{Sym}_{12}$	66
18	1-(990, 30, 2)	$[990, 65, 30]_2$	$2^{495}{:}\mathrm{M}_{11}$	66
19	1-(1584, 24, 1)	$[1584, 66, 24]_2$	$(\mathrm{Sym}_{24})^{66}{:}\mathrm{Sym}_{66}$	66
20	1-(1320, 20, 1)	$[1320, 66, 20]_2$	$(\mathrm{Sym}_{20})^{66}{:}\mathrm{Sym}_{66}$	66
21	1-(165, 13, 13)	$[165, 109, 9]_2$	M_{11}	220
22	1-(440, 8, 3)	$[440, 165, 8]_2$	$2^{220}{:}\mathrm{M}_{11}$	495
23	1-(990, 6, 1)	$[990, 165, 6]_2$	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{165}$	165
24	1-(1320, 8, 1)	$[1320, 165, 8]_2$	$(\mathrm{Sym}_8)^{165}{:}\mathrm{Sym}_{165}$	165
25	1-(990, 6, 1)	$[990, 165, 6]_2$	$(\mathrm{Sym}_6)^{165}{:}\mathrm{Sym}_{165}$	165

Table 4: *Binary codes from the designs $\mathcal{D}(x, T_i)$ given in Table 3*

Remark 6. *(i) By Remark 4, we have that* $\mathrm{Aut}(\mathcal{D}) \cong {\mathrm{Sym}_k}^b \wr \mathrm{Sym}_b = (\mathrm{Sym}_k)^b{:}\mathrm{Sym}_b$ *for the designs $\mathcal{D}$ with parameters* $1\text{-}(|x^S|, k, 1)$, *and from [26] we deduce the parameters of their codes $C = C_p(\mathcal{D})$ as $C = C_p(\mathcal{D}) = [|x^S|, b, k]_p$ for all p.*

Observe that the minimum weight of the code equals the block size of the design and that the codewords of minimum weight are the incidence vectors of the blocks of the design, i.e., the codes are spanned by the minimum weight codewords, and thus $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D})$.

(ii) Applying Remark 5, we deduce that the codes $C = C_2(\mathcal{D})$ of the $1\text{-}(|x^S|, k, 2)$*-designs of Proposition 2 have parameters $C = C_2(\mathcal{D}) = [|x^S|, b-1, k]_2$. Moreover, the minimum weight is the block size of the design and the minimum words are the rows of the incidence matrix of the design. Hence,* $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(C_2(\mathcal{D}))$.

4.4. Some 10-dimensional Δ-divisible codes

Recall that a linear code C over $\mathbb{F}_q$ is said to be Δ-divisible if the Hamming weight $\mathrm{w}(c)$ of every codeword $c \in C$ is divisible by $\Delta > 1$, and C is said to be a projective code if $d(C^{\perp}) \geq 3$.

Recently the study of Δ-divisible codes, with $\Delta = q^r$, $r \in \mathbb{N}$, of which singly-, doubly- and triply-even codes are special cases, has gained renewed interest due to many applications, see [20, 21]. In particular, the study of projective Δ-divisible codes has gained recent interest. In [20], the authors proved that there is no projective 2^3-divisible code of length 59, and listed a number of lengths for which the existence of projective 2^4-divisible binary codes are not known to exist. For example, it is not known whether a 2^4-divisible binary code of length 165 exists. In this section, we present a number of examples of projective 10-dimensional 2^r-divisible codes invariant under M_{11} of particular interest, amongst them a binary 2^3-divisible code of length 165. Our interest is motivated by the knowledge that the smallest $\mathbb{F}_2$-submodule on which M_{11} acts irreducibly and faithfully has dimension 10 [24, p. 33].

Example 1. *(i) The codes* $C_1 = [165, 11, 45]_2$ *in line 1 and* $C_6 = [165, 11, 55]_2$ *in line 6 in Table 4 are codes with* $\mathrm{Aut}(C_1) \cong \mathrm{Sym}_{11}$ *and* $\mathrm{Aut}(C_6) \cong \mathrm{M}_{11}$*, respectively. The code* $C = C_1 \cap C_6$ *is a* Δ*-divisible code, with* $\Delta = 2^3$*. In fact,* $C = [165, 10, 72]_2$ *is a 4-weight triply-even and projective code with weight distribution* $0^1 72^{55} 80^{627} 88^{330} 120^{11}$ *and* $\mathrm{Aut}(C) \cong \mathrm{Sym}_{11} \supset \mathrm{Aut}(C_6)$*. Moreover, C is the smallest faithful* $\mathbb{F}_2$*-submodule on which* M_{11} *acts irreducibly [24, p. 33].*

(ii) The code $C_2 = [440, 10, 80]_2$ *in line 2 in table 4 is a self-orthogonal* Δ*-divisible projective code, with* $\Delta = 2^4$ *and weight distribution*

$$0^1 80^{11} 144^{55} 192^{165} 224^{330} 240^{462}.$$

The code C of the reduced point- and block-primitive 1-$(55, 10, 2)$ *design discussed in the proof of Proposition 2 is a* $[55, 10, 10]_2$ *even-weight projective code with weight distribution*

$$0^1 10^{11} 18^{55} 24^{165} 28^{330} 30^{462}.$$

The dual code $C^\perp$ *of C is a* $[55, 45, 3]_2$ *code with 165 codewords of weight 3. The codimension 1 subcode* $\overline{C}$ *of* $C^\perp$ *is a* $[55, 44, 4]_2$ *code with 990 codewords of weight* 4*. It follows from [24, p. 33] that C and* $\overline{C}$ *are irreducible submodules invariant under* M_{11}*. Thus, we have shown that* $\mathbb{F}_2^{55} = \langle \jmath \rangle \oplus C \oplus C^\perp$*, so that* $\mathbb{F}_2^{55}$ *is a semisimple module.*

Observe that since $\mathbb{F}_2^{55} = C \oplus C^\perp$*, we have that C is an LCD code.*

(iii) The code $C_3 = [990, 11, 270]_2$ *in line 3 in Table 4 is a self-orthogonal code with weight distribution*

$$0^1 270^{11} 432^{55} 462^{330} 480^{627} 510^{627} 528^{330} 558^{55} 720^{11} 990^1.$$

Its subcode of codimension 1 is a Δ*-divisible* $[990, 10, 432]_2$ *code with* $\Delta = 2^4$ *with weight distribution*

$$0^1 432^{55} 480^{627} 528^{330} 720^{11}.$$

The codes related to the reduced 1-$(165, 45, 3)$*-design are those discussed in part (i) of this example.*

(iv) The code $C_4 = [1584, 11, 144]_2$ *in line 4 in Table 4 is a self-orthogonal* Δ*-divisible code, with* $\Delta = 2^4$ *and weight distribution*

$$0^1 144^{11} 288^{55} 432^{165} 576^{330} 720^{462} 864^{462} 1008^{330} 1152^{165} 1296^{55} 1440^{11} 1584^1.$$

The codimension 1 subcode C of C_4 *is a* $[1584, 10, 288]_2$ Δ*-divisible code, with* $\Delta = 2^5$ *and weight distribution*

$$0^1 288^{55} 576^{330} 864^{462} 1152^{165} 1440^{11}.$$

(v) The code $C_5 = [990, 11, 90]_2$ in line 5 inf Table 4 is a self-orthogonal Δ-divisible code, with $\Delta = 2^1$

$$0^1 90^{11} 180^{55} 270^{165} 360^{330} 450^{462} 540^{462} 630^{330} 720^{165} 810^{55} 900^{11} 990^1.$$

The codimension 1 subcode C of C_5 is a $[990, 10, 180]_2$ Δ-divisible code, with $\Delta = 2^2$ and weight distribution

$$0^1 180^{55} 360^{330} 540^{462} 720^{165} 900^{11}.$$

(vi) The code $C_8 = [1584, 11, 264]_2$ in line 8 in Table 4 is a self-orthogonal Δ-divisible code, with $\Delta = 2^4$

$$0^1 264^{12} 480^{66} 648^{220} 768^{495} 840^{792} 864^{462}.$$

The codimension 1 subcode C of C_8 is a $[1584, 10, 480]_2$ Δ-divisible code, with $\Delta = 2^5$ and weight distribution

$$0^1 480^{66} 768^{495} 864^{462}.$$

The codes of the reduced point- and block-primitive 1-$(66, 11, 2)$-design are a $[66, 11, 11]_2$ code with weight distribution

$$0^1 11^{12} 20^{66} 27^{220} 32^{495} 35^{792} 36^{462}.$$

and its codimension 1 subcode is a self-orthogonal doubly-even $[66, 10, 20]_2$ code with weight distribution

$$0^1 20^{66} 32^{495} 36^{462}.$$

5. Conclusion

We have shown that the construction of Result 1 leads to many interesting, and possibly usable, codes acted on by the Mathieu group M_{11}. We have looked particularly at those codes of small dimension. These are good candidates for permutation decoding, due to the size of the group and the large size of the check set.

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