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# Disjoint convolution sums of Stirling numbers

Wenchang Chu*

*School of Mathematics and Statistics, Zhoukou Normal University, 466 001 Henan, P. R. China*
*Department of Mathematics and Physics, University of Salento, 73 100 Lecce, Italy*

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**Abstract.** By means of the connection coefficient approach, disjoint convolution sums involving the Stirling numbers of the first and second kinds are evaluated in closed form.

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## 1. Introduction and motivation

Suppose that $\{p_k(x)\}_{k\geq 0}$ constitutes a basis for the polynomial space. Then for a polynomial $Q(x)$, it can be expanded uniquely in terms of $\{p_k(x)\}_{k\geq 0}$. Denote the connection coefficient corresponding to $p_k(x)$ by

$$\Big(p_k(x); Q(x)\Big), \quad \text{such that} \quad Q(x) = \sum_{k\geq 0} p_k(x) \; \Big(p_k(x); Q(x)\Big). \tag{1}$$

In accordance with Knuth [7], the unsigned Stirling number of the first kind and the Stirling number of the second kind will be denoted here by $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$, respectively. Define the rising and falling factorials by $(x)_0 = \langle x \rangle_0 = 1$ and

$$\begin{aligned} (x)_n &= x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}, \\ \langle x \rangle_n &= x(x-1)\cdots(x-n+1) \quad \text{for} \quad n \in \mathbb{N}. \end{aligned}$$

Then for the three polynomial bases $\{x^k\}_{k\geq 0}$, $\{(x)_k\}_{k\geq 0}$ and $\{\langle x\rangle_k\}_{k\geq 0}$, the Stirling numbers are connection coefficients among them via

$$(x)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \langle x \rangle_k,$$

where the first relation is equivalent, under $x \to -x$, to the following one:

$$\langle x \rangle_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

*Corresponding author. *Email addresses:* `chu.wenchang@unisalento.it` (W. Chu)

 

There exist numerous identities involving Stirling numbers of both kinds (see e.g. [3, Chapter V], [5, §6.1] and [1, 2, 4, 6, 8, 10]). One well–known formula is the convolution

$$\sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = \binom{n}{m} \frac{(n-1)!}{(m-1)!}.$$

As a warm-up, we reproduce its proof here as follows. Consider the expansions

$$\begin{aligned}(x)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{m=0}^{k} \begin{Bmatrix} k \\ m \end{Bmatrix} \langle x \rangle_m \\ &= \sum_{m=0}^{n} \langle x \rangle_m \sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix}.\end{aligned}$$

Then the convolution sum in question results in the connection coefficient $\Big( \langle x \rangle_m ; (x)_n \Big)$, which is known as the Lah number

$$\mathrm{L}(n,m) = \binom{n}{m} \frac{(n-1)!}{(m-1)!} = \binom{n-1}{m-1} \frac{n!}{m!}. \tag{2}$$

As remarked by an anonymous referee, the above formula can also be proved by making use of the calculus of finite differences. Denote by $\Delta_y$ the forward difference operator of unit increment with respect to $y$. Then the Newton interpolating polynomial at the points $\{0, 1, 2, \ldots, n\}$ yields the equality

$$\binom{y+n-1}{n} = \sum_{m=0}^{n} \binom{y}{m} \Delta_y^m \binom{y+n-1}{n} \Big|_{y=0}.$$

Evaluating the differences

$$\Delta_y^m \binom{y+n-1}{n} = \binom{y+n-1}{n-m} \quad \text{for} \quad 0 \le m \le n,$$

we can see that the polynomial equality is equivalent to

$$(y)_n = \sum_{m=0}^{n} \frac{n!}{m!} \binom{n-1}{n-m} \langle y \rangle_m,$$

which leads us to the same formula as in (2).

Another well-known result is the orthogonal relation

$$\sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = \chi(m = n),$$

where the logical function $\chi$ is defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. This can analogously be justified by making use of the expansions

$$\begin{aligned}\langle x\rangle_n = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}x^k &= \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\sum_{m=0}^{k}\begin{Bmatrix}k\\m\end{Bmatrix}\langle x\rangle_m\\ &= \sum_{m=0}^{n}\langle x\rangle_m\sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\begin{Bmatrix}k\\m\end{Bmatrix}\end{aligned}$$

and then determining the connection coefficient

$$\Big(\langle x\rangle_m; \langle x\rangle_n\Big) = \chi(m=n).$$

Motivated by these two identities, for an arbitrary integer parameter $\lambda$ we shall examine the following two classes of convolution sums:

$$\begin{aligned}\Phi_{m,n}(\lambda) &= \sum_{k=m}^{n-\lambda}\begin{bmatrix}n\\k+\lambda\end{bmatrix}\begin{Bmatrix}k\\m\end{Bmatrix},\\ \Psi_{m,n}(\lambda) &= \sum_{k=m}^{n-\lambda}\begin{bmatrix}n\\k+\lambda\end{bmatrix}\begin{Bmatrix}k\\m\end{Bmatrix}(-1)^{n-k}.\end{aligned} \tag{3}$$

These convolutions of Stirling numbers are called “joint” for $\lambda = 0$ and “disjoint” otherwise when $\lambda \neq 0$. In the next section, two general summation theorems will be established for $\Phi_{m,n}(\lambda)$ in accordance with $\lambda > 0$ and $\lambda < 0$. Then in Section 3, two analytic formulae will be shown for the alternating convolution sums $\Psi_{m,n}(\lambda)$. Several remarkable identities will be deduced as consequences. The first four are anticipated as follows:

$$\begin{aligned}\Phi_{m,n}(1) &= \binom{n-1}{m}\frac{(n-1)!}{m!},\\ \Phi_{m,n}(-1) &= \binom{n}{m-1}\frac{n!}{(m-1)!},\\ \Psi_{m,n}(1) &= (-1)^{n-m}\frac{(n-1)!}{m!},\\ \Psi_{m,n}(-1) &= \begin{cases}0, & 1 \le m < n\\ -n, & m = n\\ -1, & m = n+1.\end{cases}\end{aligned}$$

It should be remarked that the formula above for $\Psi_{m,n}(1)$ is equivalent to [5, Identity 6.25]. Finally, the paper concludes with a brief discussion about similar convolution sums of Lah numbers, where a few examples will be provided.

## 2. Evaluation of positive sums $\Phi_{m,n}(\lambda)$

In this section, we are going to derive two summation formulae for $\Phi_{m,n}(\lambda)$ according to $\lambda > 0$ and $\lambda < 0$. Starting with the Stirling expansions in succession

$$\begin{aligned}(x)_n &= \sum_{k=0}^{\lambda-1} \begin{bmatrix} n \\ k \end{bmatrix} x^k + \sum_{k=0}^{n-\lambda} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} x^{k+\lambda} \\ &= \sum_{k=0}^{\lambda-1} \begin{bmatrix} n \\ k \end{bmatrix} x^k + x^\lambda \sum_{k=0}^{n-\lambda} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} \sum_{m=0}^{k} \begin{Bmatrix} k \\ m \end{Bmatrix} \langle x \rangle_m \\ &= \sum_{k=0}^{\lambda-1} \begin{bmatrix} n \\ k \end{bmatrix} x^k + x^\lambda \sum_{m=0}^{n-\lambda} \langle x \rangle_m \sum_{k=m}^{n-\lambda} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix},\end{aligned}$$

we derive the following equality:

$$\sum_{m=0}^{n-\lambda} \langle x \rangle_m \sum_{k=m}^{n-\lambda} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = \frac{(x)_n}{x^\lambda} - \sum_{k=0}^{\lambda-1} \begin{bmatrix} n \\ k \end{bmatrix} x^{k-\lambda}. \tag{4}$$

The above expression on the right is a polynomial. In order to determine it explicitly, we recall that for $n \in \mathbb{N}$, there is a binomial identity

$$\binom{x+n-1}{n} = \frac{x}{n} \sum_{j=0}^{n-1} \binom{x+j-1}{j},$$

which is equivalent to

$$(x)_n = x \sum_{j=0}^{n-1} \frac{(n-1)!}{j!} (x)_j, \quad \text{where} \quad n \in \mathbb{N}. \tag{5}$$

By pulling out the initial term, we can proceed further with

$$\begin{aligned}(x)_n &= x(n-1)! + x \sum_{j_1=1}^{n-1} \frac{(n-1)!}{j_1!} (x)_{j_1} \\ &= x(n-1)! + x^2 \sum_{n>j_1>j_2\geq 0} \frac{(n-1)!}{j_1 \cdot j_2!} (x)_{j_2} \\ &= x(n-1)! + x^2 \sum_{j_1=1}^{n-1} \frac{(n-1)!}{j_1} + x^3 \sum_{n>j_1>j_2>j_3\geq 0} \frac{(n-1)!}{j_1 j_2 \cdot j_3!} (x)_{j_3}.\end{aligned}$$

For $n, \lambda \in \mathbb{N}$ with $n \geq \lambda$, we can show by induction that

$$\begin{aligned}(x)_n = \sum_{k=1}^{\lambda-1} x^k &\sum_{n>j_1>j_2>\cdots>j_{k-1}\geq 1} \frac{(n-1)!}{j_1 j_2 \cdots j_{k-1}} \\ &+ x^\lambda \sum_{n>j_1>j_2>\cdots>j_\lambda\geq 0} \frac{(n-1)!}{j_1 j_2 \cdots j_{\lambda-1} \cdot j_\lambda!} (x)_{j_\lambda}.\end{aligned}$$

Then we can state (4) equivalently as

$$\sum_{m=0}^{n-\lambda}\langle x\rangle_m \sum_{k=m}^{n-\lambda}\begin{bmatrix} n \\ k+\lambda \end{bmatrix}\begin{Bmatrix} k \\ m \end{Bmatrix} = \sum_{n>j_1>j_2>\cdots>j_\lambda\geq 0}\frac{(n-1)!}{j_1 j_2\cdots j_{\lambda-1}\cdot j_\lambda!}(x)_{j_\lambda}.$$

By reversing the summation indices, we can now evaluate the connection coefficient

$$\begin{aligned}\Phi_{m,n}(\lambda) &= \left(\langle x\rangle_m;\ \sum_{0\leq i_1<i_2<\cdots<i_\lambda<n}\frac{(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}(x)_{i_1}\right)\\ &= \sum_{0\leq i_1<i_2<\cdots<i_\lambda<n}\frac{(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}\Big(\langle x\rangle_m;(x)_{i_1}\Big)\\ &= \sum_{m\leq i_1<i_2<\cdots<i_\lambda<n}\frac{(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}\mathrm{L}(i_1,m).\end{aligned}$$

Observing further that

$$\Phi_{m,n}(\lambda) = \sum_{i_\lambda=\lambda+m-1}^{n-1}\frac{(n-1)!}{i_\lambda!}\sum_{m\leq i_1<i_2<\cdots<i_{\lambda-1}<i_\lambda}\frac{(i_\lambda-1)!}{i_1!\cdot i_2\cdots i_{\lambda-1}}\mathrm{L}(i_1,m),$$

we find the following interesting property.

**Proposition 1** ($\lambda>0$: Recurrence relation)**.**

$$\Phi_{m,n}(\lambda) = \sum_{i=\lambda+m-1}^{n-1}\frac{(n-1)!}{i!}\Phi_{m,i}(\lambda-1).$$

The same referee pointed out that this proposition can also be proved by combining (3) with the following known recurrence relation (specified by $\ell = k+\lambda$):

$$\begin{bmatrix} n \\ \ell \end{bmatrix} = \sum_{i=\ell-1}^{n-1}\frac{(n-1)!}{i!}\begin{bmatrix} i \\ \ell-1 \end{bmatrix}.$$

This identity can be found in [7, Identity 6.21] and can be shown by classifying the permutations of $\{1,2,\ldots,n\}$ with $\ell$ cycles according to the length $n-i$ of the (specific) cycle that contains the largest element '$n$'.

Define the multiple harmonic numbers by the elementary symmetric function

$$\sigma_0(m;n)\equiv 1 \quad\text{and}\quad \sigma_\lambda(m;n) = \sum_{m\leq i_1<i_2<\cdots<i_\lambda\leq n}\prod_{j=1}^{\lambda}\frac{1}{i_j}.$$

Then for $n,\lambda\in\mathbb{N}$ with $n\geq\lambda$, we can write

$$(x)_n = \sum_{k=1}^{\lambda-1}x^k(n-1)!\sigma_{k-1}(1;n-1) + x^\lambda\sum_{j=0}^{n-\lambda}(x)_j\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1).$$

From this, we can see that (4) is also equivalent to

$$\sum_{m=0}^{n-\lambda}\langle x\rangle_m\sum_{k=m}^{n-\lambda}\begin{bmatrix} n\\ k+\lambda\end{bmatrix}\begin{Bmatrix} k\\ m\end{Bmatrix}=\sum_{j=0}^{n-\lambda}(x)_j\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1).$$

Therefore, the connection coefficient can be evaluated by

$$\begin{aligned}\Phi_{m,n}(\lambda)&=\left(\langle x\rangle_m;\sum_{j=0}^{n-\lambda}(x)_j\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\right)\\&=\sum_{j=m}^{n-\lambda}\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\Big(\langle x\rangle_m;(x)_j\Big)\\&=\sum_{j=m}^{n-\lambda}\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\mathrm{L}(j,m).\end{aligned}$$

After some simplification, this is highlighted as the following theorem.

**Theorem 1** ($\lambda>0$)**.**

$$\Phi_{m,n}(\lambda)=\sum_{j=m}^{n-\lambda}\frac{(n-1)!}{m!}\binom{j-1}{m-1}\sigma_{\lambda-1}(j+1;n-1).$$

The first two formulae corresponding to $\lambda=1,2$ are recorded below.

**Corollary 1.**

($a$) $\Phi_{m,n}(1)=\binom{n-1}{m}\frac{(n-1)!}{m!}$;

($b$) $\Phi_{m,n}(2)=\sum_{i=m+1}^{n-1}\binom{i-1}{m}\frac{(n-1)!}{i\cdot m!}$.

**Proof**. The first formula (a) follows directly by letting $\lambda=1$ in Theorem 1 and then evaluating the binomial sum

$$\Phi_{m,n}(1)=\sum_{j=m}^{n-1}\frac{(n-1)!}{m!}\binom{j-1}{m-1}=\frac{(n-1)!}{m!}\binom{n-1}{m}.$$

Formula (b) follows from (a) and Proposition 1 and can also be done analogously by letting $\lambda=2$ in Theorem 1:

$$\begin{aligned}\Phi_{m,n}(2)&=\sum_{j=m}^{n-2}\sum_{i=j+1}^{n-1}\binom{j-1}{m-1}\frac{(n-1)!}{i\cdot m!}\\&=\sum_{i=m+1}^{n-1}\frac{(n-1)!}{i\cdot m!}\sum_{j=m}^{i-1}\binom{j-1}{m-1}=\sum_{i=m+1}^{n-1}\frac{(n-1)!}{i\cdot m!}\binom{i-1}{m}.\end{aligned}$$

$\square$

For $\lambda < 0$, we have another explicit formula, where $\Delta_y$ is the difference operator introduced in the introduction.

**Theorem 2** ($\lambda < 0$)**.**

$$\begin{aligned}\Phi_{m,n}(\lambda) &= \sum_{i=0}^{-\lambda}\sum_{j=0}^{i}\frac{(-1)^{\lambda+j}}{n^{\lambda+i}}\binom{-\lambda}{i}\left\{ i \atop j \right\}\mathrm{L}(j+n,m)\\ &= \sum_{j=0}^{-\lambda}(-1)^{\lambda+j}\frac{\mathrm{L}(j+n,m)}{j!}\Delta_y^j(n+y)^{-\lambda}\Big|_{y=0}.\end{aligned}$$

**Proof**. Analogously, we have the expansions

$$\begin{aligned}(x)_n &= \sum_{k=-\lambda}^{n-\lambda}\left[ n \atop k+\lambda \right] x^{k+\lambda}\\ &= x^\lambda \sum_{k=-\lambda}^{n-\lambda}\left[ n \atop k+\lambda \right]\sum_{m=0}^{k}\left\{ k \atop m \right\}\langle x\rangle_m\\ &= x^\lambda \sum_{m=0}^{n-\lambda}\langle x\rangle_m \sum_{k=m}^{n-\lambda}\left[ n \atop k+\lambda \right]\left\{ k \atop m \right\}.\end{aligned}$$

Multiplying across by $x^{-\lambda}$, we derive the expression

$$\Phi_{m,n}(\lambda) = \Big(\langle x\rangle_m; x^{-\lambda}(x)_n\Big). \tag{6}$$

Observing that

$$\begin{aligned}x^{-\lambda} = \big\{(x+n)-n\big\}^{-\lambda} &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\frac{(x+n)^i}{(-n)^{\lambda+i}}\\ &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\sum_{j=0}^{i}\frac{(-1)^{\lambda+j}}{n^{\lambda+i}}\left\{ i \atop j \right\}(x+n)_j,\end{aligned}$$

we deduce further from (6)

$$\begin{aligned}\Phi_{m,n}(\lambda) &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\sum_{j=0}^{i}\frac{(-1)^{\lambda+j}}{n^{\lambda+i}}\left\{ i \atop j \right\}\Big(\langle x\rangle_m; (x)_{n+j}\Big)\\ &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\sum_{j=0}^{i}\frac{(-1)^{\lambda+j}}{n^{\lambda+i}}\left\{ i \atop j \right\}\mathrm{L}(j+n,m),\end{aligned}$$

which gives the double sum expression in the theorem.

By interchanging the summation order, we can rewrite

$$\begin{aligned}
\Phi_{m,n}(\lambda) &= \sum_{j=0}^{-\lambda} \mathrm{L}(j+n,m) \sum_{i=0}^{-\lambda} \frac{(-1)^{\lambda+j}}{n^{\lambda+i}} \binom{-\lambda}{i} \left\{ {i \atop j} \right\} \\
&= \sum_{j=0}^{-\lambda} \frac{\mathrm{L}(j+n,m)}{j!} \sum_{i=0}^{-\lambda} \frac{(-1)^{\lambda+j}}{n^{\lambda+i}} \binom{-\lambda}{i} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^i \\
&= \sum_{j=0}^{-\lambda} \frac{\mathrm{L}(j+n,m)}{j!} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^{\lambda-k}}{n^{\lambda}} \sum_{i=0}^{-\lambda} \binom{-\lambda}{i} \left(\frac{k}{n}\right)^i \\
&= \sum_{j=0}^{-\lambda} (-1)^{\lambda} \frac{\mathrm{L}(j+n,m)}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (n+k)^{-\lambda}.
\end{aligned}$$

Expressing the rightmost sum in terms of $\Delta_y$, we obtain the second formula stated in Theorem 2. □

When $\lambda$ is a small negative integer, the sums displayed in Theorem 2 contain only a few terms and can be computed without difficulty. In particular, we have the following three elegant formulae for $\lambda = -1, -2, -3$.

**Corollary 2** ($m \geq -\lambda$).

(*a*) $\Phi_{m,n}(-1) = \binom{n}{m-1} \dfrac{n!}{(m-1)!}$;

(*b*) $\Phi_{m,n}(-2) = \binom{n+1}{m-1} \dfrac{(1+mn)n!}{(n+1)(m-1)!}$;

(*c*) $\Phi_{m,n}(-3) = \binom{n+2}{m-1} \dfrac{(1+m-n+4mn+m^2n^2)n!}{(n+1)(n+2)(m-1)!}$.

## 3. Evaluation of alternating sums $\Psi_{m,n}(\lambda)$

For the alternating convolution sums $\Psi_{m,n}(\lambda)$, two companion formulae will similarly be shown in this section. Again, we first consider the case $\lambda > 0$. In view of the Stirling expansions

$$\begin{aligned}
\langle x \rangle_n &= \sum_{k=0}^{\lambda-1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k + \sum_{k=0}^{n-\lambda} (-1)^{n-k-\lambda} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} x^{k+\lambda} \\
&= \sum_{k=0}^{\lambda-1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k + (-x)^{\lambda} \sum_{k=0}^{n-\lambda} (-1)^{n-k} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} \sum_{m=0}^{k} \left\{ {k \atop m} \right\} \langle x \rangle_m \\
&= \sum_{k=0}^{\lambda-1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k + (-x)^{\lambda} \sum_{m=0}^{n-\lambda} \langle x \rangle_m \sum_{k=m}^{n-\lambda} (-1)^{n-k} \begin{bmatrix} n \\ k+\lambda \end{bmatrix} \left\{ {k \atop m} \right\},
\end{aligned}$$

we have the following expression:

$$\sum_{m=0}^{n-\lambda}\langle x\rangle_m\Psi_{m,n}(\lambda)=\frac{\langle x\rangle_n}{(-x)^\lambda}-\sum_{k=0}^{\lambda-1}(-1)^{n-k-\lambda}\begin{bmatrix}n\\k\end{bmatrix}x^{k-\lambda}. \tag{7}$$

The above expression on the right is again a polynomial. In order to have an explicit polynomial expression, we must rewrite (5) by making the replacement $x\to -x$:

$$\langle x\rangle_n=(-x)\sum_{j=0}^{n-1}(-1)^{n-j}\frac{(n-1)!}{j!}\langle x\rangle_j,\quad \text{where}\quad n\in\mathbb{N}.$$

By iterating this relation, we can proceed further with

$$\begin{aligned}\langle x\rangle_n&=(-x)(n-1)!(-1)^n+(-x)\sum_{j_1=1}^{n-1}(-1)^{n-j_1}\frac{(n-1)!}{j_1!}\langle x\rangle_{j_1}\\&=(-x)(n-1)!(-1)^n+(-x)^2\sum_{n>j_1>j_2\geq 0}(-1)^{n-j_2}\frac{(n-1)!}{j_1\cdot j_2!}\langle x\rangle_{j_2}\\&=(-x)(n-1)!(-1)^n+(-x)^2(-1)^n\sum_{j_1=1}^{n-1}\frac{(n-1)!}{j_1}\\&\quad+(-x)^3\sum_{n>j_1>j_2>j_3\geq 0}(-1)^{n-j_3}\frac{(n-1)!}{j_1j_2\cdot j_3!}\langle x\rangle_{j_3}.\end{aligned}$$

In general, for $n,\lambda\in\mathbb{N}$ with $n\geq\lambda$, we can prove by induction that

$$\begin{aligned}\langle x\rangle_n=&\sum_{k=1}^{\lambda-1}(-x)^k\sum_{n>j_1>j_2>\cdots>j_{k-1}\geq 1}\frac{(-1)^n(n-1)!}{j_1j_2\cdots j_{k-1}}\\&+(-x)^\lambda\sum_{n>j_1>j_2>\cdots>j_\lambda\geq 0}\frac{(-1)^{n-j_\lambda}(n-1)!}{j_1j_2\cdots j_{\lambda-1}\cdot j_\lambda!}\langle x\rangle_{j_\lambda}.\end{aligned}$$

By combining this with (7), we get the expression

$$\sum_{m=0}^{n-\lambda}\langle x\rangle_m\Psi_{m,n}(\lambda)=\sum_{n>j_1>j_2>\cdots>j_\lambda\geq 0}\frac{(-1)^{n-j_\lambda}(n-1)!}{j_1j_2\cdots j_{\lambda-1}\cdot j_\lambda!}\langle x\rangle_{j_\lambda}.$$

After reversing the summation indices, we can manipulate the sum

$$\begin{aligned}\Psi_{m,n}(\lambda) &= \Bigg(\langle x\rangle_m;\sum_{0\le i_1<i_2<\cdots<i_\lambda<n}\frac{(-1)^{n-i_1}(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}\langle x\rangle_{i_1}\Bigg)\\ &= \sum_{0\le i_1<i_2<\cdots<i_\lambda<n}\frac{(-1)^{n-i_1}(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}\Big(\langle x\rangle_m;\langle x\rangle_{i_1}\Big)\\ &= \sum_{0\le i_1<i_2<\cdots<i_\lambda<n}\frac{(-1)^{n-i_1}(n-1)!}{i_1!\cdot i_2\cdots i_\lambda}\chi(i_1=m)\\ &= \sum_{m<i_2<\cdots<i_\lambda<n}\frac{(-1)^{n-m}(n-1)!}{m!\cdot i_2\cdots i_\lambda}.\end{aligned}$$

By reformulating the last multiple sum in two different manners

$$\sum_{i_2=m+1}^{n-\lambda+1}(-1)^{i_2-m}\frac{(i_2-1)!}{m!}\sum_{i_2<i_3<\cdots<i_\lambda<n}\frac{(-1)^{n-i_2}(n-1)!}{i_2!\cdot i_3\cdots i_\lambda}$$

and

$$\sum_{i_\lambda=m+\lambda-1}^{n-1}(-1)^{n-i_\lambda}\frac{(n-1)!}{i_\lambda!}\sum_{m<i_2<i_3<\cdots<i_\lambda}\frac{(-1)^{i_\lambda-m}(i_\lambda-1)!}{m!\cdot i_2\cdots i_{\lambda-1}},$$

we find the following two remarkable equalities for $\Psi_{m,n}(\lambda)$ when $\lambda>1$, which can be seen to hold also in the case $\lambda=1$.

**Proposition 2** ($\lambda>0$: Recurrence relations)**.**

$$\begin{aligned}\Psi_{m,n}(\lambda) &= \sum_{i=m+1}^{n-\lambda+1}(-1)^{i-m}\frac{(i-1)!}{m!}\Psi_{i,n}(\lambda-1)\\ &= \sum_{i=m+\lambda-1}^{n-1}(-1)^{n-i}\frac{(n-1)!}{i!}\Psi_{m,i}(\lambda-1).\end{aligned}$$

By making use of the $\sigma$-symbol, we also have a compact expression

$$\begin{aligned}\langle x\rangle_n = (-1)^n(n-1)!&\sum_{k=1}^{\lambda-1}(-x)^k\sigma_{k-1}(1;n-1)\\ &+(-x)^\lambda\sum_{j=0}^{n-\lambda}(-1)^{n-j}\langle x\rangle_j\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1).\end{aligned}$$

Then we get the following equality equivalent to (7):

$$\sum_{m=0}^{n-\lambda}\langle x\rangle_m\Psi_{m,n}(\lambda)=\sum_{j=0}^{n-\lambda}(-1)^{n-j}\langle x\rangle_j\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1).$$

Hence the connection coefficient can alternatively be evaluated by

$$\begin{aligned}\Psi_{m,n}(\lambda) &= \left(\langle x\rangle_m; \sum_{j=0}^{n-\lambda}(-1)^{n-j}\langle x\rangle_j \frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\right)\\ &= \sum_{j=m}^{n-\lambda}(-1)^{n-j}\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\left(\langle x\rangle_m;\langle x\rangle_j\right)\\ &= \sum_{j=m}^{n-\lambda}(-1)^{n-j}\frac{(n-1)!}{j!}\sigma_{\lambda-1}(j+1;n-1)\chi(m=j),\end{aligned}$$

which becomes the formula stated in the following theorem.

**Theorem 3** ($\lambda > 0$)**.**

$$\Psi_{m,n}(\lambda) = (-1)^{n-m}\frac{(n-1)!}{m!}\sigma_{\lambda-1}(m+1;n-1).$$

As a consequence, we have the following formulae corresponding to $\lambda = 1, 2, 3$.

**Corollary 3.**

(*a*) $\Psi_{m,n}(1) = (-1)^{n-m}\dfrac{(n-1)!}{m!}$;

(*b*) $\Psi_{m,n}(2) = (-1)^{n-m}\dfrac{(n-1)!}{m!}\sigma_1(m+1;n-1)$;

(*c*) $\Psi_{m,n}(3) = (-1)^{n-m}\dfrac{(n-1)!}{m!}\sigma_2(m+1;n-1)$.

When $\lambda < 0$, the corresponding formula is as follows.

**Theorem 4** ($\lambda < 0$)**.**

$$\Psi_{m,n}(\lambda) = \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\left\{{i \atop m-n}\right\}\frac{(-1)^{\lambda}}{n^{\lambda+i}}.$$

**Proof**. Alternatively, we have the expansions

$$\begin{aligned}\langle x\rangle_n &= \sum_{k=-\lambda}^{n-\lambda}(-1)^{n-k-\lambda}\left[{n \atop k+\lambda}\right]x^{k+\lambda}\\ &= (-x)^{\lambda}\sum_{k=-\lambda}^{n-\lambda}(-1)^{n-k}\left[{n \atop k+\lambda}\right]\sum_{m=0}^{k}\left\{{k \atop m}\right\}\langle x\rangle_m\\ &= (-x)^{\lambda}\sum_{m=0}^{n-\lambda}\langle x\rangle_m\sum_{k=m}^{n-\lambda}(-1)^{n-k}\left[{n \atop k+\lambda}\right]\left\{{k \atop m}\right\}.\end{aligned}$$

Multiplying across by $(-x)^{-\lambda}$, we derive the expression

$$\Psi_{m,n}(\lambda) = \Big(\langle x\rangle_m; (-x)^{-\lambda}\langle x\rangle_n\Big). \tag{8}$$

Observing that

$$\begin{aligned} x^{-\lambda} = \big\{(x-n)+n\big\}^{-\lambda} &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\frac{(x-n)^i}{n^{\lambda+i}} \\ &= \sum_{i=0}^{-\lambda}\binom{-\lambda}{i}\sum_{j=0}^{i}\frac{\langle x-n\rangle_j}{n^{\lambda+i}}\left\{ {i \atop j} \right\}, \end{aligned}$$

we deduce from (8) the double sum expression

$$\begin{aligned} \Psi_{m,n}(\lambda) &= \sum_{i=0}^{-\lambda}\frac{(-1)^\lambda}{n^{\lambda+i}}\binom{-\lambda}{i}\sum_{j=0}^{i}\left\{ {i \atop j} \right\}\Big(\langle x\rangle_m; \langle x\rangle_{n+j}\Big) \\ &= \sum_{i=0}^{-\lambda}\frac{(-1)^\lambda}{n^{\lambda+i}}\binom{-\lambda}{i}\sum_{j=0}^{i}\left\{ {i \atop j} \right\}\chi(m=n+j), \end{aligned}$$

which becomes the formula displayed in Theorem 4. □

When $\lambda$ is a small negative integer, we can easily compute $\Psi_{m,n}(\lambda)$ by means of Theorem 4. Here are the first two examples.

**Corollary 4.**

$$(a)\ \ \Psi_{m,n}(-1) = \begin{cases} 0, & 1 < m < n \\ -n, & m = n \\ -1, & m = n+1; \end{cases}$$

$$(b)\ \ \Psi_{m,n}(-2) = \begin{cases} 0, & 1 \le m < n \\ n^2, & m = n \\ 2n+1, & m = n+1 \\ 1, & m = n+2. \end{cases}$$

## 4. Convolution sums of Lah numbers

By employing the approach of connection coefficients, convolution sums of Lah numbers can also be examined. However, the resulting expressions are quite involved. Here we give five initial formulae as examples, which can be confirmed by using the Chu–Vandermonde convolution formula. The informed reader will notice that among these five identities recorded below, (9) is well-known and equivalent to the equation appearing in [9, p. 44].

$$
\begin{aligned}
&\sum_{k=m}^{n} \mathrm{L}(n,k)\mathrm{L}(k,m) = 2^{n-m}\mathrm{L}(n,m),\\
&\sum_{k=m}^{n+1} \mathrm{L}(n,k-1)\mathrm{L}(k,m) = 2^{n-m}\frac{(m+n)^2-m+n}{2n(n+1)}\mathrm{L}(n+1,m),\\
&\sum_{k=m}^{n} (-1)^{k-m}\mathrm{L}(n,k)\mathrm{L}(k,m) = \chi(m=n),\\
&\sum_{k=m}^{n-1} (-1)^{k-m}\mathrm{L}(n,k+1)\mathrm{L}(k,m) = \frac{(n-m)(n-1)!}{m!},\\
&\sum_{k=m}^{n+1} (-1)^{k-m}\mathrm{L}(n,k-1)\mathrm{L}(k,m) = \begin{cases} 0, & 1 \le m < n-1\\ n(n-1), & m=n-1\\ -2n, & m=n\\ 1, & m=n+1. \end{cases}
\end{aligned} \tag{9}
$$


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