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Hermite spatial variations for the solution to the stochastic heat equation

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Abstract. In this paper, we study Hermite spatial variations for the solution to the stochastic heat equation with space-time white noise. We prove that these variations satisfy the central limit theorem and we obtain the almost sure central limit theorem.
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1. Introduction

In probability theory, the study of stochastic partial differential equations (SPDE) is an important topic of research. Precisely, the linear stochastic heat equation driven by white noise is one of the most studied SPDEs. This is due to its mathematical simplicity and its application in several areas. The solution to this stochastic partial differential equation has been of great interest to researchers in the last years. It is known that a mild solution exists if and only if the spatial dimension is one, it is also Hölder continuous of order $0 < \delta < 1/4$ with respect to its time coordinate and of order $0 < \delta < 1/2$ with respect to its space coordinate and it has a close relation with the fractional Brownian motion and the bifractional Brownian motion, among other interesting properties (for more details, see section 2.1 [16]).

Spatial and temporal quadratic variations of this process can be found in [17], and in [15], respectively. In relation to parameter estimation for the heat equation with white noise using variations, we can mention [13, 19, 2] and [3], and the references therein. Other approaches to the problem of the limit behavior of variations could be used (see, for example, [4, 5, 6, 19]).

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It is known that the covariance of this process is stationary with respect to the space variable, but dependent on time. Thus, for the purpose of observing the influence of the time variable in the behavior in space of higher variations of this process, a new parameter α must be defined. This new parameter allows us to consider the behavior of the variation when the time variable is either fixed or big or small. This approach has been introduced in [8], based on the tools developed in [12], to study the influence of the time variable on the spatial quadratic variation of the solution to the heat equation.

The aim of this paper is to study higher order spatial variations of the solution to the linear heat equation driven by space-time white noise. Precisely, we will focus on the Hermite spatial variations with moving time.

For every $N \geq 1$ and for every $i = 0, \dots, N$, we denote $x_i = \frac{i}{N}$. So, the Hermite variation statistic with moving time, over the unit interval $[0, 1]$, is defined in the following way:

$$V_N = \sum_{i=0}^{N-1} H_q \left(\frac{u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)}{\sqrt{\mathbf{E}\left(u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)\right)^2}} \right),$$

where u is the solution to the linear heat equation, H_q is a Hermite polynomial of order q and $\alpha \in \mathbb{R}$.

We prove that, under suitable normalization depending on α, this sequence converges in law to a normal random variable. More precisely, we have

$$d(F_N, \mathcal{N}(0,1)) \leq C \frac{1}{\sqrt{N}}.$$

Here, $F_N = V_N / v_N$ with $v_N^2 = \mathbf{E}(V_N^2)$, where d could be either the Kolmogorov, the Wasserstein, or total variation distance. Furthermore, we prove an almost sure central limit theorem for F_N.

We organized our paper as follows. In Section 2 we give a brief introduction to Malliavin calculus and the heat equation driven by space-time white noise. In Section 3, we introduce Hermite variations and estimate their second moment. In Section, 4 we study the asymptotic distribution of Hermite variations. Finally, in Section 5, an almost sure central limit theorem is obtained.

2. Preliminaries

This section is dedicated to presenting some definitions and results that are used in this paper, related to Malliavin calculus and the heat equation driven by space-time white noise.

2.1. Elements of Malliavin calculus

Here, we briefly recall some elements of stochastic analysis; for an in-depth introduction we refer the reader to [11]. Consider $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space (Ω, F, P), which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (a symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of random variables $H_q(B(\varphi))$, where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = \frac{(-1)^q}{q!} \exp\left(\frac{x^2}{2}\right) \frac{d^q}{dx^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as:

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

It also holds that

$$I_q(f) = I_q\big(\tilde{f}\big),$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}).$$

We recall that any square-integrable random variable F, which is measurable with respect to the σ-algebra generated by B, can be expanded into an orthogonal sum of multiple stochastic integrals:

$$F = \sum_{q \geq 0} I_q(f_q);$$

here the series converges in $L^2(\Omega)$-sense, the kernels f_q, belonging to $\mathcal{H}^{\odot q}$, are uniquely determined by F and $I_0(f_0) = \mathbf{E}\left(F\right)$.

We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}$, and that is defined as follows:

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}\left(\mathcal{H}\right)$ into $\mathbb{D}^{\alpha-1,p}\left(\mathcal{H}\right)$.

We will need a general formula for calculating products of Wiener chaos integrals of any order p, q, so, for any symmetric integrand $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, it is:

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2}$$

If $\mathcal{H}$ is the space $L^2([0,T])$, then the contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by:

$$(f \otimes_r g)(s_1, \dots, s_{p-r}, t_1, \dots, t_{q-r})$$
$$= \int_{[0,T]^r} f(s_1, \dots, s_{p-r}, u_1, \dots, u_r) g(t_1, \dots, t_{q-r}, u_1, \dots, u_r) \mathrm{d}u_1 \dots \mathrm{d}u_r.$$

We will use the following result that characterizes the convergence in the distribution of a sequence of multiple integrals toward the Gaussian law (see [10] or [9] for details).

Theorem 1. *Let $I_q(f)$ be a multiple integral of order $q \geq 1$. Assume $\mathbf{E}[I_q(f)^2] = \sigma^2$. Then*

$$d(I_q(f), \mathcal{N}(0, \sigma^2)) \leq c \sqrt{Var\left[\left(\frac{1}{q}\|DI_q(f)\|^2_{\mathcal{H}}\right)^2\right]},$$

where D stands for the Malliavin derivative with respect to u and $c = 1/\sigma^2$, when d is the Kolmogorov distance, and $c = 1/\sigma$, when d is the Wasserstein distance, and finally $c = 2/\sigma^2$, when d is the total variation distance.

2.2. Heat equation with space-time white noise

In this subsection, we recall some known facts concerning the solution to the linear stochastic heat equation driven by white noise in time and space (our main reference is [16]).

The linear stochastic heat equation is given by the following expression:

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u + \dot{W} \\ u(0,x) &= 0, \quad \text{for every } x \in \mathbb{R}; \end{aligned} \tag{3}$$

here Δ is the Laplacian on $\mathbb{R}$, $u_t := \frac{\partial u}{\partial t}$ and $W = \{W(t,A) : t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ is space-time Gaussian white noise, that is, W is a Gaussian process with mean zero and covariance given by

$$\mathbf{E}(W(t,A)W(s,B)) = (t \wedge s)\lambda(A \cap B), \quad s,t \geq 0,$$

where λ denotes the Lebesgue measure.

Process $u(t,x)$ is the solution to (3) (in the mild sense) if

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) W(ds, dy), \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{4}$$

where G is the fundamental solution of the corresponding deterministic heat equation, that is,

$$G(t,x) = (2\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{2t}\right) 1_{(0,\infty)}(t)$$

for $t, x \in \mathbb{R}$. Process (4) is well-defined as a square integrable random variable if and only if the spatial dimension d is equal to 1.

2.2.1. The spatial correlation structure

The covariance of the process u given by (4) can be obtained by (see e.g. formula (7) in [8]):

$$\begin{aligned}
&\mathbf{E}(u(t,x)u(s,y))\\
&= \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s}e^{-\frac{|y-x|^2}{2(t+s)}} - \sqrt{t-s}e^{-\frac{|y-x|^2}{2(t-s)}}\right) - \frac{1}{\sqrt{\pi}}|y-x|\int_{\frac{|y-x|}{\sqrt{2(t+s)}}}^{\frac{|y-x|}{\sqrt{2(t-s)}}} e^{-z^2}dx\\
&= \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s}e^{-\frac{|y-x|^2}{2(t+s)}} - \sqrt{t-s}e^{-\frac{|y-x|^2}{2(t-s)}}\right)\\
&\quad - \sqrt{2}|x-y|\mathrm{erf}\left(\frac{|x-y|}{\sqrt{2(t-s)}}\right) + \sqrt{2}|x-y|\mathrm{erf}\left(\frac{|x-y|}{\sqrt{2(t+s)}}\right),
\end{aligned}$$

where erf denotes the error function

$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-z^2}dz.$$

In particular, when $t=s$,

$$\mathbf{E}(u(t,x)u(t,y)) = \frac{1}{\sqrt{\pi}}\sqrt{t}e^{-\frac{|y-x|^2}{4t}} + \sqrt{2}|x-y|\mathrm{erf}\left(\frac{|x-y|}{2\sqrt{t}}\right) - \frac{1}{2}|y-x|, \qquad (5)$$

while for $x=y$ we find, as in [15] and [16],

$$\mathbf{E}(u(t,x)u(s,x)) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right).$$

Hence, $u(t,x)$ is a normal random variable with mean zero and variance $\sqrt{\frac{t}{\pi}}$.

In the sequel we will denote $x_i = \frac{i}{N}$ for every $i = 0,..,N$ and for every $N \geq 1$. Due to [8], the following results will play an important role in the study of the Hermite variations V_N:

Lemma 1 (see [8]). *For $\alpha \in \mathbb{R}$, let $g_\alpha(N)$ be defined by*

$$g_\alpha(N) = \mathbf{E}\left|u(N^\alpha, x_{j+1}) - u(N^\alpha, x_j)\right|^2.$$

1. *If $\alpha > -2$, then*
$$N g_\alpha(N) \xrightarrow[N\to\infty]{} 1/2.$$

2. *If $\alpha = -2$, then*
$$N g_\alpha(N) \xrightarrow[N\to\infty]{} \frac{2}{\sqrt{\pi}}(1-e^{-1/4}) + \frac{1}{2\sqrt{\pi}}\int_{1/4}^{\infty} b^{-1/2}e^{-b}db.$$

3. *If $\alpha < -2$, then*
$$N^{-\frac{\alpha}{2}} g_\alpha(N) \xrightarrow[N\to\infty]{} \frac{2}{\sqrt{\pi}}.$$

Lemma 2 (see [8]). *Let u be given by (4), $\alpha \in \mathbb{R}$ and*

$$r_N(i-j) := \mathbf{E}\left(u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)\right)\left(u(N^\alpha, x_{j+1}) - u(N^\alpha, x_j)\right),$$

where, for $k \in \mathbb{Z}$,

$$\begin{aligned} r_N(k) =& \frac{1}{\sqrt{\pi}} N^{\alpha/2}\left[2\exp\left(-\frac{k^2}{4N^{2+\alpha}}\right) - \exp\left(-\frac{(k+1)^2}{4N^{2+\alpha}}\right) - \exp\left(-\frac{(k-1)^2}{4N^{2+\alpha}}\right)\right] \\ & + \sqrt{2}\left[2\frac{|k|}{N} erf\left(\frac{|k|}{2N^{1+\alpha/2}}\right) - \frac{|k+1|}{N} erf\left(\frac{|k+1|}{2N^{1+\alpha/2}}\right)\right. \\ & \left. - \frac{|k-1|}{N} erf\left(\frac{|k-1|}{2N^{1+\alpha/2}}\right)\right]. \end{aligned} \tag{6}$$

Then for $k = 1, \dots N-1$, we have

$$r_N(k) \le C\frac{1}{N^{2+\frac{\alpha}{2}}} h_N(k), \tag{7}$$

where

$$h_N(k) = \left(\frac{(k+1)^2}{N^{2+\alpha}} + 1\right) e^{-\frac{(k-1)^2}{4N^{2+\alpha}}}.$$

In particular, if $\alpha > 0$

$$r_N(k) \le C\frac{1}{N^{2+\frac{\alpha}{2}}}.$$

3. Hermite variation with moving time

Now we focus on the asymptotic behavior of Hermite spatial variations for the solution to the heat equation; our main tools are the estimates given by Lemma 1, Lemma 2 and the Stein-Malliavin theory (see [9]).

Let us recall that for every $N \ge 1$ and for every $i = 0, \dots, N$, we denoted $x_i = \frac{i}{N}$. So, the Hermite variation statistic over the unit interval $[0,1]$, is defined in the following way:

$$V_N = \sum_{i=0}^{N-1} H_q\left(\frac{u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)}{\sqrt{\mathbf{E}\left(u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)\right)^2}}\right),$$

for $\alpha \in \mathbb{R}$.

We denote by $\mathcal{H}$ the canonical Hilbert space associated to the Gaussian solution process $(u(N^\alpha, x))_{x\in\mathbb{R}}$. This Hilbert space is defined as the closure of the set ξ of indicator functions $1_{[0,x]}$, $x > 0$, with respect to the inner product:

$$\langle 1_{[0,x]}, 1_{[0,y]}\rangle_{\mathcal{H}} = \mathbf{E}\Big(u(N^\alpha, x)u(N^\alpha, y)\Big).$$

The above inner product has the explicit form given by (5).

From now on, we denote by I_q^u, $q \geq 1$, the multiple Wiener-integral with respect to the Gaussian process $(u(N^\alpha, x))_{x\in[0,1]}$, so the increment $u(N^\alpha, y) - u(N^\alpha, x)$ can be expressed as $I_1^u(1_{[x,y]})$, for every $x < y$.

Therefore, we can easily rewrite

$$V_N = \sum_{i=0}^{N-1} H_q\left(g_\alpha(N)^{-1/2} I_1^u(1_{[x_i, x_{i+1}]})\right),$$

with $g_\alpha(N) = \mathbf{E}\left|u(N^\alpha, x_{i+1}) - u(N^\alpha, x_i)\right|^2$. Let us note that $g_\alpha(N)$ does not depend on i (stationary in space). So

$$V_N = \frac{1}{q!}\sum_{i=0}^{N-1} g_\alpha(N)^{-q/2} I_q^u(1_{[x_i, x_{i+1}]}^{\otimes q}) = I_q^u(s_N),$$

where

$$s_N := \frac{1}{q!}\sum_{i=0}^{N-1} g_\alpha(N)^{-q/2} 1_{[x_i, x_{i+1}]}^{\otimes q} \in \mathcal{H}^{\otimes q}.$$

In order to apply the fourth moment theorem we need to consider the following normalized sequence:

$$F_N = \frac{V_N}{v_N} = \frac{I_q^u(s_N)}{v_N} \quad \text{with} \quad v_N^2 = \mathbf{E}(V_N^2). \tag{8}$$

Lemma 3. *Let v_N be given by (8). Then*

$$\frac{1}{N} v_N^2 \xrightarrow[N\to\infty]{} \begin{cases} \dfrac{1}{q!} & \text{if } \alpha + 2 > 0 \\[2ex] \dfrac{1}{q!} + \dfrac{1}{2^{q-1} q!} & \text{if } \alpha + 2 < 0 \\[2ex] \dfrac{1}{q!} + c_3 & \text{if } \alpha + 2 = 0, \end{cases}$$

where

$$\begin{aligned} c_3 =& \frac{2}{q!} C_0^{-q} \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{\pi}} \left[2\exp\left(-\frac{k^2}{4}\right) - \exp\left(-\frac{(k+1)^2}{4}\right) - \exp\left(-\frac{(k-1)^2}{4}\right) \right] \right. \\ &\left. + \sqrt{2}\left[2|k| erf\left(\frac{|k|}{2}\right) - |k+1| erf\left(\frac{|k+1|}{2}\right) - |k-1| erf\left(\frac{|k-1|}{2}\right) \right] \right)^q . \end{aligned}$$

Proof. By isometry formula (1), we get

$$\begin{aligned} v_N^2 &= \frac{1}{q!} g_\alpha(N)^{-q} \sum_{i,j=0}^{N-1} \langle 1_{[x_i,x_{i+1}]}^{\otimes q}, 1_{[x_j,x_{j+1}]}^{\otimes q} \rangle_{\mathcal{H}^{\otimes q}} \\ &= \frac{1}{q!} g_\alpha(N)^{-q} \sum_{i,j=0}^{N-1} \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle_{\mathcal{H}}^q \\ &= \frac{N}{q!} + \frac{2}{q!} g_\alpha(N)^{-q} \sum_{i,j=0, i>j}^{N-1} \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle_{\mathcal{H}}^q . \\ &= R_N^{(1)} + R_N^{(2)}. \end{aligned}$$

Clearly, for every α, we have

$$\frac{1}{N} R_N^{(1)} \xrightarrow[N\to\infty]{} \frac{1}{q!}.$$

For r_N given by (6),

$$\frac{1}{N} R_N^{(2)} = \frac{2}{q!} \frac{1}{N} g_\alpha(N)^{-q} \sum_{k=1}^{N-1} (N-k)(r_N(k))^q. \tag{9}$$

The limit behavior of $R^{(2)}$ is divided into three cases:

Case 1: $\alpha + 2 > 0$. By Lemma 1, we know that $g_\alpha(N)$ behaves as $\frac{1}{2N}$; then by inequality (7) we have:

$$\begin{aligned} \frac{1}{N} R_N^{(2)} &\leq \frac{2}{q!} C \frac{1}{N} g_\alpha(N)^{-q} \frac{1}{N^{(2+\alpha/2)q}} \sum_{k=1}^{N-1} (N-k)\,(h_N(k))^q \\ &\leq C \frac{2^{q+1}}{q!} N^{q-1} \frac{1}{N^{(2+\alpha/2)q}} \sum_{k=1}^{N-1} (N-k)\,(h_N(k))^q \\ &\leq C \frac{1}{N^{(1+\alpha/2)q}} \sum_{k=1}^{N-1} (h_N(k))^q , \end{aligned}$$

and following the same steps in the proof of Proposition 1 in [8], we obtain

$$\frac{1}{N} R_N^{(2)} \to 0.$$

Case 2: $\alpha + 2 < 0$. By Lemma 1, $g_\alpha(N) \sim \frac{2}{\sqrt{\pi}} N^{\alpha/2}$. Furthermore, by Proposition 2 in [8], the only contribution to the expression of $R_N^{(2)}/N$ in (9) is given when

$k=1$. Therefore, by (6)

$$\begin{aligned}\frac{1}{N}R_N^{(2)} &\sim \frac{2}{q!}\frac{1}{N}N^{-q\alpha/2}\left(\frac{2}{\sqrt{\pi}}\right)^{-q}(N-1)\times\left(N^{\alpha/2}\frac{1}{\sqrt{\pi}}\left(2e^{-\frac{1}{4}N^{-2-\alpha}}-e^{-N^{-2-\alpha}}-1\right)\right.\\ &\quad\left.+\frac{2\sqrt{2}}{N}\left(erf\left(\frac{1}{2N^{1+\alpha/2}}\right)-erf\left(\frac{1}{N^{1+\alpha/2}}\right)\right)\right)^q\\ &\xrightarrow[N\to\infty]{}\frac{1}{2^{q-1}q!}.\end{aligned}$$

Case 3: $\alpha+2=0$. By Lemma 1, we know that $g_\alpha(N)$ behaves as C_0N^{-1}. Therefore,

$$\frac{1}{N}R_N^{(2)}\sim\frac{2}{q!}C_0^{-q}N^{q-1}\sum_{k=1}^{N-1}(N-k)(r_N(k))^q,$$

given that $\alpha=-2$ and (6), we get

$$r_N(k)=N^{-1}(r_N^{(1)}(k)+r_N^{(2)}(k)),$$

where

$$r_N^{(1)}(k)=\frac{1}{\sqrt{\pi}}\left[2\exp\left(-\frac{k^2}{4}\right)-\exp\left(-\frac{(k+1)^2}{4}\right)-\exp\left(-\frac{(k-1)^2}{4}\right)\right]$$

and

$$r_N^{(2)}(k)=\sqrt{2}\left[2|k|erf\left(\frac{|k|}{2}\right)-|k+1|erf\left(\frac{|k+1|}{2}\right)-|k-1|erf\left(\frac{|k-1|}{2}\right)\right].$$

So

$$\frac{1}{N}R_N^{(2)}\sim\frac{2}{q!}C_0^{-q}\sum_{k=1}^{N-1}(r_N^{(1)}(k)+r_N^{(2)}(k))^q$$

and

$$\lim_{N\to\infty}\frac{1}{N}R_N^{(2)}=c_3,$$

where

$$\begin{aligned}c_3=&\frac{2}{q!}C_0^{-q}\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{\pi}}\left[2\exp\left(-\frac{k^2}{4}\right)-\exp\left(-\frac{(k+1)^2}{4}\right)-\exp\left(-\frac{(k-1)^2}{4}\right)\right]\right.\\ &\left.+\sqrt{2}\left[2|k|erf\left(\frac{|k|}{2}\right)-|k+1|erf\left(\frac{|k+1|}{2}\right)-|k-1|erf\left(\frac{|k-1|}{2}\right)\right]\right)^q.\end{aligned}$$

Let us note that the constant c_3 is finite. □

4. Central limit theorem

In this section, we will prove the convergence of the sequence F_N toward the Gaussian law. In fact, we can apply Theorem 1 to the sequence F_N given by (8) since it is a multiple integral of order q, obtaining the following normal convergence result.

Theorem 2. *Let F_N be given by (8) with $q \geq 2$. Then we have*

$$d(F_N, \mathcal{N}(0,1)) \leq C \frac{1}{\sqrt{N}}.$$

Proof. Let us compute the Malliavin derivative of F_N

$$DF_N = \frac{1}{(q-1)!} \frac{1}{v_N} g_\alpha(N)^{-q/2} \sum_{i=0}^{N-1} I^u_{q-1}(1^{\otimes q-1}_{[x_i,x_{i+1}]}) 1_{[x_i,x_{i+1}]},$$

so

$$\begin{aligned}\|DF_N\|^2_{\mathcal{H}} =& \frac{1}{(q-1)!^2} \frac{1}{v_N^2} g_\alpha(N)^{-q} \sum_{i,j=0}^{N-1} I^u_{q-1}(1^{\otimes q-1}_{[x_i,x_{i+1}]}) I^u_{q-1}(1^{\otimes q-1}_{[x_j,x_{j+1}]}) \\ & \times \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle_{\mathcal{H}}.\end{aligned}$$

Product formula (2) implies

$$\begin{aligned}\|DF_N\|^2_{\mathcal{H}} =& \frac{1}{(q-1)!^2} \frac{1}{v_N^2} g_\alpha(N)^{-q} \sum_{r=0}^{q-1} r!(C^{q-1}_r)^2 \sum_{i,j=0}^{N-1} I^u_{2q-2-2r}(1^{\otimes q-1-r}_{[x_i,x_{i+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_j,x_{j+1}]}) \\ & \times \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle^{r+1}_{\mathcal{H}},\end{aligned}$$

where $C^{q-1}_r = \binom{q-1}{r}$ and $\tilde{\otimes}$ denotes the symmetric tensor product. Furthermore, following [14], we can write $\|DF_N\|^2_{\mathcal{H}}$ as

$$\|DF_N\|^2_{\mathcal{H}} = S_{1,N} + S_{2,N}, \tag{10}$$

where

$$S_{1,N} = \frac{1}{(q-1)!} \frac{1}{v_N^2} g_\alpha(N)^{-q} \sum_{i,j=0}^{N-1} \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle^q_{\mathcal{H}}$$

and

$$\begin{aligned}S_{2,N} =& \frac{1}{(q-1)!^2} \frac{1}{v_N^2} g_\alpha(N)^{-q} \sum_{r=0}^{q-2} r!(C^{q-1}_r)^2 \sum_{i,j=0}^{N-1} I^u_{2q-2-2r}(1^{\otimes q-1-r}_{[x_i,x_{i+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_j,x_{j+1}]}) \\ & \times \langle 1_{[x_i,x_{i+1}]}, 1_{[x_j,x_{j+1}]} \rangle^{r+1}_{\mathcal{H}}.\end{aligned}$$

By Theorem 1, we get

$$d(F_N, \mathcal{N}(0,1)) \leq C \left(Var \|DF_N\|^2_{L^2([0,1])} \right)^{1/2} = C \left(\mathbf{E}(S^2_{2,N}) \right)^{1/2}.$$

Clearly, $\mathcal{H} = L^2([0,1])$. Now we will prove that $S_{2,N}$ converges to zero in $L^2(\Omega)$. In fact, by (1),

$$\begin{aligned}
\mathbf{E}(S_{2,N}^2) =& \frac{1}{(q-1)!^4}\frac{1}{v_N^4} g_\alpha(N)^{-2q} \sum_{r_1,r_2=0}^{q-2} r_1! r_2! (C_{r_1}^{q-1})^2 (C_{r_2}^{q-1})^2 \sum_{i_1,i_2,j_1,j_2=0}^{N-1} \\
& \mathbf{E}\left[I^u_{2q-2-2r_1}(1^{\otimes q-1-r_1}_{[x_{i_1},x_{i_1+1}]} \tilde{\otimes} 1^{\otimes q-1-r_1}_{[x_{j_1},x_{j_1+1}]}) I^u_{2q-2-2r_2}(1^{\otimes q-1-r_2}_{[x_{i_2},x_{i_2+1}]} \tilde{\otimes} 1^{\otimes q-1-r_2}_{[x_{j_2},x_{j_2+1}]}) \right] \\
& \times \langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \rangle_{\mathcal{H}}^{r_1+1} \langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \rangle_{\mathcal{H}}^{r_2+1} \\
=& \frac{1}{(q-1)!^4}\frac{1}{v_N^4} g_\alpha(N)^{-2q} \sum_{r=0}^{q-2} r!^2 (C_r^{q-1})^4 \sum_{i_1,i_2,j_1,j_2=0}^{N-1} (2q-2-2r)! \\
& \times \left\langle 1^{\otimes q-1-r}_{[x_{i_1},x_{i_1+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_{j_1},x_{j_1+1}]}, 1^{\otimes q-1-r}_{[x_{i_2},x_{i_2+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}^{\otimes 2(q-r-l)}} \\
& \times \langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \rangle_{\mathcal{H}}^{r+1} \langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \rangle_{\mathcal{H}}^{r+1}.
\end{aligned}$$

We will use the fact that (see [14])

$$\begin{aligned}
& \left\langle 1^{\otimes q-1-r}_{[x_{i_1},x_{i_1+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_{j_1},x_{j_1+1}]}, 1^{\otimes q-1-r}_{[x_{i_2},x_{i_2+1}]} \tilde{\otimes} 1^{\otimes q-1-r}_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}^{\otimes 2(q-r-l)}} \\
&\quad = \sum_{\alpha+\beta=q-1-r} C(r,q,\alpha,\beta) \left\langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \right\rangle_{\mathcal{H}}^{\alpha} \\
&\quad \times \left\langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}}^{\beta} \left\langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \right\rangle_{\mathcal{H}}^{\beta} \\
&\quad \times \left\langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}}^{\alpha};
\end{aligned}$$

here $C(\cdot)$ is a generic constant that does not depend on N. This implies

$$\begin{aligned}
\mathbf{E}(S_{2,N}^2) =& \frac{1}{(q-1)!^4}\frac{1}{v_N^4} g_\alpha(N)^{-2q} \sum_{r=0}^{q-2} r!^2 (C_r^{q-1})^4 (2q-2-2r)! \\
& \times \sum_{\alpha+\beta=q-1-r} C(r,q,\alpha,\beta) a_N(r,q,\alpha,\beta),
\end{aligned}$$

where

$$\begin{aligned}
a_N(r,q,\alpha,\beta) :=& \sum_{i_1,i_2,j_1,j_2=0}^{N-1} \left\langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \right\rangle_{\mathcal{H}}^{\alpha} \left\langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}}^{\beta} \\
& \times \left\langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \right\rangle_{\mathcal{H}}^{\beta} \left\langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \right\rangle_{\mathcal{H}}^{\alpha} \\
& \times \langle 1_{[x_{i_1},x_{i_1+1}]}, 1_{[x_{j_1},x_{j_1+1}]} \rangle_{\mathcal{H}}^{r+1} \langle 1_{[x_{i_2},x_{i_2+1}]}, 1_{[x_{j_2},x_{j_2+1}]} \rangle_{\mathcal{H}}^{r+1} \\
=& \sum_{i_1,i_2,j_1,j_2=0}^{N-1} r_N^{\beta}(|j_1-i_2|) r_N^{\alpha+r+1}(|i_1-j_1|) r_N^{\alpha+r+1}(|i_2-j_2|) r_N^{\beta}(|j_2-i_1|).
\end{aligned}$$

At this point we need to study the limit behavior of $\mathbf{E}(S_{2,N}^2)$ in three different cases:

Case 1: $\alpha+2>0$. Taking into account the behavior of $g_\alpha(N)$ and v_N^2 (Lemma 1 and Lemma 3), we get

$$\begin{aligned}\mathbf{E}(S_{2,N}^2) \sim CN^{2q-2}\sum_{r=0}^{q-2} r!^2 (C_r^{q-1})^4(2q-2-2r)!\\ \times \sum_{\alpha+\beta=q-1-r} C(r,q,\alpha,\beta)a_N(r,q,\alpha,\beta).\end{aligned} \tag{11}$$

Therefore, we need to study the rate of convergence of the following expression:

$$\begin{aligned}&N^{2q-2}a_N(r,q,\alpha,\beta)\\ &\quad = N^{2q-2}\sum_{i_1,i_2,j_1,j_2=0}^{N-1} r_N^{\beta}(|j_1-i_2|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|j_2-i_1|).\end{aligned}$$

We decompose $a_N(r,q,\alpha,\beta)$ as

$$\sum_{i_1,i_2,j_1,j_2} = \sum_{i_1=i_2=j_1=j_2} + \sum_{i_1=i_2=j_1\neq j_2} + \sum_{i_1=i_2\neq j_1\neq j_2} + \sum_{i_1,i_2,j_1,j_2 \text{ distinct}}.$$

Therefore, we can write

$$\mathbf{E}(S_{2,N}^2) = S_{2,1,N}+S_{2,2,N}+S_{2,3,N}+S_{2,4,N}. \tag{12}$$

We will prove that each term in (12) is majorized by $C\frac{1}{N}$.

$$S_{2,1,N} = CN^{2q-2}\sum_{i=0}^{N-1} r_N^{2q}(0) \sim CN^{2q-2}\sum_{i=0}^{N-1}(N^{-1})^{2q} \le C\frac{1}{N},$$

Since we have different bounds for $h_N(k)$ for $-2<\alpha<0$ and $\alpha\ge 0$, we will handle these separately.

$$\begin{aligned}S_{2,2,N} =& CN^{2q-2}\sum_{i,j=0;i>j}^{N-1} r_N^q(0)r_N^q(i-j)\\ \le& CN^{q-1}\frac{1}{N^{q(2+\alpha/2)}}\sum_{k=1}^{N-1} h_N^q(k).\end{aligned}$$

If $-2<\alpha<0$, for N large enough and for $\varepsilon\ge 0$ such that $\epsilon<-\alpha/2$, we have

$$h_N(k) \le \begin{cases} CN^{2\varepsilon} & \text{if } \ k\le [N^{1+\frac{\alpha}{2}+\varepsilon}],\\ Ce^{-N^{2\varepsilon}}N^{-\alpha} & \text{if } \ k\ge [N^{1+\frac{\alpha}{2}+\varepsilon}]+1.\end{cases}$$

Then for ε close to 0 and N close to ∞,

$$
\begin{aligned}
S_{2,2,N} \leq & CN^{q-1}\frac{1}{N^{q(2+\alpha/2)}}\left(\sum_{k=1}^{[N^{1+\alpha/2+\epsilon}]} N^{2q\varepsilon} + \sum_{[N^{1+\alpha/2+\varepsilon}]+1}^{N-1} e^{-qN^{2\varepsilon}}N^{-q\alpha}\right)\\
\leq & CN^{q}\frac{1}{N^{q(2+\alpha/2)}}\sum_{k=1}^{[N^{1+\alpha/2+\epsilon}]} N^{2q\epsilon-2\epsilon}\\
\leq & C\frac{1}{N^{(q-1)(2+\alpha/2)+\epsilon(1-2q)}} \leq C\frac{1}{N},
\end{aligned}
$$

by choosing a very small ϵ. By a similar technique we can handle the term $j_1 = i_2 \neq i_1 \neq j_2$. Therefore,

$$
\begin{aligned}
& S_{2,3,N}\\
& = CN^{2q-2}\sum_{i_1=i_2\neq j_1\neq j_2} r_N^{\beta}(|j_1-i_2|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|j_2-i_1|)\\
& = CN^{2q-2}\sum_{i>j>k} r_N^{\beta}(i-j)r_N^{\alpha+r+1}(i-j)r_N^{\alpha+r+1}(i-k)r_N^{\beta}(j-k)\\
& \leq CN^{2q-2}\frac{1}{N^{(2+\alpha/2)2q}}\sum_{i>j>k} h_N^{\beta}(i-j)h_N^{\alpha+r+1}(i-j)h_N^{\alpha+r+1}(i-k)h_N^{\beta}(j-k).
\end{aligned}
$$

Again, for $\alpha+2>0$, the dominant part of h_N is the one between 1 and $[N^{1+\alpha/2+\epsilon}]$, this implies

$$
S_{2,3,N} \leq C\frac{1}{N^{q(2+\alpha)-2(1+\alpha)-12\epsilon}},
$$

for every $\epsilon>0$, such that $\epsilon<-\alpha/2$. So, by taking ϵ close to zero we can bound the term for every $q\geq 2$ by $\frac{1}{N}$. For the last term in the case $\alpha+2>0$, we have

$$
\begin{aligned}
& S_{2,4,N}\\
& = CN^{2q-2}\sum_{i_1,i_2,j_1,j_2 \text{ distinct}} r_N^{\beta}(|j_1-i_2|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|j_2-i_1|)\\
& \leq CN^{2q-2}\sum_{i_1,i_2,j_1,j_2 \text{ distinct}} r_N^{\beta}(|i_2-j_1|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|i_1-j_2|)\\
& \leq CN^{2q-2}\sum_{i_2>i_1>j_1>j_2} r_N^{\beta}(i_2-j_1)r_N^{\alpha+r+1}(i_1-j_1)r_N^{\alpha+r+1}(i_2-j_2)r_N^{\beta}(i_1-j_2)\\
& \leq CN^{2q-2}\frac{1}{N^{(2+\alpha/2)2q}}\sum_{i_2>i_1>j_1>j_2} h_N^{\beta}(i_2-j_1)h_N^{\alpha+r+1}(i_1-j_1)h_N^{\alpha+r+1}(i_2-j_2)h_N^{\beta}(i_1-j_2).
\end{aligned}
$$

As before, we can get

$$
S_{2,4,N} \leq CN^{2q-2}\frac{1}{N^{(2+\alpha/2)2q}}N^{4(1+\alpha/2+\epsilon)}N^{8q\epsilon} \leq C\frac{1}{N}.
$$

For the case $\alpha \geq 0$, we will use the bound

$$r_N(k) \leq C\frac{1}{N^{2+\alpha/2}}$$

and (11), consequently

$$\mathbf{E}(S_{2,N}^2) \leq CN^{2q-2}N^{-2q(2+\alpha/2)}N^4 \leq N^{2-2q(1+\alpha/2)} \leq C\frac{1}{N}.$$

Case 2: $\alpha + 2 < 0$. From Proposition 2 in [8], we have that

$$r_N(k) \leq Ce^{-N^{-2-\alpha}} \quad \text{if } k \geq 2$$

and

$$r_N(0) \sim CN^{\alpha/2}, \qquad r_N(1) \sim CN^{\alpha/2}.$$

Taking into account (11) and the behavior of $g_\alpha(N)$ and v_N^2 (Lemma 1 and Lemma 3), we obtain

$$\begin{aligned}
&\mathbf{E}(S_{2,N}^2)\\
&\sim C\frac{1}{N^2}N^{-\alpha q}\sum_{i_1,i_2,j_1,j_2=0}^{N-1} r_N^{\beta}(|i_2-j_1|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|i_1-j_2|)\\
&\leq C\frac{1}{N^2}N^{-\alpha q}\sum_k (N^{\alpha/2})^{2q} \leq C\frac{1}{N}.
\end{aligned}$$

Case 3: $\alpha + 2 = 0$. For this case the proof of Proposition 3 in [8] implies that

$$r_N(k) \leq C\frac{1}{N}e^{-\frac{(k-1)^2}{2}},$$

for every $k \geq 1$, hence

$$\begin{aligned}
&\mathbf{E}(S_{2,N}^2)\\
&\sim CN^{2q-2}\sum_{i_1,i_2,j_1,j_2=0}^{N-1} r_N^{\beta}(|j_1-i_2|)r_N^{\alpha+r+1}(|i_1-j_1|)r_N^{\alpha+r+1}(|i_2-j_2|)r_N^{\beta}(|i_1-j_2|)\\
&\leq C\frac{1}{N^2}\sum_{i_1>j_1>i_2>j_2} e^{\frac{\beta(j_1-i_2-1)^2}{2}}e^{\frac{(\alpha+r+1)(i_1-j_1-1)^2}{2}}e^{\frac{(\alpha+r+1)(i_2-j_2-1)^2}{2}}e^{\frac{\beta(i_1-j_2-1)^2}{2}}.
\end{aligned}$$

Now taking $a = j_1 - i_2 - 1$, $b = i_1 - j_1 - 1$, $c = i_2 - j_2 - 1$ and using the bound $e^{-x^2} \leq 1$, we get

$$\mathbf{E}(S_{2,N}^2) \leq C\frac{1}{N}\sum_{a,b,c=1}^{N} e^{\frac{\beta(j_1-i_2-1)^2}{2}}e^{\frac{(\alpha+r+1)(i_1-j_1-1)^2}{2}}e^{\frac{(\alpha+r+1)(i_2-j_2-1)^2}{2}} \leq C\frac{1}{N}. \tag{13}$$

Finally, the result is achieved from (10) to (13). □

5. Almost sure central limit theorem (ASCLT)

Here, we establish and prove our result concerning the ASCLT for Hermite variations as to the solution to the heat equation with moving time. Let us begin with the following definition.

Definition 1. *Let $(G_N)_{N\geq 1}$ be a sequence of real valued random variables defined on a common probability space (Ω, F, P). The sequence $(G_N)_{N\geq 1}$ satisfies an almost sure central limit theorem (ASCLT), if, almost surely, for every bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have*

$$\frac{1}{\log N} \sum_{i=1}^{N} \frac{\varphi(G_i)}{i} \to \mathbf{E}(\varphi(Z)), \quad \text{as } N \to \infty,$$

where Z is an $N(0,1)$ random variable.

In order to prove the main result of this section, since we work with multiple stochastic integrals, we use the following theorem:

Theorem 3 (see [1]). *Fix $q \geq 2$, and let $(F_N)_{N\geq 1}$ be a sequence of random variables defined by $F_N := (I_q(f_N))_{N\geq 1}$ with $f_N \in \mathcal{H}^{\odot q}$, such that for all $N \geq 1$, $\mathbf{E}(F_N^2) = q!\|f_N\|^2_{\mathcal{H}^{\otimes q}} = 1$ and $\|f_N \otimes_r f_N\|^2_{\mathcal{H}^{\otimes 2(q-r)}}$ goes to zero as N goes to ∞, for every $r = 1, \ldots, q-1$. Then, $F_N \xrightarrow[N\to\infty]{Law} Z \sim \mathcal{N}(0,1)$. Moreover, if the following two conditions are satisfied:*

$$(H1) \ \sum_{N\geq 2} \frac{1}{N \log^2 N} \sum_{l=1}^{N} \frac{1}{l} \|f_N \otimes_r f_N\|^2_{\mathcal{H}^{\otimes 2(q-r)}} < \infty, \quad \text{for every } 1 \leq r \leq q-1,$$

$$(H2) \ \sum_{N\geq 2} \frac{1}{N \log^3 N} \sum_{m,l=1}^{N} \frac{\mathbf{E}(F_m F_l)}{ml} < \infty,$$

then $(F_N)_{N\geq 1}$ satisfies an ASCLT.

The previous theorem allows us to provide the following result.

Theorem 4. *The sequence F_N given by (8) with $q \geq 2$ satisfies the ASCLT as $N \to \infty$.*

Proof. In order to prove Theorem 4, we need to check hypotheses $(H1)$ and $(H2)$ in Theorem 3 since by Theorem 2 we know that the sequence $(F_N)_{N\geq 1}$ satisfies a CLT. By relation (8), we have that $F_N = I_q(f_N)$, with

$$f_N = \frac{1}{v_N} \frac{1}{q!} g_\alpha(N)^{-q/2} \sum_{i=0}^{N-1} 1^{\otimes q}_{[x_i, x_{i+1}]}.$$

Now, using the definition of contraction of order r, we obtain

$$\begin{aligned}
&f_l \otimes_r f_l \\
&= \frac{1}{v_l^2}\frac{1}{q!^2}\sum_{j,k=0}^{l-1} g_\alpha(l)^{-q} 1_{[j/l,(j+1)/l]}^{\otimes q} \otimes_r 1_{[k/l,(k+1)/l]}^{\otimes q} \\
&= \frac{1}{v_l^2}\frac{1}{q!^2}\sum_{j,k=0}^{l-1} g_\alpha(l)^{-q} \left\langle 1_{[j/l,(j+1)/l]}^{\otimes r}, 1_{[k/l,(k+1)/l]}^{\otimes r}\right\rangle_{\mathcal{H}^{\otimes r}} 1_{[j/l,(j+1)/l]}^{\otimes (q-r)} \otimes 1_{[k/l,(k+1)/l]}^{\otimes (q-r)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|f_l \otimes_r f_l\|^2_{\mathcal{H}^{\otimes 2(q-r)}} \\
&= \frac{1}{v_l^4}\frac{1}{q!^4}\sum_{j,k,l,m=0}^{l-1}\Big[g_\alpha(l)^{-2q}\left\langle 1_{[j/l,(j+1)/l]}^{\otimes r}, 1_{[k/l,(k+1)/l]}^{\otimes r}\right\rangle_{\mathcal{H}^{\otimes r}} \\
&\quad \times \left\langle 1_{[l/l,(l+1)/l]}^{\otimes r}, 1_{[m/l,(m+1)/l]}^{\otimes r}\right\rangle_{\mathcal{H}^{\otimes r}} \\
&\quad \times \left\langle 1_{[j/l,(j+1)/l]}^{\otimes (q-r)}\tilde{\otimes} 1_{[k/l,(k+1)/l]}^{\otimes (q-r)}, 1_{[l/l,(l+1)/l]}^{\otimes (q-r)}\tilde{\otimes} 1_{[m/l,(m+1)/l]}^{\otimes (q-r)}\right\rangle_{\mathcal{H}^{\otimes 2(q-r)}}\Big].
\end{aligned}$$

Computations similar to the ones in the last section allow us to get

$$\begin{aligned}
&\|f_l \otimes_r f_l\|^2_{\mathcal{H}^{\otimes 2(q-r)}} \\
&= \frac{1}{(q)!^4}\frac{1}{v_l^4} g_\alpha(l)^{-2q}\sum_{r=0}^{q-2} r!^2 (C_r^{q-1})^4 (2(q-r))! \\
&\quad \times \sum_{\alpha,\beta\ge 0;\alpha+\beta=q-r}\ \sum_{\gamma,\delta\ge 0;\gamma+\delta=q-r} C(r,q,\alpha,\beta,\gamma,\delta) z_N(r,q,\alpha,\beta,\gamma,\delta),
\end{aligned}$$

where

$$\begin{aligned}
&z_N(r,q,\alpha,\beta,\gamma,\delta) \\
&= \sum_{i_1,i_2,j_1,j_2=0}^{N-1} \langle 1_{[j_1/N,(j_1+1)/N]}, 1_{[i_2/N,(i_2+1)/N]}\rangle_{\mathcal{H}}^{\beta} \\
&\quad \times \langle 1_{[i_1/N,(i_1+1)/N]}, 1_{[j_1/N,(j_1+1)/N]}\rangle_{\mathcal{H}}^{\alpha} \\
&\quad \times \langle 1_{[i_1/N,(i_1+1)/N]}, 1_{[j_1/N,(j_1+1)/N]}\rangle_{\mathcal{H}}^{r} \langle 1_{[i_2/N,(i_2+1)/N]}, 1_{[j_2/N,(j_2+1)/N]}\rangle_{\mathcal{H}}^{r} \\
&\quad \times \langle 1_{[i_1/N,(i_1+1)/N]}, 1_{[j_2/N,(j_2+1)/N]}\rangle_{\mathcal{H}}^{\beta} \langle 1_{[i_2/N,(i_2+1)/N]}, 1_{[j_2/N,(j_2+1)/N]}\rangle_{\mathcal{H}}^{\alpha} \\
&= \sum_{i_1,i_2,j_1,j_2=0}^{N-1} r_N^{\beta}(|j_1-i_2|) r_N^{\alpha+r}(|i_1-j_1|) r_N^{\alpha+r}(|i_2-j_2|) r_N^{\beta}(|j_2-i_1|).
\end{aligned}$$

Following arguments similar to those in the previous section, we can obtain

$$\|f_l \otimes_r f_l\|^2_{\mathcal{H}^{\otimes 2(q-r)}} \le \frac{C}{l}.$$

This implies

$$\begin{aligned}\sum_{N\geq 2}\frac{1}{N\log^2 N}\sum_{l=1}^{N}\frac{1}{l}\|f_l\otimes_r f_l\|^2_{\mathcal{H}^{\otimes 2(q-r)}} \leq & C\sum_{N\geq 2}\frac{1}{N\log^2 N}\sum_{l=1}^{\infty}\frac{1}{l^2}\\ \leq & C\sum_{N\geq 2}\frac{1}{N\log^2 N};\end{aligned}$$

consequently, condition (H1) is satisfied.

With respect to (H2), we have

$$\mathbf{E}(F_m F_l) = \frac{1}{v_m}\frac{1}{v_l}\frac{1}{q!}g_\alpha(m)^{-q/2}g_\alpha(l)^{-q/2}\sum_{i=0}^{m-1}\sum_{j=0}^{l-1}\langle 1_{[i/m,i+1/m]}, 1_{[j/l,j+1/l]}\rangle^q_{\mathcal{H}}.$$

Assuming that $l < m$ and following the lines of lemmas 1 and 2, for every α, we can get

$$\mathbf{E}(F_m F_l) \leq C\sqrt{\frac{l}{m}}.$$

According to Remark 3.3 in [1], this last inequality implies condition (H2), and the proof is complete. □

Remark 1. *The tools from Malliavin calculus and Stein's method (see [10, 11, 12]) applied in this article can be used for other SPDEs where the Green function is known (see [7, 16, 20], among others).*

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