Neke zanimljive jednakosti u vezi radijusa pripisanih kružnica trokuta i njihove posljedice

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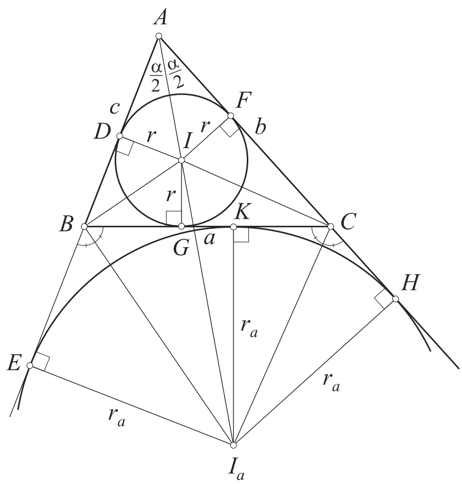
U ovom prilogu ćemo se baviti pripisanim kružnicama trokuta ABC, odnosno jednakostima u vezi polumjera tih kružnica te posljedicama tih jednakosti. Promatrajmo trokut ABC i njemu pripisane kružnice čiji su radijusi r_a, r_b i r_c.



Slika 1.

Najprije ćemo dokazati sljedeće jednakosti:

$$r_a = \frac{P}{s-a}, \quad r_b = \frac{P}{s-b}, \quad r_c = \frac{P}{s-c}, \tag{1}$$

gdje je P površina trokuta ABC, a s njegov poluopseg, tj. $s = \dfrac{a+b+c}{2}$. Dat ćemo dva razna dokaza ovih jednakosti.

Dokaz 1. Promatrajmo trokute ABI_a, ACI_a, BCI_a, gdje je točka I_a središte pripisane kružnice koja dodiruje stranicu $\overline{BC}$ i produžetke stranica $\overline{AB}$ i $\overline{AC}$ $\triangle ABC$. Za njihove površine vrijedi:

$$P_{\triangle ABC} = P_{\triangle ABI_a} + P_{\triangle ACI_a} - P_{\triangle BCI_a},$$

tj.

$$P = \frac{cr_a}{2} + \frac{br_a}{2} - \frac{ar_a}{2}, \qquad (P = P_{\triangle ABC})$$

a odavde

$$P = \frac{r_a}{2}(c+b-a), \qquad r_a = \frac{P}{s-a}.$$

Analogno se dokazuju i druge dvije jednakosti u (1).

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Dokaz 2. Sa slike imamo

$$|AD| = |AF|, \quad |BD| = |BG| \quad \text{i} \quad |CF| = |CG|$$

kao duljine tangentnih dužina upisane kružnice u $\triangle ABC$. Neka je $|AD| = |AF| = x$, $|BD| = |BG| = y$ i $|CF| = |CG| = z$. Vrijedi $x+y = c$, $y+z = a$, $z+x = b$, a odavde nakon zbrajanja ovih jednakosti dobivamo

$$x + y + z = s,$$

a odavde

$$x = s - a, \quad y = s - b, \quad z = s - c,$$

tj.

$$|AD| = |AF| = s - a, \quad |BD| = |BG| = s - b \quad \text{i} \quad |CF| = |CG| = s - c.$$

Očigledno je $|AE| = |AH|$ kao duljine tangentnih dužina iz vrha A na pripisanu kružnicu radijusa r_a. Iz istoga razloga je $|BE| = |BK|$ i $|CH| = |CK|$. Zato imamo:

$$\begin{aligned}2|AE| = 2|AH| &= |AE| + |AH| = |AB| + |BE| + |AC| + |CH| \\ &= |AB| + |BK| + |AC| + |CK| = |AB| + |AC| + (|BK| + |CK|) \\ &= |AB| + |AC| + |BC| = a + b + c = 2s,\end{aligned}$$

a odavde

$$|AE| = |AH| = s.$$

Sada iz sličnih trokuta ADI i AEI_a imamo:

$$|ID| : |AD| = |I_aE| : |AE|$$

ili

$$r : (s - a) = r_a : s,$$

a odavde

$$rs = r_a(s - a)$$

odnosno, zbog $P = rs$,

$$r_a = \frac{P}{s - a}.$$

Sada ćemo dokazati nekoliko jednakosti u vezi r_a, r_b, r_c i iz njih izvesti neke posljedice.

Primjer 1. U svakom trokutu vrijedi jednakost

$$r_a + r_b + r_c = 4R + r, \tag{2}$$

gdje su R i r polumjeri opisane i upisane kružnice $\triangle ABC$.

Dokaz. Iz (1) imamo:

$$\begin{aligned}&r_a + r_b + r_c - r \\ &\quad= \frac{P}{s-a} + \frac{P}{s-b} + \frac{P}{s-c} - \frac{P}{s} \\ &\quad= P \cdot \frac{[s(s-b)(s-c) + s(s-a)(s-c) + s(s-a)(s-b) - (s-a)(s-b)(s-c)]}{s(s-a)(s-b)(s-c)} \\ &\quad= \frac{1}{P}[s(s-a)(s-b) + s(s-b)(s-c) + s(s-a)(s-c) - (s-a)(s-b)(s-c)] \\ &\quad= \frac{1}{P}(2s^3 - as^2 - bs^2 - cs^2 + abc) = \frac{1}{P}[2s^3 - s^2(a+b+c) + abc] \\ &\quad= \frac{abc}{P} = 4R,\end{aligned}$$

odakle slijedi (2).

Posljedica 1. Iz poznate Eulerove nejednakosti $R \geqslant 2r$, imamo $4R+r \geqslant 9r$, odnosno $4R+r \leqslant \frac{9}{2}R$, te iz (2) dobivamo

$$9r \leqslant r_a + r_b + r_c \leqslant \frac{9}{2}R. \tag{3}$$

Jednakost u (3) vrijedi ako i samo ako je $R = 2r \iff r_a = r_b = r_c$, tj. ako i samo ako je trokut jednakostraničan.

Primjer 2. U svakom trokutu vrijedi jednakost

$$\left(\frac{r_a}{r} - 1\right)\left(\frac{r_b}{r} - 1\right)\left(\frac{r_c}{r} - 1\right) = \frac{4R}{r}. \tag{4}$$

Dokaz. Zbog (1) imamo

$$\frac{r_a}{r} = \frac{\frac{P}{s-a}}{\frac{P}{s}} = \frac{s}{s-a} \implies \frac{r_a}{r} - 1 = \frac{s}{s-a} - 1 = \frac{a}{s-a},$$

te analogno

$$\frac{r_b}{r} - 1 = \frac{b}{s-b} \quad \text{i} \quad \frac{r_c}{r} - 1 = \frac{c}{s-c}.$$

Sada dobivamo

$$\begin{aligned}\left(\frac{r_a}{r} - 1\right)\left(\frac{r_b}{r} - 1\right)\left(\frac{r_c}{r} - 1\right) &= \frac{abc}{(s-a)(s-b)(s-c)} \\ &= \frac{abcs}{s(s-a)(s-b)(s-c)} \\ &= \frac{4RPs}{P^2} = \frac{4Rs}{P} = \frac{4Rs}{rs} = \frac{4R}{r}.\end{aligned}$$

Posljedica 2. Iz Eulerove nejednakosti $R \geqslant 2r$ slijedi $\frac{4R}{r} \geqslant 8$ pa iz (4) dobivamo

$$\left(\frac{r_a}{r} - 1\right)\left(\frac{r_b}{r} - 1\right)\left(\frac{r_c}{r} - 1\right) \geqslant 8,$$

gdje jednakost vrijedi ako i samo ako je $R = 2r \iff r_a = r_b = r_c$, tj. ako je trokut jednakostraničan.

Primjer 3. U svakom trokutu vrijedi jednakost

$$a(r_a + r) + b(r_b + r) + c(r_c + r) = 4Rs. \tag{5}$$

Dokaz. Iz (1) imamo

$$r_a + r = \frac{P}{s-a} + \frac{P}{s} = P \cdot \frac{2s-a}{s(s-a)} = P \cdot \frac{b+c}{s(s-a)},$$

te analogno

$$r_b + r = P \cdot \frac{c+a}{s(s-b)}, \qquad r_c + r = P \cdot \frac{a+b}{s(s-c)}.$$

Sada je

$$a(r_a+r)+b(r_b+r)+c(r_c+r)=P\Big[\frac{a(b+c)}{s(s-a)}+\frac{b(c+a)}{s(s-b)}+\frac{c(a+b)}{s(s-c)}\Big]$$
$$=P\cdot\frac{a(b+c)(s-b)(s-c)+b(c+a)(s-a)(s-c)+c(a+b)(s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$
$$=\frac{1}{P}\{ab[(s-b)(s-c)+(s-a)(s-c)]+bc[(s-a)(s-c)+(s-a)(s-b)]$$
$$+ca[(s-b)(s-c)+(s-a)(s-b)]\}$$
$$=\frac{1}{P}\{ab[(s-c)(2s-b-a)]+bc[(s-a)(2s-c-b)]+ca[(s-b)(2s-c-a)]\}$$
$$=\frac{1}{P}[abc(s-c)+abc(s-a)+abc(s-b)]$$
$$=\frac{abc}{P}[3s-(a+b+c)]=\frac{4RP}{P}\cdot s=4Rs.$$

Posljedica 3. Koristeći nejednakosti Mitrinovića $s\leqslant\dfrac{3\sqrt{3}R}{2}$ i $s\geqslant 3\sqrt{3}r$ imamo nejednakosti

$$3\sqrt{3}r\leqslant s\leqslant\frac{3\sqrt{3}R}{2},$$

a odavde

$$12\sqrt{3}Rr\leqslant 4Rs\leqslant 6\sqrt{3}R^2.$$

Sada iz (5) dobivamo

$$12\sqrt{3}Rr\leqslant a(r_a+r)+b(r_b+r)+c(r_c+r)\leqslant 6\sqrt{3}R^2,$$

gdje jednakost vrijedi ako i samo ako je $R=2r \iff r_a=r_b=r_c$, tj. $a=b=c$.

Primjer 4. U svakom trokut vrijedi jednakost

$$\frac{a^2}{r_br_c}+\frac{b^2}{r_cr_a}+\frac{c^2}{r_ar_b}=4\Big(\frac{R}{r}-1\Big). \tag{6}$$

Dokaz. Sa slike 1 iz $\triangle AEI_a$ vidimo

$$r_a=s\cdot\operatorname{tg}\frac{\alpha}{2},$$

te analogno

$$r_b=s\cdot\operatorname{tg}\frac{\beta}{2}\quad\text{i}\quad r_c=s\cdot\operatorname{tg}\frac{\gamma}{2}.$$

Zato koristeći teorem o sinusima imamo

$$\begin{aligned}\frac{a^2}{r_br_c}&=\frac{4R^2\sin^2\alpha}{s^2\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}}=\frac{4R^2\cdot 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{s^2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\\&=\frac{4R^2\sin^2\frac{\alpha}{2}\cos\frac{\alpha}{2}\cdot 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{s^2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}.\end{aligned}\tag{7}$$

Nadalje,

$$\begin{aligned}
s &= \frac{a+b+c}{2} = R(\sin\alpha + \sin\beta + \sin\gamma) \\
&= R\Big(2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\Big) \\
&= 2R\Big[\sin\Big(\frac{\pi}{2}-\frac{\gamma}{2}\Big)\cos\frac{\alpha-\beta}{2} + 2\sin\Big(\frac{\pi}{2}-\frac{\alpha+\beta}{2}\Big)\cos\frac{\gamma}{2}\Big] \\
&= 2R\Big(\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma}{2}\Big) \\
&= 2R\cos\frac{\gamma}{2}\Big(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\Big) \\
&= 2R\cos\frac{\gamma}{2}\cdot 2\cos\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}}{2}\cos\frac{\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}}{2} \\
&= 4R\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\Big(-\frac{\beta}{2}\Big) = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.
\end{aligned}$$

Dakle,

$$4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = s. \tag{8}$$

Sada iz (7) i (8) dobivamo

$$\frac{a^2}{r_b r_c} = \frac{4R\sin^2\frac{\alpha}{2}\cos\frac{\alpha}{2}}{s\cdot\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} \tag{9}$$

i analogno

$$\frac{b^2}{r_c r_a} = \frac{4R\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}}{s\cdot\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}} \quad \text{i} \quad \frac{c^2}{r_a r_b} = \frac{4R\sin^2\frac{\gamma}{2}\cos\frac{\gamma}{2}}{s\cdot\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}. \tag{10}$$

Nakon zbrajanja (9) i (10), dobivamo

$$\begin{aligned}
S &= \frac{a^2}{r_b r_c} + \frac{b^2}{r_c r_a} + \frac{c^2}{r_a r_b} = \frac{4R}{s}\left(\frac{\sin^2\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} + \frac{\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}}{\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}} + \frac{\sin^2\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\right) \\
&= \frac{4R\cdot 4R\Big(\sin^3\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin^3\frac{\beta}{2}\cos\frac{\beta}{2} + \sin^3\frac{\gamma}{2}\cos\frac{\gamma}{2}\Big)}{s\cdot 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}.
\end{aligned} \tag{11}$$

Iz poznatih jednakosti

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \quad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

dobivamo

$$\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}=\frac{(s-b)(s-c)}{bc}\cdot\frac{(s-c)(s-a)}{ca}\cdot\frac{(s-a)(s-b)}{ab}$$
$$=\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2},$$

a odavde

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{\frac{P^2}{s}}{4RP}=\frac{P}{4Rs}=\frac{rs}{4Rs}=\frac{r}{4R},$$
$$r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}. \qquad (12)$$

Iz (11) i (12) slijedi

$$S=\frac{8R^2\Big(\sin^2\frac{\alpha}{2}\sin\alpha+\sin^2\frac{\beta}{2}\sin\beta+\sin^2\frac{\gamma}{2}\sin\gamma\Big)}{sr}$$
$$=\frac{8R^2}{sr}\Big(\sin\alpha\cdot\frac{1-\cos\alpha}{2}+\sin\beta\cdot\frac{1-\cos\beta}{2}+\sin\gamma\cdot\frac{1-\cos\gamma}{2}+\Big)$$
$$=\frac{4R^2}{sr}\Big[(\sin\alpha+\sin\beta+\sin\gamma)-\frac{1}{2}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)\Big]$$
$$=\frac{4R^2}{sr}\Big(\frac{s}{R}-\frac{sR}{R^2}\Big),$$

jer je

$$\sin\alpha+\sin\beta+\sin\gamma=\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}=\frac{1}{2R}(a+b+c)=\frac{s}{R},$$

te

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\cdot\sin\beta\cdot\sin\gamma$$
$$=4\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}$$
$$=\frac{abc}{2R^3}=\frac{4RP}{2R^3}$$
$$=\frac{2P}{R^2}=\frac{2rs}{R^2}.$$

Najzad dobivamo

$$S=4\Big(\frac{R}{r}-1\Big).$$

Posljedica 4. Iz Eulerove nejednakosti $R\geqslant 2r$, tj. $\frac{R}{r}\geqslant 2$ te $4\Big(\frac{R}{r}-1\Big)\geqslant 4$, iz jednakosti (6) dobivamo

$$\frac{a^2}{r_br_c}+\frac{b^2}{r_cr_a}+\frac{c^2}{r_ar_b}\geqslant 4,$$

pri čemu jednakost vrijedi ako i samo ako je $R=2r \iff r_a=r_b=r_c \iff a=b=c$.

Primjer 5. Trokut *ABC* je pravokutan ako i samo ako je $r+r_a+r_b+r_c=2s$.

Dokaz. 1° Ako je trokut pravokutan, tada je

$$r=\frac{a+b-c}{2} \qquad \text{i} \qquad c=2R.$$

Iz (2) imamo

$$r_a+r_b+r_c=4R+r.$$

Zato je

$$r+r_a+r_b+r_c=r+4R+r=2r+4R=a+b-c+2c=a+b+c=2s.$$

2° Dokazat ćemo i obrat ove tvrdnje, tj. ako vrijedi $r+r_a+r_b+r_c=2s$, tada je trokut pravokutan. Iz (2) imamo

$$\begin{aligned} r+r_a+r_b+r_c=2s &\Longrightarrow r+r+4R=2s \Longrightarrow r+2R=s \\ &\Longrightarrow (s-c)\operatorname{tg}\frac{\gamma}{2}+\frac{c}{\sin\gamma}=s. \end{aligned}$$

Uz supstituciju $\operatorname{tg}\frac{\gamma}{2}=t$, imamo $\sin\gamma=\dfrac{2\operatorname{tg}\frac{\gamma}{2}}{1+\operatorname{tg}\frac{\gamma}{2}}=\dfrac{2t}{1+t^2}$, te dalje redom:

$$\begin{aligned} &(s-c)t+c\frac{1+t^2}{2t}=s \\ &(2s-c)t^2-2st+c=0 \\ &2st(t-1)-c(t^2-1)=0 \\ &(t-1)(2st-ct-c)=0 \\ &t-1=0 \quad \text{ili} \quad (2s-c)t=c, \end{aligned}$$

tj.

$$t_1=1 \quad \text{ili} \quad t_2=\frac{c}{2s-c}.$$

Za $t_1=1$, tj. $\operatorname{tg}\frac{\gamma}{2}=1$ slijedi $\gamma=90°$ što znači da je trokut pravokutan s pravim kutom pri vrhu tog trokuta.

Za $\operatorname{tg}\frac{\gamma}{2}=\dfrac{c}{2s-c}$ imamo:

$$\begin{aligned} \operatorname{tg}\frac{\gamma}{2}&=\frac{c}{a+b}=\frac{2R\sin\gamma}{2R(\sin\alpha+\sin\beta)}=\frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} \\ &=\frac{\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\sin\left(90°-\frac{\gamma}{2}\right)\cos\frac{\alpha-\beta}{2}}=\frac{\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}}=\frac{\sin\frac{\gamma}{2}}{\cos\frac{\alpha-\beta}{2}}, \end{aligned}$$

tj.

$$\operatorname{tg}\frac{\gamma}{2}=\frac{\sin\frac{\gamma}{2}}{\cos\frac{\alpha-\beta}{2}} \;\Big/ \;:\left(\sin\frac{\gamma}{2}\neq 0\right)$$

$$\Longrightarrow \cos\frac{\gamma}{2} = \cos\frac{\alpha-\beta}{2} \Longrightarrow \frac{\gamma}{2} = \pm\frac{\alpha-\beta}{2} \Longrightarrow \gamma = \pm(\alpha-\beta).$$

Može biti:

$$\gamma = \alpha - \beta \Longrightarrow \alpha = \beta + \gamma \Longrightarrow \alpha = 90^\circ,$$

ili

$$\gamma = \beta - \alpha \Longrightarrow \beta = \alpha + \gamma \Longrightarrow \beta = 90^\circ.$$

U svakom slučaju trokut je pravokutan.

Napomena 1. Nakon duljeg računanja pokazuje se da je jednakost $r + 2R = s$, gdje je

$$R = \frac{abc}{4P},$$

$$r = \frac{P}{s},$$

$$P = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^3)}$$

ekvivalentna jednakosti

$$(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) = 0,$$

a odavde slijedi

$$a^2 + b^2 - c^2 = 0 \quad \text{ili} \quad b^2 + c^2 - a^2 = 0 \quad \text{ili} \quad c^2 + a^2 - b^2 = 0,$$

tj.

$$a^2 + b^2 = c^2 \quad \text{ili} \quad b^2 + c^2 = a^2 \quad \text{ili} \quad c^2 + a^2 = b^2,$$

što znači da je trokut pravokutan s pravim kutom pri vrhu C ili pri vrhu A ili pri vrhu B.

Napomena 2. Budući da je $r + r_a + r_b + r_c = 2s \iff r + 2R = s$, primjer 5 možemo iskazati u obliku: Trokut je pravokutan ako i samo ako je $r + 2R = s$.

Navodimo još niz zanimljivih jednakosti u trokutu.

1. $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$,
2. $r_ar_b + r_br_c + r_cr_a = s^2$,
3. $r_ar_br_c = sP$,
4. $r_a + r_b = \dfrac{c(s-c)}{r} = \dfrac{sc}{rc}$, $c = \dfrac{r_c(r_a + r_b)}{s}$,
5. $(r_a + r_b)(r_b + r_c)(r_c + r_a) = 4Rs^2$,
6. $r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2$,
7. $\dfrac{1}{r_a^2} + \dfrac{1}{r_b^2} + \dfrac{1}{r_c^2} = \dfrac{s^2 - 2r^2 - 8Rr}{P^2}$,
8. $r_a - r = \dfrac{ar_a}{s}$, $r_b - r = \dfrac{br_b}{s}$, $r_c - r = \dfrac{cr_c}{s}$,

9. $ar_a + br_b + cr_c = 2s(2R - r)$,

10. $\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = \frac{2}{s}(4R + r)$,

11. $ar_br_c + br_ar_c + cr_ar_b = 2P(4R + r)$,

12. $a^2r_a + b^2r_b + c^2r_c = 4s^2(R - r)$,

13. $ar_a^2 + br_b^2 + cr_c^2 = 2s(8R^2 + 2Rr - s^2)$,

14. $a^2r_a^2 + b^2r_b^2 + c^2r_c^2 = 2s^2(8R^2 + r^2 - s^2)$,

15. $ar_ar_b + br_br_c + cr_cr_a = s(s^2 - r^2 - 4Rr)$,

16. $\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{(4R + r)^2 + s^2}{4Rs}$,

17. $\frac{1}{ar_a} + \frac{1}{br_b} + \frac{1}{cr_c} = \frac{s^2 + r^2 - 8Rr}{4Rsr^2}$,

18. $\frac{a^2}{r_a} + \frac{b^2}{r_b} + \frac{c^2}{r_c} = 4(R + r)$,

19. $\frac{r_a + r_b}{c} + \frac{r_b + r_c}{a} + \frac{r_c + r_a}{b} = \frac{s}{r}$,

20. $r_a + r_b = \frac{4R}{ab} r_ar_b$,

21. $h_c = \frac{2r_ar_b}{r_a + r_b}$, $h_b = \frac{2r_ar_c}{r_a + r_c}$, $h_a = \frac{2r_br_c}{r_b + r_c}$,

22. $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$,

23. $\frac{a}{r_a^2} + \frac{b}{r_b^2} + \frac{c}{r_c^2} = \frac{2(2R - r)}{sr}$,

24. $\frac{ab}{r_ar_b} + \frac{bc}{r_br_c} + \frac{ac}{r_ar_c} = 1 + \left(\frac{4R + r}{s}\right)^2$,

25. $\frac{a^2}{r_a^2} + \frac{b^2}{r_b^2} + \frac{c^2}{r_c^2} = 2\left[\left(\frac{4R + r}{s}\right)^2 - 1\right]$.

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