Designs and binary codes from maximal subgroups and conjugacy classes of the Mathieu group $M_{11}$

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Abstract. By using a method of constructing block-primitive and point-transitive 1-designs, in this paper we determine all block-primitive and point-transitive 1-$(v,k,\lambda)$-designs from the maximal subgroups and the conjugacy classes of elements of the small Mathieu group $M_{11}$. We examine the properties of 1-$(v,k,\lambda)$-designs and construct the codes defined by the binary row span of the incidence matrices of the designs. Furthermore, we present a number of interesting $\Delta$-divisible binary codes invariant under $M_{11}$.

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Key words: primitive designs, linear code, Mathieu group $M_{11}$

1. Introduction

One of the main problems in design theory is the classification of structures with given parameters and/or with a prescribed automorphism group. The construction of primitive designs from finite simple groups gives additional information on the group acting on a design, which is interesting from both the group theoretical and the combinatorial points of view. There are a number of known constructions of 1-designs invariant under finite groups. However, two methods proposed by Key and Moori in [25] and [26], respectively, stand out in that they provide the construction of point- and block-primitive 1-designs invariant under finite simple groups.

The technique presented in [26] outlines the construction of block-primitive and point-transitive 1-designs from the maximal subgroups and conjugacy classes of elements of a finite non-abelian simple group. Note that the said 1-designs are not necessarily symmetric.

Examples of the application of this method to some families of finite simple groups of Lie type are given in [26, 27, 28, 30, 31, 32]. In this paper, taking the simple group $M_{11}$ of Mathieu as an example of application, and using the method

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described in [26], we determine all block-primitive and point-transitive 1-(v,k,λ)-
designs invariant under M_{11}. To our knowledge, this is the first instance of an appli-
cation of the said method to obtaining enumeration results which provide a structure
description of automorphism groups of the designs and of the corresponding binary
codes, on which a simple group of sporadic type acts as a permutation group of
automorphisms. With this study we attempt to gain further understanding of the
structure description of the automorphism group of 1-(v,k,λ) designs for λ ≥ 2.

For each design we construct the associated binary linear code and the automor-
phism group. A number of interesting Δ-divisible binary codes are constructed from
the incidence matrices of these designs or of their reduced designs. In particular,
we construct some irreducible 10-dimensional Δ-divisible codes invariant under M_{11}.

A variety of structures preserved by the Mathieu group M_{11} have been studied.
These include combinatorial designs and strongly regular graphs [13], arc-transitive
graphs [15], residually-primitive geometries [7], and linear codes [12, 14], to name
but a few.

The paper is organized as follows: in Section 2, we give the terminology and
in Section 3, we give a brief but complete overview of the construction method
used to obtain the designs and codes examined in the paper. In Section 4, we give
some background to the Mathieu group M_{11} and some results of the permutation
characters defined by the action of M_{11} on the sets of conjugates of its distinct
maximal subgroups, and in Sections 4.1, 4.2, 4.3 and 4.4, we present our results on
the designs and codes. The latter are obtained as binary row spans of incidence
matrices of the designs.

2. Terminology and notation

Our notation is standard, and it is as in [1, 2] for designs, and in [9, 23] for group
theory and character theory. For the structure of finite simple groups and their
maximal subgroups, as well as for the structure of conjugacy classes, we follow the
ATLAS notation [9]. The notations G.H, G : H, and G · H are used to denote a
general extension, a split extension and a non-split extension, respectively. For a
prime p, the symbol p^n denotes an elementary abelian group of that order. If p is
an odd prime, p^{1+2n} and p^{1+2n} denote the extraspecial groups of order p^{1+2n}
and exponent p or p^2 respectively. Let Sym_{Ω} denote the full symmetric permutation
group of the set Ω and we write Sym_n for the symmetric group acting on {1, ..., n}
and A_n for the alternating group on {1, ..., n}.

An incidence structure D = (P, B, I) with point set P, block set B and incidence
I is a t-(v,k,λ) design, if |P| = v, every block B ∈ B is incident with precisely k
points, and every t distinct points are together incident with precisely λ blocks. We
say that D is symmetric if it has the same number of points and blocks.

The code C_F of the design D over the finite field F is the space spanned by the
incidence vectors of the blocks over F. We take F to be a prime field F_p, in which
case we also write C_p for C_F, and refer to the dimension of C_p as the p-rank of D.
In the general case of a t-design, the prime must divide the order of the design,
i.e. r − λ, where r is the replication number for the design, that is, the number of
blocks through a point. If the point set of D is denoted by P and the block set by
B, and if Q is any subset of P, then we will denote the incidence vector of Q by v^Q. Thus, \( C_F = \{ v^B \mid B \in B \} \) and is a subspace of \( F^P \), the full vector space of functions from \( P \) to \( F \). For any code \( C \), the dual or orthogonal code \( C^\perp \) is the orthogonal under the standard inner product. The hull of a design’s code over some field is the intersection \( C \cap C^\perp \). If a linear code over a field of order \( q \) is of length \( n \), dimension \( k \), and minimum weight \( d \), then we write \([n, k, d]_q\) to represent this information. A linear code \( C \) over any field is a linear code with a complementary dual (LCD) code if \( C \cap C^\perp = \{0\} \).

The all-one vector will be denoted by \( \mathbf{1} \), and it is the constant vector of weight \( n \) of the code, whose coordinate entries consist entirely of 1’s. A linear code \( C \) over \( \mathbb{F}_q \) is said to be \( \Delta \)-divisible if the Hamming weight \( w(c) \) of every codeword \( c \in C \) is divisible by \( \Delta \). Binary doubly-even and triply-even codes are special cases of \( \Delta \)-divisible codes, where \( \Delta = 2^r \) and \( r = 2, 3 \). The weight distribution of a code \( C \) is the sequence \( \{ A_i \mid i = 0, 1, \ldots, n \} \), where \( A_i \) is the number of codewords of weight \( i \). The polynomial \( W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i \) is called the weight enumerator of \( C \). The weight enumerator of a code \( C \) and its dual \( C^\perp \) are related via MacWilliams’ identity.

Two linear codes of the same length and over the same field are equivalent if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are isomorphic if they can be obtained from one another by permuting the coordinate positions. An automorphism of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of \( C \).

The graphs, \( \Gamma = (V, E) \) with vertex set \( V \) and edge set \( E \), discussed here are undirected with no loops, apart from the case where all loops are included, in which case the graph is called reflexive. A graph is regular if all the vertices have the same valency. An adjacency matrix \( A \) of a graph of order \( n \) is an \( n \times n \) matrix with entries \( a_{ij} \) such that \( a_{ij} = 1 \) if vertices \( v_i \) and \( v_j \) are adjacent, and \( a_{ij} = 0 \) otherwise. An incidence matrix of \( \Gamma \) is an \( n \times |E| \) matrix \( B \) with \( b_{ij} = 1 \) if the vertex labelled by \( i \) is on the edge labelled by \( j \), and \( b_{ij} = 0 \) otherwise. If \( \Gamma \) is regular with valency \( k \), then the 1-\((|E|, k, 2)\) design with incidence matrix \( B \) is called the incidence design of \( \Gamma \). The neighbourhood design of a regular graph is the 1-design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix as an incidence matrix for the design.

3. Construction of the designs

In this section, we outline a method of constructing block-primitive and point-transitive 1-designs (see Result 1) given by Key and Moori in \[26\].

We start by recalling the notion of permutation character, for it is useful in determining one of the parameters of these designs.

Let \( G \) be a finite group and \( \Omega \) a finite \( G \)-set. The map \( \pi_\Omega : G \rightarrow \mathbb{N} \) given by \( \pi_\Omega(g) = |\text{Fix}_\Omega(g)| = |\{ \omega \in \Omega : g \cdot \omega = \omega \}| \) is called the permutation character of the action of \( G \) on \( \Omega \). A transitive permutation character is the character of a transitive action. If the subgroup \( M \) of \( G \) is the stabilizer of a point in a transitive \( G \)-set, we
Lemma 1. Let $M$ be a subgroup of $G$, $x \in G$ and $x^G$, the conjugacy class of $x$, and let $1^G_M$ be the permutation character of the action of $G$ on the right cosets of $M$. Then

$$\frac{1^G_M(x) \times |x^G|}{1^G_M(1)} = |x^G \cap M|.$$ 

Moreover, if $M$ is self-normalizing, then the number of conjugates of $M$ containing $x$ is $1^G_M(x)$.

Proof. Recall that the permutation character of $G$ on the right cosets of $M$ is the induced character obtained from the principal character $1^G_M$ of $M$. Applying the formula for induced characters yields:

$$1^G_M(x) = \frac{1}{|M|} \sum_{g \in G} 1^G_M(g^{-1}xg)$$

$$= \frac{|x^G \cap M|}{|M|} \cdot |C_G(x)|$$

$$= \frac{|x^G \cap M||G|}{|x^G|}$$

$$= \frac{|x^G \cap M||G:M|}{|x^G|}$$

$$= \frac{|x^G \cap M| 1^G_M(1)}{|x^G|}.$$ 

From this we obtain the result. If $M$ is self-normalizing, then the actions of $G$ on the right cosets of $M$ by right multiplication and on the set of conjugates of $M$ by conjugation are equivalent and thus have the same character. 

Remark 1. (i) Transitive permutation characters are a tool to deduce information about the subgroups of a group $G$ from the character table of $G$. From the permutation character $1^G_M$ one can obtain information about the conjugacy classes of $M$; for example, one can compute the number of elements of a certain order in $M$.

(ii) Lemma 1 has numerous applications amongst which are determining the covering number of some finite simple groups [17]; finding possible permutation characters of a finite group [4]; determining multiplicity-free permutation characters [5]; and the construction of combinatorial designs [26]. Restricted to finite simple groups, the results of Lemma 1 are used in [26], where Key and Moori proposed a method of constructing point-transitive and block-primitive $1-(v,k,\lambda)$ designs from the maximal subgroups and conjugacy classes of elements of the group. This construction is outlined in Result 1 below and will constitute the main tool used for our investigations.

Result 1. Let $S$ be a finite simple group, $T$ a maximal subgroup of $S$ and $x^S$ a conjugacy class of elements of order $n$ in $S$ such that $T \cap x^S \neq \emptyset$. Also, let $\chi_T$ be the permutation character afforded by the action of $S$ on the set of conjugates of $T$ in $S$. 

Let \( B = \{(T \cap x^S)^y \mid y \in S\} \). Then we have a \( 1-\{(x^S), |T \cap x^S|, \chi_T(x)\} \) design \( D \). The group \( S \) acts as an automorphism group on \( D \), primitive on blocks and transitive (but not necessarily primitive) on points of \( D \).

**Proof.** See [26, Theorem 4].

**Remark 2.** (i) Observe that in Result 1 and elsewhere \( 1_G^G \) is denoted by \( \chi_T \).

(ii) The 1-design \( D \) of Result 1 is an \( S_r(1,k,v) \) 1-design called a tactical configuration (see [2, Definition 3.5]).

The following lemma shows that if we obtain two of the three parameters of the design, the third may be directly computed.

**Lemma 2.** Let \( D = (v,k,\lambda) \) be a design obtained by Result 1. Then

\[
1_G^G(1) = \frac{\lambda v}{k} \quad \text{and} \quad r = \lambda.
\]

**Proof.** This follows by Lemma 1 by noticing that \( 1_G^G(1) = b = [S:T] \). The remainder follows since \( D \) is a tactical configuration, in which case we obtain by [2, Proposition 1.1, Chapter II] that \( b = \frac{\lambda v}{k} \). Now, \( r = \lambda \) is a direct application of [2, Corollary 1.4, Chapter II].

In the sequel we consider \( S \) to be the simple group \( M_{11} \) of Mathieu and apply Result 1 to construct all point- and block-primitive 1-(\( v,k,\lambda \)) designs from its maximal subgroups, and conjugacy classes of elements. We give a brief description of \( M_{11} \), and encourage the reader to consult some standard group theory textbooks such as, for example, [8] for further details.

### 4. Some results of the Mathieu group \( M_{11} \)

The small Mathieu groups \( M_{11}, M_{12} \) were discovered by the French mathematician Émile Mathieu (1835–1890), who also discovered the large Mathieu groups \( M_{22}, M_{23} \) and \( M_{24} \). They are remarkable groups; for example, apart from the symmetric and alternating groups, \( M_{12} \) and \( M_{24} \) are the only 5-transitive permutation groups.

Here, we consider the simple group \( M_{11} \) of Mathieu, see \( ATLAS \) (see also [33, Section 5.3.8]). The Mathieu group \( M_{11} \) is defined as a stabilizer of a point in \( M_{12} \) and this definition shows that it has order \( 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11 \). \( M_{11} \) has five classes of maximal subgroups up to conjugation, namely of degrees 11, 12, 55, 66 and 165. In Table 1 below, the first column depicts the ordering of the primitive representations as given by the \( ATLAS \) [9] and as used in our computations; the second gives the maximal subgroups; and the third gives the degrees (the number of cosets of the point stabilizer).

In the next five lemmas, we determine the values of \( \lambda = \chi_T(x) \) for \( x \in G \) and using Lemma 2, we obtain the remaining parameters of the designs as given in Table 3.
Table 1: Maximal subgroups of $M_{11}$

<table>
<thead>
<tr>
<th>Line</th>
<th>Max. sub. structure</th>
<th>degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>$A_6 \cdot 2_3$</td>
<td>11</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$L_2(11)$</td>
<td>12</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$3^2:Q_8 \cdot 2$</td>
<td>55</td>
</tr>
<tr>
<td>$T_4$</td>
<td>$\text{Sym}_5$</td>
<td>66</td>
</tr>
<tr>
<td>$T_5$</td>
<td>$2\cdot\text{Sym}_4$</td>
<td>165</td>
</tr>
</tbody>
</table>

Lemma 3. Let $T_1$ be a maximal subgroup of $M_{11}$ isomorphic to $A_6 \cdot 2_3$ and suppose that $\chi_{T_1}$ is the permutation character afforded by the action of $M_{11}$ on the set of conjugates of $T_1$. Then the following holds:

$$\chi_{T_1}(x) = \begin{cases} 
3, & \text{if } o(x) = 2 \text{ or } o(x) = 4; \\
2, & \text{if } o(x) = 3; \\
1, & \text{if } o(x) = 5 \text{ or } o(x) = 8; \\
0, & \text{if } o(x) = 6 \text{ or } o(x) = 11.
\end{cases}$$

Proof. The action of $M_{11}$ on the set of conjugates of $T_1$ is doubly transitive [6, Lemma 2.1.1(i)] of degree 11. So, the decomposition of the permutation character of $M_{11}$ with respect to this action is $\pi_{\Omega} = 1 + \varphi$, where $\varphi$ is an irreducible character of $M_{11}$ of degree 10. Hence, $\varphi(1) = 11 - 1 = 10$. An inspection of the character table of $M_{11}$ [9, p. 18] shows that $M_{11}$ has three irreducible characters of degree 10, namely $\chi_2, \chi_3$ and $\chi_4$. In particular, notice in this case that $\varphi = \chi_2$, since the character values are real, (see also [6, Lemma 2.1.1(i)]). Thus, the values of $\varphi$ on conjugacy classes of $M_{11}$ are $-1, 0, 1$ and 2. In fact, if $x^{M_{11}}$ is a conjugacy class of $M_{11}$ we have

$$\varphi(x^{M_{11}}) = \begin{cases} 
2, & \text{if } o(x) = 2 \text{ or } o(x) = 4; \\
1, & \text{if } o(x) = 3; \\
0, & \text{if } o(x) = 5 \text{ or } o(x) = 8; \\
-1, & \text{if } o(x) = 6 \text{ or } o(x) = 11.
\end{cases}$$

Notice that the value $\varphi(x^{M_{11}}) = -1$ occurs if $x^{M_{11}} \cap T_1 = \emptyset$. This implies that there are no elements of orders 6 and 11, respectively, in $A_6 \cdot 2_3$. Direct application of Lemma 1 shows this.

Lemma 4. Let $T_2$ be a maximal subgroup of $M_{11}$ isomorphic to $L_2(11)$ and suppose that $\chi_{T_2}$ is the permutation character afforded by the action of $M_{11}$ on the set of conjugates of $T_2$. Then the following holds:

$$\chi_{T_2}(x) = \begin{cases} 
4, & \text{if } o(x) = 2; \\
3, & \text{if } o(x) = 3; \\
2, & \text{if } o(x) = 5; \\
1, & \text{if } o(x) = 6 \text{ or } o(x) = 11; \\
0, & \text{if } o(x) = 4 \text{ or } o(x) = 8.
\end{cases}$$

Proof. The proof follows by using arguments similar to those used in the proof of Lemma 3 bearing in mind that the action of $M_{11}$ on the set of conjugates of $T_2$.
is doubly transitive of degree 12 [6, Lemma 2.1.1(ii)], and that there is a unique irreducible character of degree 11, namely $\varphi = \chi_5$.

**Lemma 5.** Let $T_3$ be a maximal subgroup of $M_{11}$ isomorphic to $3^2:Q_8:2$ and suppose that $\chi_{T_3}$ is the permutation character afforded by the action of $M_{11}$ on the set of conjugates of $T_3$. Then the following holds:

$$\chi_{T_3}(x) = \begin{cases} 7, & \text{if } o(x) = 2; \\ 3, & \text{if } o(x) = 4; \\ 1, & \text{if } o(x) = 3, o(x) = 6 \text{ or } o(x) = 8; \\ 0, & \text{if } o(x) = 5 \text{ or } o(x) = 11. \end{cases}$$

**Proof.** Notice first that the action of $M_{11}$ on the set of conjugates of $T_3$ is rank 3 of degree 55 with subdegrees 1, 18, and 36. In particular, it follows from [6, Lemma 2.1.1(iii)] that the permutation character is $\varphi = 1 + 10a + 44a$. We obtain that the values of $\varphi$ on conjugacy classes of $M_{11}$ are $-1, 0, 1, 2$ and 4, and the result follows.

**Lemma 6.** Let $T_4$ be a maximal subgroup of $M_{11}$ isomorphic to $\text{Sym}_5$ and suppose that $\chi_{T_4}$ is the permutation character afforded by the action of $M_{11}$ on the set of conjugates of $T_4$. Then the following holds:

$$\chi_{T_4}(x) = \begin{cases} 10, & \text{if } o(x) = 2; \\ 3, & \text{if } o(x) = 3; \\ 2, & \text{if } o(x) = 4; \\ 1, & \text{if } o(x) = 5 \text{ or } o(x) = 6; \\ 0, & \text{if } o(x) = 8 \text{ or } o(x) = 11. \end{cases}$$

**Proof.** Omitted.

**Lemma 7.** Let $T_5$ be a maximal subgroup of $M_{11}$ isomorphic to $2 \cdot \text{Sym}_4$ and suppose that $\chi_{T_5}$ is the permutation character afforded by the action of $M_{11}$ on the set of conjugates of $T_5$. Then the following holds:

$$\chi_{T_5}(x) = \begin{cases} 13, & \text{if } o(x) = 2; \\ 3, & \text{if } o(x) = 3; \\ 1, & \text{if } o(x) = 4, o(x) = 6 \text{ or } o(x) = 8; \\ 0, & \text{if } o(x) = 5 \text{ or } o(x) = 11. \end{cases}$$

**Proof.** Omitted.

Let $H$ be a subgroup of $G$. We say that $H$ controls $G$-fusion in itself if each pair of elements in $H$ which are conjugates in $G$ are also conjugates in $H$, or equivalently, if for $x \in H$ we have $x^G \cap H = x^H$.

We adopt the following definition from [30]:

**Definition 1.** Let $H \leq G$ and $k$ be a positive integer. We define

$$cn_H^G(k) := |\{x^G | x \in H, o(x) = k\}|.$$

Also, we write $cn_H^H(k) := cn_H^G(k)$. It is easy to see that $cn_H^G(k) \leq cn_H^H(k)$ and, if the equality holds, then, for every $x \in H$ with $o(x) = k$, we have $x^G \cap H = x^H$. 
We need the following result on the maximal subgroups of $M_{11}$ that satisfies the control fusion property.

**Lemma 8.** Let $T_i$ be a maximal subgroup of $M_{11}$. Then $T_i$ controls $M_{11}$-fusion in itself in the following cases:

(a) $T_i$ is any maximal subgroup of $M_{11}$ and $o(x) = 3$;
(b) $T_i \cong A_6 \cdot 2_3$ for all $x \in T_i$;
(c) $T_i \cong L_2(11)$ and $o(x) = 2$ or $o(x) = 6$;
(d) $T_i \cong 3^2:Q_8 \cdot 2$ and $o(x) = 6, 8$ or $o(x) = 11$;
(e) $T_i \cong \text{Sym}_5$ and $o(x) = 4, 5$ or $o(x) = 6$;
(f) $T_i \cong 2 \cdot \text{Sym}_4$ and $o(x) = 4$ or $o(x) = 8$.

**Proof.** We will illustrate case (a). The remaining cases follow by inspecting the ATLAS [9] or by direct computations with Magma [3] or GAP [19].

(a) Let $x \in T_i$ be a non-trivial element and suppose that $o(x) = 3$. Then we obtain $\text{cn}_{T_i}(3) = 1$, and so we have $x^{M_{11}} \cap T_i = x^{T_i}$. In particular, if $T_i \cong A_6 \cdot 2_3$ or $T_i \cong L_2(11)$, this follows by a direct inspection of [9, p. 4] and [9, p. 7], where the reader will notice that each such maximal subgroup of $M_{11}$ has exactly one conjugacy class of elements of order 3.

We end this section with the following useful result which deals with designs constructed using Result 1 when the subgroup $T_i$ controls $M_{11}$-fusion in itself.

**Lemma 9.** Let $S$ be a simple group with a maximal subgroup $T$ and assume that $T$ controls $S$-fusion in itself. Then the designs constructed by Result 1 are 1-$(|xS|, |xT|, [C_S(x) : C_T(x)])$ designs, where $x$ is an element of $T$.

**Proof.** See [32, Proposition 3.4].

### 4.1. The designs

Using results of Section 4 and Section 3 we are able to state and prove our main result for designs in Proposition 1. In Table 2, we reproduce from the ATLAS the information on the element orders, conjugacy class sizes and centralizer orders for the elements of $M_{11}$.

<table>
<thead>
<tr>
<th>Cycle type</th>
<th>$1^3$</th>
<th>$1^22^1$</th>
<th>$1^33^0$</th>
<th>$1^33^1$</th>
<th>$1^33^2$</th>
<th>$2^33^0$</th>
<th>$1^22^03^1$</th>
<th>$1^22^03^2$</th>
<th>$1^12^17^1$</th>
<th>$1^12^07^2$</th>
<th>$1^12^03^1$</th>
<th>$1^02^03^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centralizer order</td>
<td>7920</td>
<td>185</td>
<td>440</td>
<td>990</td>
<td>1584</td>
<td>1584</td>
<td>990</td>
<td>990</td>
<td>720</td>
<td>720</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2:** Conjugacy classes of elements of $M_{11}$
By \( \mathcal{D}(x,T_i) \) we denote the design obtained from each maximal subgroup \( T_i \) of \( M_{11} \) and a non-trivial element \( x \) of order \( n \) in \( x^{M_{11}} \).

**Proposition 1.** Let \( T_i \) be a maximal subgroup of \( M_{11} \) as given in lines 1 to 5 in Table 1 and let \( x \in M_{11} \) be a non-trivial element given in Table 2. Then the parameters of all non-trivial 1-designs \( \mathcal{D}(x,T_i) = (v,k,\lambda) \) are as given in Table 3.

**Proof.** By Result 1, \( \mathcal{D}(x,T_i) = ([x^{M_{11}}],|T_i \cap x^{M_{11}}|,\chi_{T_i}(x)) \). The first parameter \( |x^{M_{11}}| \) is given in Table 2. By Lemma 2 we can deduce the value of \( |T_i \cap x^{M_{11}}| \) and by Lemmas 3 - 7 we obtain the value of \( \chi_{T_i}(x) \). Alternatively, once two of the parameters are known, the third can be calculated directly using Lemma 2. The result now follows.

In Table 3 below, we give the parameters of the designs constructed from all maximal subgroups and conjugacy classes of elements of \( M_{11} \).

**Remark 3.** The designs \( \mathcal{D} \) constructed by using Result 1 are not symmetric in general. In fact, \( \mathcal{D} \) is symmetric if and only if

\[
\begin{align*}
|T_i| &= |\mathcal{D}(x,T_i)| = \frac{|x^{M_{11}}|}{|T_i \cap x^{M_{11}}|} = |\chi_{T_i}(x)|, \\
|T_i| &= \text{Sym}^{|T_i|}.
\end{align*}
\]

Observe that \( \mathcal{D}(x,T_5) = (165,13,13) \) (in Line 21) is the only symmetric design in Table 3.

| Line | Max | \( t = o(x) \) | \( v = |x^{M_{11}}| \) | \( k = |T_i \cap x^{M_{11}}| \) | \( \lambda = \chi_{T_i}(x) \) | \( b = |\mathcal{D}(x,T_i)| \) | \( \text{Aut}(\mathcal{D}(x,T_i)) \) |
|------|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1    | \( T_1 \) | 2    | 205  | 45  | 3  | 11  | \( \text{Sym}^{11} \) |
| 2    | \( T_1 \) | 3    | 440  | 80  | 2  | 11  | \( \text{Sym}^{20} \) |
| 3    | \( T_1 \) | 4    | 990  | 270 | 3  | 11  | \( \text{Sym}^{11} \) |
| 4    | \( T_1 \) | 5    | 1584 | 14  | 1  | 11  | \( \text{Sym}^{11} \) |
| 5    | \( T_1 \) | 6    | 359  | 90  | 1  | 11  | \( \text{Sym}^{11} \) |
| 6    | \( T_2 \) | 7    | 165  | 55  | 4  | 12  | \( \text{Sym}^{11} \) |
| 7    | \( T_2 \) | 8    | 440  | 30  | 2  | 12  | \( \text{Sym}^{11} \) |
| 8    | \( T_2 \) | 9    | 1320 | 110 | 1  | 12  | \( \text{Sym}^{11} \) |
| 9    | \( T_2 \) | 10   | 720  | 60  | 1  | 12  | \( \text{Sym}^{11} \) |
| 10   | \( T_2 \) | 11   | 440  | 8   | 1  | 55  | \( \text{Sym}^{11} \) |
| 11   | \( T_2 \) | 12   | 1320 | 24  | 1  | 55  | \( \text{Sym}^{11} \) |
| 12   | \( T_2 \) | 13   | 990  | 54  | 3  | 55  | \( \text{Sym}^{11} \) |
| 13   | \( T_3 \) | 14   | 1320 | 24  | 1  | 55  | \( \text{Sym}^{11} \) |
| 14   | \( T_3 \) | 15   | 990  | 30  | 2  | 55  | \( \text{Sym}^{11} \) |
| 15   | \( T_3 \) | 16   | 359  | 90  | 1  | 12  | \( \text{Sym}^{11} \) |
| 16   | \( T_3 \) | 17   | 165  | 55  | 4  | 12  | \( \text{Sym}^{11} \) |
| 17   | \( T_3 \) | 18   | 440  | 30  | 2  | 12  | \( \text{Sym}^{11} \) |
| 18   | \( T_3 \) | 19   | 1320 | 110 | 1  | 12  | \( \text{Sym}^{11} \) |
| 19   | \( T_3 \) | 20   | 720  | 60  | 1  | 12  | \( \text{Sym}^{11} \) |
| 20   | \( T_4 \) | 21   | 440  | 8   | 1  | 55  | \( \text{Sym}^{11} \) |
| 21   | \( T_4 \) | 22   | 1320 | 24  | 1  | 55  | \( \text{Sym}^{11} \) |
| 22   | \( T_4 \) | 23   | 990  | 54  | 3  | 55  | \( \text{Sym}^{11} \) |
| 23   | \( T_4 \) | 24   | 359  | 90  | 1  | 55  | \( \text{Sym}^{11} \) |
| 24   | \( T_4 \) | 25   | 165  | 55  | 4  | 12  | \( \text{Sym}^{11} \) |
| 25   | \( T_4 \) | 26   | 440  | 30  | 2  | 12  | \( \text{Sym}^{11} \) |
| 26   | \( T_4 \) | 27   | 1320 | 110 | 1  | 12  | \( \text{Sym}^{11} \) |
| 27   | \( T_4 \) | 28   | 720  | 60  | 1  | 12  | \( \text{Sym}^{11} \) |
| 28   | \( T_5 \) | 29   | 440  | 8   | 1  | 165 | \( \text{Sym}^{11} \) |
| 29   | \( T_5 \) | 30   | 1320 | 24  | 1  | 165 | \( \text{Sym}^{11} \) |
| 30   | \( T_5 \) | 31   | 990  | 54  | 3  | 165 | \( \text{Sym}^{11} \) |
| 31   | \( T_5 \) | 32   | 359  | 90  | 1  | 165 | \( \text{Sym}^{11} \) |
| 32   | \( T_5 \) | 33   | 165  | 55  | 4  | 165 | \( \text{Sym}^{11} \) |
| 33   | \( T_5 \) | 34   | 440  | 30  | 2  | 165 | \( \text{Sym}^{11} \) |
| 34   | \( T_5 \) | 35   | 1320 | 110 | 1  | 165 | \( \text{Sym}^{11} \) |
| 35   | \( T_5 \) | 36   | 720  | 60  | 1  | 165 | \( \text{Sym}^{11} \) |

Table 3: Non-trivial 1-(v,k,\( \lambda \))-designs from \( M_{11} \) constructed using Result 1
4.2. Automorphism groups of the designs

As stated in Section 1, a study of the structure of the automorphism group Aut(D) of a 1-(v, k, λ)-design D constructed using Result 1 was carried out in [29]. In that paper, the notion of a reduced design was introduced with a view to providing a description of the structure of Aut(D) when λ ≥ 2.

For the sake of completeness, in what follows we restate the definition of reduced designs and some subsequent results.

Definition 2 ([29], Definition 2.10). For a point x ∈ P of the 1-(v, k, λ) design D(x, T_i), let B_1, ..., B_λ be all distinct blocks containing x. Define

\[ I_x = \bigcap_{s=1}^{\lambda} B_s. \]

Lemma 10. Let D = (P, B) be a 1-(v, k, λ) design. Then we can construct a 1-(v/|I_x|, k/|I_x|, λ) design D_I called the reduced design of D.

In this section, we examine the structure of Aut(D) of the designs D given in Table 3, placing emphasis on those cases where λ ≥ 2. To this end we state the following known observations:

Remark 4. If λ = 1, then D is a 1-(|x^S|, k, 1) design with b pairwise disjoint blocks. It follows from [26, Remark 3], (see also [27, Remark 4.3]) that Aut(D) ∼= Sym_k^b ⋊ Sym_b = (Sym_k)^b ⋊ Sym_b.

If r = λ = 2, then D is a 1-(|x^S|, k, 2) design with b blocks. The study of t-(v, k, 2)-designs for t = 2 is of current interest and many papers have devoted attention thereto. We refer the reader to [16] (see also some of the references therein), which examines the classification of flag-transitive 2-(v, k, 2) designs.

In the case of a 1-(|x^S|, k, 2) design, a regular graph can be defined, where the vertices are the blocks of the design, and two vertices labelled by the blocks b_i and b_j are adjacent if b_i and b_j meet. Moreover, the incidence matrix for the design is an incidence matrix for the graph. In particular, when the graph is an undirected graph without multiple edges, the following result can be used.

Result 2. ([18, Lemma 1]). Let Γ = (V, E) be a regular graph with |V| = N, |E| = e and valency v. Let \( G \) be the 1-(e, v, 2) incidence design from an incidence matrix A for Γ. Then Aut(Γ) = Aut(G).

Remark 5. If the graph Γ is also connected, then using an inductive argument it can be shown that rank_p(A) ≥ |V| - 1 for all p. Equality holds when p = 2. If, in addition, the minimum weight is the valency of the graph, and the words of this weight are the scalar multiples of the rows of the incidence matrix, then it follows that Aut(C_p(G)) = Aut(G).

As an application of the preceding results to the study of PSL(2, q)-invariant 1-((|x^S|, k, 2)-designs in [27, Section 5.2] (see also [26, Section 6]), the authors showed that Aut(D) is isomorphic to 2^q(q + 1) ⋊ Sym_{q+1} or to Sym_{q+1}, depending on whether
or not the non-trivial element $x \in x^S$ is an involution. However, this does not seem to provide a general structure description for the automorphism groups of the $1-(|x^S|, k, 2)$-designs constructed using Result 1.

The reader will notice in Proposition 2 below that the automorphism groups of the $1-(|x^S|, k, 2)$-designs constructed from the maximal subgroups and conjugacy classes of elements of $M_{11}$ possess a different structure description.

**Proposition 2.** Let $D$ be a block-primitive and point-transitive $1-(v, k, \lambda)$ design constructed from a maximal subgroup $T_i$ and a conjugacy class $x^{M_{11}}$ of elements of order $n$ in $M_{11}$. Then $D$ is a $1-(v, k, 2)$ design if and only if $D$ is constructed from $T_i \cong A_6 \cdot 2_4$, $S_5$ and $L_2(11)$, and $x^S = 3A, 4A$ and $5A$, respectively. Further, $\text{Aut}(D)$ is isomorphic to $(\text{Sym}_8)^{55}:\text{Sym}_{11}$, $2^{495}:M_{11}$, and $(\text{Sym}_{24})^{66}:\text{Sym}_{12}$, respectively.

**Proof.** Suppose that $T_i \cong A_6 \cdot 2_4$. By Proposition 1, it follows that $D$ is a $1$-(440, 80, 2) design with $b = 11$ blocks. By results of [29, Sections 2, 4], we observe that for every point $x \in 3A$, the intersection of the two blocks of $D$ containing $x$ has size 8, i.e., $|I_x| = 8$. Thus, the reduced design $D_1$ is a $1$-(55, 10, 2) design with 11 blocks and the automorphism group isomorphic to $\text{Sym}_{11}$ and $S(I) = (\text{Sym}_8)^{55} \leq \text{Aut}(D)$ and so $\text{Aut}(D) \cong (\text{Sym}_8)^{55}:\text{Sym}_{11}$. From this, it follows that $\text{Aut}(D_1) \cong \text{Aut}(D)/S(I) \cong \text{Sym}_{11}$. Arguing similarly, it can be shown that the automorphism groups of the $1$-(990, 30, 2)-design constructed from a maximal subgroup isomorphic to $\text{Sym}_3$ and an element of order 4, and a $1$-(1584, 264, 2)-design obtained from a maximal subgroup isomorphic to $L_2(11)$ and an element of order 3, are isomorphic to $2^{495}:M_{11}$, and $(\text{Sym}_{24})^{66}:\text{Sym}_{12}$, respectively. \qed

Next, we examine the structure of the automorphism group of the $1-(v, k, \lambda)$ designs $D$ for which $r = \lambda \geq 3$. Table 3 shows that these designs are constructed from all maximal subgroups $T_i$ of $M_{11}$ and every conjugacy class $x^{M_{11}}$ of non-trivial elements.

**Proposition 3.** Let $T_i$ be a maximal subgroup of $M_{11}$. Let $x \in T_i$ be a non-trivial element with $o(x) \neq 2$. Then $D(x, T_i)$ is a block-primitive and point-transitive $1-(v, k, 3)$ design and

(a) $v = 990$, $k = 270$ for $T_i \cong A_6 \cdot 2$;
(b) $v = 440$, $k = 110$ for $T_i \cong L_2(11)$;
(c) $v = 990$, $k = 54$ for $T_i \cong 3Q_8 \cdot 2$;
(d) $v = 440$, $k = 20$ for $T_i \cong \text{Sym}_5$;
(e) $v = 440$, $k = 8$ for $T_i \cong 2\text{Sym}_4$.

Moreover, $|I_x| \geq 2$, and $\text{Aut}(D)$ is as in lines 3, 7, 13, 17 and 22 in Table 3.

In what follows, we consider those cases of a $1-(v, k, \lambda)$ design for $\lambda \geq 3$ and $|I_x| = 1$.

**Lemma 11.** Let $T_i$ be a maximal subgroup of $M_{11}$. Let $x \in T_i$ be an element of order 2. Then $D(x, T_i)$ is a point- and block-primitive $1-(v, k, \lambda)$ design with parameters
(a) $v = 165$, $k = 45$ and $\lambda = 3$ for $T_i \cong A_6 \cdot 2$;

(b) $v = 165$, $k = 55$ and $\lambda = 4$ for $T_i \cong L_2(11)$;

(c) $v = 165$, $k = 21$ and $\lambda = 7$ for $T_i \cong 3Q_8 \cdot 2$;

(d) $v = 165$, $k = 25$ and $\lambda = 10$ for $T_i \cong S_4$;

(e) $v = 165$, $k = 13$ and $\lambda = 13$ for $T_i \cong 2S_4$.

Moreover, $|I_x| = 1$, and $\text{Aut}(D)$ is as in lines 1, 6, 11, 16 and 21 in Table 3.

**Proof.** (a) Suppose that $T_i \cong A_6 \cdot 3$. Since by Lemma 8(b), $T_i$ controls $M_{11}$-fusion in itself, we obtain $x^{M_{11}} \cap T_i = x^{T_i}$. Furthermore, $|x^{M_{11}} \cap T_i| = 45 = |x^{T_i}|$. Now let $\chi_{T_i}$ be the permutation character of $M_{11}$ with respect to the action of $M_{11}$ on the conjugates of $T_i$. Then, by [27, Corollary 2.4], we have $\chi_{T_i}(x) = \sum_{i=1}^{k} \frac{|C_{M_{11}}(x)|}{|C_{T_i}(x_i)|}$, where $x_1, x_2, \ldots, x_k$ are representatives of the conjugacy classes of $T_i$ that fuse to the class $x^{M_{11}}$ in $M_{11}$. Since $T_i$ controls $M_{11}$-fusion in itself, we get $k = 1$ and $\chi_{T_i}(x) = |C_{M_{11}}(x) : C_{T_i}(x)|$.

Observe from Table 2 that $|C_{M_{11}}(x)| = 48$, and from Table 1 that $C_{M_{11}}(x) \cong 2S_4$. Moreover, from [9, p. 4] we deduce that $C_{T_i}(x)$ is a maximal subgroup of $T_i$, isomorphic to $8:2$ and $|C_{T_i}(x)| = 16$. Thus, $\chi_{T_i}(x) = |C_{M_{11}}(x) : C_{T_i}(x)| = 3$. A direct application of Lemma 9 yields a 1-(165, 45, 3) design $D$ with 11 blocks. As for the automorphism group of $D$, notice that $C_{M_{11}}(x)$ is maximal in $M_{11}$ and so $|I_x| = 1$, i.e., for every point $x \in x^{M_{11}}$, the intersection of the three blocks containing $x$ is of size one. Thus, $D$ is itself a reduced design. Now, the structure of $\text{Aut}(D)$ can be determined directly as an application of [29, Remark 4.3]. From the abovementioned, we deduce that $\text{Aut}(D) \cong \text{Sym}_{11}$.

(b) Let $T_i \cong L_2(11)$. According to Lemma 8(c), $T_i$ controls $M_{11}$-fusion. So we have $x^{M_{11}} \cap T_i = x^{T_i}$. The remainder follows arguing similarly to (a). The structure of $\text{Aut}(D)$ can be determined by recalling that $\text{Aut}(M_{11}) = M_{11}$. As an application of [29, Remark 4.3], it can be shown that $\text{Aut}(D) \cong M_{11}$.

(c) Consider $T_i \cong 3Q_8 \cdot 2$. The group $T_i$ has two conjugacy classes of elements of order 2, say $2A$ and $2B$, with $|2A| = 9$ and $|2B| = 12$, and thus $|x^{M_{11}} \cap T_i| = 21 = |x^{T_i}|$. Observe that $|C_{T_i}(2A)| = 16$ and $|C_{T_i}(2B)| = 12$, and $C_{T_i}(2A) \cong 8:2$, while $C_{T_i}(2B) \cong A_4$, see [10, 11]. Now, $\chi_{T_i}(x) = \sum_{i=1}^{2} \frac{|C_{M_{11}}(x)|}{|C_{T_i}(x_i)|} = \frac{|C_{M_{11}}(x)|}{|C_{T_i}(2A)|} + \frac{|C_{M_{11}}(x)|}{|C_{T_i}(2B)|} = \frac{48}{16} + \frac{48}{24} = 7$. Thus, $D$ is a 1-(165, 21, 7) design with 55 blocks. Again notice that $C_{M_{11}}(x)$ is maximal in $M_{11}$, and moreover $|I_x| = 1$, i.e., for every point $x \in x^{M_{11}}$, the intersection of the seven blocks containing $x$ has size one. Direct calculations using [29, Theorem 4.2] show that $\text{Aut}(D) \cong M_{11}$.

Parts (d) and (e) follow arguing similarly to (c).

4.3. Binary codes of the designs

In Table 4, we list all codes $C_2(D(x, T_i))$ defined by the binary row span of the incidence matrices of the designs $D(x, T_i)$ given in Table 3. The second column lists the parameters of the design, the third column gives the parameters of the
corresponding binary codes, followed by the automorphism groups and the number of codewords of minimum weight.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Line} & |D(x, T_i)| & |C_2(D(x, T_i))| & \text{#minwords} \\
\hline
1 & 1-(165, 45, 3) & [165, 11, 45] & \text{Sym}_{11} & 11 \\
2 & 1-(1440, 80, 2) & [1440, 10, 80] & \text{Sym}_{10}\wr\text{Sym}_{11} & 11 \\
3 & 1-(990, 270, 3) & [990, 11, 270] & \text{Sym}_{10}\wr\text{Sym}_{11} & 11 \\
4 & 1-(1584, 144, 1) & [1584, 11, 144] & \text{Sym}_{11}\wr\text{Sym}_{11} & 11 \\
5 & 1-(990, 90, 1) & [990, 11, 90] & \text{Sym}_{10}\wr\text{Sym}_{11} & 11 \\
6 & 1-(1320, 330, 2) & [1320, 11, 330] & \text{M}_{11} & 12 \\
7 & 1-(440, 110, 3) & [440, 12, 110] & 2^{10}\text{Sym}_{12} & 12 \\
8 & 1-(1584, 264, 2) & [1584, 11, 264] & 2^{12}\text{Sym}_{12} & 12 \\
9 & 1-(1320, 110, 3) & [1320, 11, 110] & \text{Sym}_{11}\wr\text{Sym}_{11} & 12 \\
10 & 1-(720, 60, 1) & [720, 12, 60] & \text{Sym}_{11}\wr\text{Sym}_{12} & 12 \\
11 & 1-(165, 21, 3) & [165, 55, 20] & \text{M}_{11} & 55 \\
12 & 1-(440, 8, 1) & [440, 55, 8] & \text{Sym}_{10}\wr\text{Sym}_{55} & 55 \\
13 & 1-(990, 54, 3) & [990, 45, 54] & \text{Sym}_{10}\wr\text{Sym}_{11} & 55 \\
14 & 1-(1320, 24, 1) & [1320, 55, 24] & \text{Sym}_{11}\wr\text{Sym}_{55} & 55 \\
15 & 1-(990, 18, 1) & [990, 55, 18] & \text{Sym}_{11}\wr\text{Sym}_{55} & 55 \\
16 & 1-(165, 25, 10) & [165, 45, 25] & \text{M}_{11} & 66 \\
17 & 1-(440, 20, 3) & [440, 55, 20] & 2^{10}\text{Sym}_{12} & 66 \\
18 & 1-(990, 30, 2) & [990, 60, 30] & 2^{10}\text{M}_{11} & 66 \\
19 & 1-(1584, 24, 1) & [1584, 66, 24] & 2^{12}\text{Sym}_{12} & 66 \\
20 & 1-(1320, 20, 1) & [1320, 60, 20] & 2^{12}\text{M}_{11} & 66 \\
21 & 1-(1320, 13, 1) & [1320, 66, 13] & \text{M}_{11} & 102 \\
22 & 1-(440, 8, 3) & [440, 165, 8] & 2^{10}\text{M}_{11} & 455 \\
23 & 1-(990, 6, 1) & [990, 165, 6] & \text{Sym}_{11}\wr\text{Sym}_{115} & 165 \\
24 & 1-(1320, 8, 1) & [1320, 165, 8] & \text{Sym}_{11}\wr\text{Sym}_{115} & 165 \\
25 & 1-(990, 6, 1) & [990, 165, 6] & \text{Sym}_{11}\wr\text{Sym}_{115} & 165 \\
\hline
\end{array}
\]

Table 4: Binary codes from the designs $D(x, T_i)$ given in Table 3

Remark 6. (i) By Remark 4, we have that $\text{Aut}(D) \cong \text{Sym}_k \wr \text{Sym}_b = (\text{Sym}_k)^b \text{Sym}_b$ for the designs $D$ with parameters $1-|C^2|, k, 1|$, and from [26] we deduce the parameters of their codes $C = C_p(D)$ as $C = C_p(D) = |C^2|, b, k_p$ for all $p$.

Observe that the minimum weight of the code equals the block size of the design and that the codewords of minimum weight are the incidence vectors of the blocks of the design, i.e., the codes are spanned by the minimum weight codewords, and thus $\text{Aut}(C) = \text{Aut}(D)$.

(ii) Applying Remark 5, we deduce that the codes $C = C_2(D)$ of the $1-|C^2|, k, 2|$-designs of Proposition 2 have parameters $C = C_2(D) = |C^2|, b, k, 2$. Moreover, the minimum weight is the block size of the design and the minimum words are the rows of the incidence matrix of the design. Hence, $\text{Aut}(D) = \text{Aut}(C_2(D))$.

4.4. Some 10-dimensional $\Delta$-divisible codes

Recall that a linear code $C$ over $\mathbb{F}_q$ is said to be $\Delta$-divisible if the Hamming weight $w(c)$ of every codeword $c \in C$ is divisible by $\Delta > 1$, and $C$ is said to be a projective code if $d(C^{-1}) \geq 3$.

Recently the study of $\Delta$-divisible codes, with $\Delta = q^r$, $r \in \mathbb{N}$, of which singly-, doubly- and triply-even codes are special cases, has gained renewed interest due to many applications, see [20, 21]. In particular, the study of projective $\Delta$-divisible codes has gained recent interest. In [20], the authors proved that there is no projective $2^3$-divisible code of length 59, and listed a number of lengths for which the existence of projective $2^4$-divisible binary codes are not known to exist. For example, it is not known whether a $2^4$-divisible binary code of length 165 exists. In this...
section, we present a number of examples of projective 10-dimensional codes invariant under $M_{11}$ of particular interest, amongst them a binary $2^3$-divisible code of length 165. Our interest is motivated by the knowledge that the smallest $F_2$-submodule on which $M_{11}$ acts irreducibly and faithfully has dimension 10 [24, p. 33].

**Example 1.** (i) The codes $C_1 = [165, 11, 45]_2$ in line 1 and $C_6 = [165, 11, 55]_2$ in line 6 in Table 4 are codes with $\text{Aut}(C_1) \cong \text{Sym}_{11}$ and $\text{Aut}(C_6) \cong M_{11}$, respectively. The code $C = C_1 \cap C_6$ is a $\Delta$-divisible code, with $\Delta = 2^3$. In fact, $C = [165, 10, 72]_2$ is a $4$-weight triply-even and projective code with weight distribution $0^11^32^55^88^227^8330^{12}20^{11}$ and $\text{Aut}(C) \cong \text{Sym}_{11} \supset \text{Aut}(C_6)$. Moreover, $C$ is the smallest faithful $F_2$-submodule on which $M_{11}$ acts irreducibly [24, p. 33].

(ii) The code $C_2 = [440, 10, 80]_2$ in line 2 in Table 4 is a self-orthogonal $\Delta$-divisible projective code, with $\Delta = 2^4$ and weight distribution $0^11^18^12^45^526^830^{10}2^{11}$. The code $C$ of the reduced point- and block-primitive $1$-$(55, 10, 2)$ design discussed in the proof of Proposition 2 is a $[55, 10, 10]_2$ even-weight projective code with weight distribution $0^11^55^88^227^8330^{12}20^{11}$. The dual code $C^\perp$ is a $[55, 45, 3]_2$ code with 165 codewords of weight 3. The codimension 1 subcode $C$ of $C^\perp$ is a $[55, 44, 4]_2$ code with 990 codewords of weight 4. It follows from [24, p. 33] that $C$ and $C^\perp$ are irreducible submodules invariant under $M_{11}$. Thus, we have shown that $F_5^{55} = \langle \mathbf{j} \rangle \oplus C \oplus C^\perp$, so that $F_5^{55}$ is a semisimple module.

Observe that since $F_5^{55} = C \oplus C^\perp$, we have that $C$ is an LCD code.

(iii) The code $C_3 = [990, 11, 270]_2$ in line 3 in Table 4 is a self-orthogonal code with weight distribution $0^11^{11}5^524^{16}5^830^{10}2^{11}$. Its subcode of codimension 1 is a $\Delta$-divisible $[990, 10, 432]_2$ code with $\Delta = 2^4$ with weight distribution $0^143^25^58^627^528^{10}5^830^{12}720^{11}$. The codes related to the reduced $1$-$(165, 45, 3)$-design are those discussed in part (i) of this example.

(iv) The code $C_4 = [1584, 11, 144]_2$ in line 4 in Table 4 is a self-orthogonal $\Delta$-divisible code, with $\Delta = 2^4$ and weight distribution $0^1144^{11}5^830^{12}720^{10}28^64^26^227^8330^{15}1152^{16}5^81296^{11}5^{56}1440^{11}$. The codimension 1 subcode $C$ of $C_4$ is a $[1584, 10, 288]_2$ $\Delta$-divisible code, with $\Delta = 2^5$ and weight distribution $0^128^{11}576^{30}8^64^26^21152^{16}5^81296^{11}5^{56}1440^{11}$. 


(v) The code $C_5 = [990, 11, 90]_2$ in line 5 in Table 4 is a self-orthogonal $\Delta$-divisible code, with $\Delta = 2^3$

$$0^190^{11}180^{55}270^{165}360^{330}450^{462}540^{165}630^{330}720^{165}810^{55}900^{11}990^1.$$  

The codimension 1 subcode $C$ of $C_5$ is a $[990, 10, 180]_2$ $\Delta$-divisible code, with $\Delta = 2^2$ and weight distribution

$$0^1180^{55}360^{330}450^{462}720^{165}900^{11}.$$  

(vi) The code $C_8 = [1584, 11, 264]_2$ in line 8 in Table 4 is a self-orthogonal $\Delta$-divisible code, with $\Delta = 2^4$

$$0^1264^{12}480^{66}648^{220}768^{405}840^{792}864^{462}.$$  

The codimension 1 subcode $C$ of $C_8$ is a $[1584, 10, 480]_2$ $\Delta$-divisible code, with $\Delta = 2^5$ and weight distribution

$$0^1480^{66}768^{405}840^{792}864^{462}.$$  

The codes of the reduced point- and block-primitive 1-(66, 11, 2)-design are a $[66, 11, 11]_2$ code with weight distribution

$$0^111^{12}20^{66}27^{220}32^{495}35^{792}36^{462}.$$  

and its codimension 1 subcode is a self-orthogonal doubly-even $[66, 10, 20]_2$ code with weight distribution

$$0^120^{66}32^{495}36^{462}.$$  

5. Conclusion

We have shown that the construction of Result 1 leads to many interesting, and possibly usable, codes acted on by the Mathieu group $M_{11}$. We have looked particularly at those codes of small dimension. These are good candidates for permutation decoding, due to the size of the group and the large size of the check set.

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