Polialternativne konačne sume u eksplicitnom obliku, 1. dio

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U ovom članku su dane sume polialternativnih konačnih redova u eksplicitnom obliku. Izvedene su formule za ove sume:

$$1+2-3-4+5+6-\ldots+f_2(n)n$$
$$1+2+3-4-5-6+\ldots+f_3(n)n$$
$$1^2+2^2-3^2-4^2+5^2+6^2-\ldots+f_2(n)n^2$$
$$1^3+2^3-3^3-4^3+5^3+6^3-\ldots+f_2(n)n^3$$
$$1^2+2^2+3^2-4^2-5^2-6^2+\ldots+f_3(n)n^2$$
$$1\cdot 2+2\cdot 3-3\cdot 4-4\cdot 5+5\cdot 6+6\cdot 7-\ldots+f_2(n)n\,(n+1)\,,\ldots$$

gdje je $f_k(n)=\operatorname{sgn}\left(\sin\dfrac{2n-1}{2k}\pi\right)$ *za* $k,\ n\in\mathbb{N}$*, a dani su primjeri suma konvergentnih bialternativnih redova, npr.*

$$\frac{1}{1}+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\ldots=\frac{\pi}{4}+\frac{1}{2}\ln 2.$$

Ponekad treba generirati predznak u konačnim ili beskonačnim sumama. Tako npr. ako imamo sumu $\sum\limits_{m=1}^{n\ (\text{ili }\infty)} a_m$, tada je alternativna suma $\sum\limits_{m=1}^{n\ (\text{ili }\infty)}(-1)^{m+1}a_m$. Naime, tu se pojavljuje preslikavanje, koje određuje predznak člana alternativne sume, koji se izmjenično ponavlja. Ta se funkcija može definirati s

$$f_1:\mathbb{N}\to\{-1,1\},\tag{1}$$

gdje je $\mathbb{N}=\{1,2,3,\ldots\}$ domena, a dvočlani skup $\{-1,1\}$ kodomena. Preslikavanje elemenata je dano formulom

$$f_1(n)=(-1)^{n+1},\tag{2}$$

gdje je $n\in\mathbb{N}$. Očito da se tim preslikavanjem dobiva signativni niz $1,\ -1,\ 1,\ -1,\ 1,\ -1,\ldots$

Sada ćemo definiciju signativnosti članova niza i reda poopćiti.

Definicija 1. Niz

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ldots\tag{3}$$

se zove *alternativni signativni niz* ili *monoalternativni signativni niz*; niz

$$1,\ 1,\ -1,\ -1,\ 1,\ 1,\ -1,\ -1,\ldots\tag{4}$$

se zove *bialternativni signativni niz*; a niz

$$\underbrace{1,1,\ldots,1,}_{k}\ \underbrace{-1,-1,\ldots,-1,}_{k}\ \underbrace{1,1,\ldots,1,}_{k}\ \underbrace{-1,-1,\ldots,-1,}_{k}\ \ldots\tag{5}$$

se zove *polialternativni signativni niz* ili k*-alternativni signativni niz*, gdje je $k\in\mathbb{N}$.

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Postavlja se pitanje kako naći funkciju koja će generirati ove nizove. Ako je $k=2$, formula za generiranje bialternativnog signativnog niza (4) mogla bi se dati s

$$f_2(n)=\operatorname{sgn}\left(\sin\frac{2n-1}{4}\pi\right). \tag{6}$$

Dakle, $f_2(1)=1$, $f_2(2)=1$, $f_2(3)=-1$, $f_2(4)=-1$, $f_2(5)=1$, $f_2(6)=1,\dots$

Teorem 1. Dokazati da funkcija

$$f_k:\mathbb{N}\to\{-1,1\} \tag{7}$$

gdje je

$$f_k(n)=\operatorname{sgn}\left(\sin\frac{2n-1}{2k}\pi\right),\quad k,\ n\in\mathbb{N} \tag{8}$$

generira *k-alternativni signativni niz.*

Dokaz. Nađimo rješenje nejednadžbe

$$\sin\frac{2n-1}{2k}\pi>0. \tag{9}$$

Dakle,

$$\sin\frac{2n-1}{2k}\pi>0 \implies 0<\frac{2n-1}{2k}\pi<\pi \quad \text{tj.} \quad n=1,2,\dots,k. \tag{10}$$

Analognim razmatranjem za nejednadžbu,

$$\sin\frac{2n-1}{2k}\pi<0 \tag{11}$$

uz iste uvjete, dobivamo

$$\sin\frac{2n-1}{2k}\pi<0 \implies \pi<\frac{2n-1}{2k}\pi<2\pi \quad \text{tj.} \quad n=k+1,k+2,\dots,2k. \tag{12}$$

Iz definicije sinusa iz (10) i (12) slijedi (8), pa je time teorem dokazan.

Definicija 2. Ako je $f_k(n)=\operatorname{sgn}\left(\sin\frac{2n-1}{2k}\pi\right)$, $k,\ n\in\mathbb{N}$ tada nizu

$$a_1,\ a_2,\ a_3,\ a_4,\dots \tag{13}$$

pripadaju sljedeći nizovi: *alternativni signativni niz* ili *monoalternativni signativni niz*

$$a_1,\ -a_2,\ a_3,\ -a_4,\dots \tag{14}$$

ili u oznaci

$$f_1(1)a_1,\ f_1(2)a_2,\ f_1(3)a_3,\ f_1(4)a_4,\dots \tag{15}$$

bialternativni signativni niz

$$a_1,\ a_2,\ -a_3,\ -a_4,\ a_5,\ a_6,\ -a_7,\ -a_8,\dots \tag{16}$$

ili u oznaci

$$f_2(1)a_1,\ f_2(2)a_2,\ f_2(3)a_3,\ f_2(4)a_4,\dots \tag{17}$$

i općenito *polialternativni signativni niz* (*k-alternativni signativni niz*)

$$\underbrace{a_1,a_2,\dots,a_k}_{k},\ \underbrace{-a_{k+1},-a_{k+2},\dots,-a_{2k}}_{k},\ \underbrace{a_{2k+1},a_{2k+2},\dots,a_{3k}}_{k},\dots \tag{18}$$

ili u oznaci

$$f_k(1)a_1,\ f_k(2)a_2,\ f_k(3)a_3,\ f_k(4)a_4,\dots \tag{19}$$

Definicija 3. Ako je $f_k(n) = \operatorname{sgn}\left(\sin \dfrac{2n-1}{2k}\pi\right)$, k, $n \in \mathbb{N}$, redu

$$a_1 + a_2 + a_3 + a_4 + \ldots \tag{20}$$

pripadaju sljedeći redovi: *monoalternativni signativni red*

$$a_1 - a_2 + a_3 - a_4 + \ldots \tag{21}$$

ili u oznaci

$$f_1(1)a_1 + f_1(2)a_2 + f_1(3)a_3 + f_1(4)a_4 + \ldots \tag{22}$$

bialternativni signativni red

$$a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - \ldots \tag{23}$$

ili u oznaci

$$f_2(1)a_1 + f_2(2)a_2 + f_2(3)a_3 + f_2(4)a_4 + \ldots \tag{24}$$

i općenito *polialternativni signativni red* (k*-alternativni signativni red*)

$$\underbrace{a_1 + a_2 + \ldots + a_k}_{k} \underbrace{-a_{k+1} - a_{k+2} - \ldots - a_{2k}}_{k} \underbrace{+a_{2k+1} + a_{2k+2} + \ldots + a_{3k}}_{k} - \ldots \tag{25}$$

ili u oznaci

$$f_k(1)a_1 + f_k(2)a_2 + f_k(3)a_3 + f_k(4)a_4 + \ldots \tag{26}$$

Definicija 4. Ako je $p = 0, 1, 2, 3, \ldots$ definirat ćemo ove oznake za sume:

$$S_p(n) = 1^p + 2^p + 3^p + \ldots + n^p, \tag{27}$$

$$\overline{S}_p(n) = 1^p - 2^p + 3^p - \ldots + (-1)^{n+1}n^p, \tag{28}$$

$$\overline{S}_{k,p}(n) = f_k(1)1^p + f_k(2)2^p + f_k(3)3^p + \ldots + f_k(n)n^p. \tag{29}$$

Iz D4 je

$$S_0(n) = n, \tag{30}$$

$$\overline{S}_0(n) = \frac{1-(-1)^n}{2}, \tag{31}$$

$$\overline{S}_{1,p}(n) = \overline{S}_p(n). \tag{32}$$

Nadalje ćemo se prvenstveno baviti konačnim redovima rješavajući zadatke, koji proizlaze iz danih definicija.

Zadatak 1. Dokažimo

$$\overline{S}_{2,1}(4n) = \sum_{m=1}^{4n} f_2(m)m = 1 + 2 - 3 - 4 + \ldots - (4n-1) - (4n) = -4n. \tag{33}$$

Rješenje. Imamo

$$\overline{S}_1(n) = 1 - 2 + 3 - \ldots + (-1)^{n+1}n = \frac{1+(-1)^{n+1}(2n+1)}{4}. \tag{34}$$

Koristeći (34) promatrajmo sumu

$$\begin{aligned}\overline{S}_{2,1}(4n) &= \sum_{m=1}^{4n} m \cdot \operatorname{sgn}\left(\sin \frac{2m-1}{4}\pi\right)\\ &= 1 + 2 - 3 - 4 + \ldots - (4n-1) - (4n) = (1 - 3 + 5 - \ldots - (4n-1))\\ &\quad + 2(1 - 2 + 3 - \ldots - (2n))\\ &= ((2\cdot 1 - 1) - (2\cdot 2 - 1) + (2\cdot 3 - 1) - \ldots - (2(2n) - 1) + 2\overline{S}_1(2n)\end{aligned}$$

$$= 2\overline{S}_1(2n) - \underbrace{(1-1+1-\ldots-1)}_{2n} + 2\overline{S}_1(2n) = 4\overline{S}_1(2n)$$
$$= 1 - 4n - 1 = -4n,$$
dakle (33) vrijedi.

Provjera. Ako je $n=1$, izravnim računom se dobije $\overline{S}_{2,1}(4\cdot 1) = 1+2-3-4 = -4$, a ista vrijednost se dobije i iz (33): $\overline{S}_{2,1}(4\cdot 1) = -4\cdot 1 = -4$.

Ako je $n=2$, izravnim računom se dobije $\overline{S}_{2,1}(4\cdot 2) = 1+2-3-4+5+6-7-8 = -8$, a ista vrijednost se dobije i iz (33): $\overline{S}_{2,1}(4\cdot 2) = -4\cdot 2 = -8$, što smo i očekivali.

Napomena 1. Sada ćemo koristiti $\overline{S}_{2,1}(4n)$ za izvođenje općenitije formule $\overline{S}_{2,1}(n)$. I u sljedećim zadatcima ćemo prije izvoditi specijalne slučajeve, a onda poopćenja. To znači, da moramo prije izvesti formulu za sumu $\overline{S}_{k,p}(2kn)$, a onda izvoditi poopćenje $\overline{S}_{k,p}(n)$.

Napomena 2. Suvišno je napominjati, da barem jedna netočna vrijednost, koja se dobije provjeravanjem izvedene sume negira njezinu točnost. No, ako provjerom dobijemo za $2k$ uzastopnih točnu vrijednost sume, izvedena formula je vjerojatno točna, ali i ne mora biti. Ta vrijednost se računa izravnom provjerom sume, sumiranjem član po član, i primjenom izvedene formule. No, ponekad ćemo provjeriti formulu samo za jednu vrijednost varijable, a točnost tog ishoda daje malu vjerojatnost za opću točnost. Sve to radimo jer se zbog glomaznih formula i njihovog izvoda lako može napraviti greška.

Napomena 3. Sume za opći oblik $\overline{S}_{k,p}(n)$ će sadržavati *funkciju pod* (*function floor*) $\lfloor\ \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, koja realnom broju x pridružuje najveći cijeli broj koji nije veći od x. Svakako, da se sumiranje malog broja članova imenovanog reda može uraditi i bez ove formule, ali se veliki broj članova ne bi mogao sumirati bez ove formule u razumnom vremenu.

Zadatak 2. Dokažimo
$$\begin{aligned}\overline{S}_{2,1}(n) &= 1+2-3-4+5+6-\ldots+f_2(n)n\\ &= 1\cdot\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right)\\ &\quad+(n+1)\cdot\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right)\\ &\quad+\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)-n\cdot\left(1-\operatorname{sgn}\left(\frac{n}{4}-\left\lfloor\frac{n}{4}\right\rfloor\right)\right)\end{aligned} \tag{35}$$

Rješenje. Iz (33) slijedi:
$$1+2-3-4+\ldots-(4n-4)+(4n-3) = 1, \tag{36}$$
$$1+2-3-4+\ldots+(4n-3)+(4n-2) = (4n-2)+1, \tag{37}$$
$$1+2-3-4+\ldots+(4n-2)-(4n-1) = 0, \tag{38}$$
$$1+2-3-4+\ldots-(4n-1)-4n = -4n. \tag{39}$$
Dakle, iz ovih formula dobivamo
$$1+2-3-4+\ldots+f_2(n)n = 1\cdot\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right), \tag{40}$$

$$1+2-3-4+\ldots+f_2(n)n=(n+1)\cdot\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right), \tag{41}$$

$$1+2-3-4+\ldots+f_2(n)n=\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right), \tag{42}$$

$$1+2-3-4+\ldots+f_2(n)n=(-n)\cdot\left(1-\operatorname{sgn}\left(\frac{n}{4}-\left\lfloor\frac{n}{4}\right\rfloor\right)\right). \tag{43}$$

Ako zbrojimo desne strane od (40) do (43), dobivamo (35), što smo i trebali pokazati.

Provjera. Sada ćemo napraviti provjeru, koja potvrđuje vjerojatnu, ne i sigurnu, točnost (35).

Ako je $n=1$, izravnim računom dobijemo, $\overline{S}_{2,1}(1)=1$, a istu vrijednost dobijemo i iz (35): $\overline{S}_{2,1}(1)=1\cdot\left(1-\operatorname{sgn}\left(\frac{1+3}{4}-\left\lfloor\frac{1+3}{4}\right\rfloor\right)\right)+0+0-0=1$.

Ako je $n=2$, izravnim računom dobijemo, $\overline{S}_{2,1}(2)=1+2=3$, a istu vrijednost dobijemo i iz (35): $\overline{S}_{2,1}(2)=0+(2+1)\cdot\left(1-\operatorname{sgn}\left(\frac{2+2}{4}-\left\lfloor\frac{2+2}{4}\right\rfloor\right)\right)+0-0=3$, a to smo i očekivali.

Ako je $n=3$, izravnim računom dobijemo, $\overline{S}_{2,1}(3)=1+2-3=0$, a istu vrijednost dobijemo i iz (35): $\overline{S}_{2,1}(3)=0+0+\operatorname{sgn}\left(\frac{3+1}{4}-\left\lfloor\frac{3+1}{4}\right\rfloor\right)-0=0$, a to smo i očekivali.

Ako je $n=4$, izravnim računom dobijemo, $\overline{S}_{2,1}(4)=1+2-3-4=-4$, a istu vrijednost dobijemo i iz (35): $\overline{S}_{2,1}(4)=0+0+0-4\cdot\left(1-\operatorname{sgn}\left(\frac{4}{4}-\left\lfloor\frac{4}{4}\right\rfloor\right)\right)=-4$, a to smo i očekivali.

Zadatak 3. Dokažimo

$$\overline{S}_{3,1}(6n)=\sum_{m=1}^{6n}f_3(m)\,m=-9n. \tag{44}$$

Rješenje. Iz (34) imamo

$$\overline{S}_1(2n)=1-2+3-\ldots+(-1)^{2n+1}(2n)=\frac{1+(-1)^{2n+1}(4n+1)}{4}=-n. \tag{45}$$

Iz (44), dobijemo

$$\begin{aligned}
&1+2+3-4-5-6+\ldots-(6n-2)-(6n-1)-6n\\
&=\frac{3}{2}(1+3)-\frac{3}{2}(4+6)+\frac{3}{2}(7+9)-\ldots-\frac{3}{2}((6n-2)+6n)\\
&=3\,[2-5+8-\ldots-(6n-1)]\\
&=3\,[(3\cdot1-1)-(3\cdot2-1)+(3\cdot3-1)-\ldots-(3\cdot2n-1)]\\
&=9\,(1-2+3-\ldots-(2n))-3\,\underbrace{(1-1+1-\ldots+1-1)}_{2n}=9(-n)-3(0)=-9n.
\end{aligned}$$

Provjera. Ako $n=1$, izravno je $1+2+3-4-5-6=-9$, a pomoću (44): $\overline{S}_{3,1}(6\cdot1)=-9\cdot1=-9$, a to smo i očekivali.

Zadatak 4. Dokažimo

$$
\begin{aligned}
\overline{S}_{3,1}(n) &= 1+2+3-4-5-6+\ldots+f_3(n)n=\sum_{m=1}^{n} f_3\,(m)\,m\\
&= \frac{-n+3}{2}\left(1-\operatorname{sgn}\left(\frac{n+5}{6}-\left\lfloor\frac{n+5}{6}\right\rfloor\right)\right)\\
&\quad+\frac{n+4}{2}\left(1-\operatorname{sgn}\left(\frac{n+4}{6}-\left\lfloor\frac{n+4}{6}\right\rfloor\right)\right)\\
&\quad+\frac{3(n+1)}{2}\left(1-\operatorname{sgn}\left(\frac{n+3}{6}-\left\lfloor\frac{n+3}{6}\right\rfloor\right)\right)\\
&\quad+\frac{n}{2}\left(1-\operatorname{sgn}\left(\frac{n+2}{6}-\left\lfloor\frac{n+2}{6}\right\rfloor\right)\right)\\
&\quad-\frac{-n-1}{2}\left(1-\operatorname{sgn}\left(\frac{n+1}{6}-\left\lfloor\frac{n+1}{6}\right\rfloor\right)\right)+\frac{3n}{2}\left(1-\operatorname{sgn}\left(\frac{n}{6}-\left\lfloor\frac{n}{6}\right\rfloor\right)\right)
\end{aligned} \tag{46}
$$

Rješenje. Redom imamo:

$$1+2+3-4-5-6+\ldots-(6n-7)-(6n-6)+(6n-5)=-3n+4=-\frac{1}{2}\,(6n-5)+\frac{3}{2}, \tag{47}$$

$$1+2+3-4-5-6+\ldots-(6n-6)+(6n-5)+(6n-4)=3n=\frac{1}{2}\,(6n-4)+2, \tag{48}$$

$$1+2+3-4-5-6+\ldots+(6n-5)+(6n-4)+(6n-3)=9n-3=\frac{3}{2}\,(6n-3)+\frac{3}{2}, \tag{49}$$

$$1+2+3-4-5-6+\ldots+(6n-4)+(6n-3)-(6n-2)=3n-1=\frac{1}{2}\,(6n-2)\,, \tag{50}$$

$$1+2+3-4-5-6+\ldots+(6n-3)-(6n-2)-(6n-1)=-3n=-\frac{1}{2}\,(6n-1)-\frac{1}{2}, \tag{51}$$

$$1+2+3-4-5-6+\ldots-(6n-2)-(6n-1)-6n=-9n=\frac{3}{2}\,(-6n)\,. \tag{52}$$

Jednostavnije je ići od (52) do (47).

Ako u (47) izraz $(6n-5)$ zamijenimo s $f_3(n)n$, ali uz uvjet $(n+5)\equiv 0 \pmod 6$, tada je $\sum_{m=1}^{n} f_3\,(m)\,m=\frac{-n+3}{2}\left(1-\operatorname{sgn}\left(\frac{n+5}{6}-\left\lfloor\frac{n+5}{6}\right\rfloor\right)\right)$. Analognim zaključivanjem i u ostalim slučajevima nakon sumiranja dobivamo (46).

Provjera. Ako $n=5$, izravno je $1+2+3-4-5=-3$, a pomoću (46):

$$
\begin{aligned}
\overline{S}_{3,1}(5) &= \frac{-5+3}{2}\left(1-\operatorname{sgn}\left(\frac{5+5}{6}-\left\lfloor\frac{5+5}{6}\right\rfloor\right)\right)\\
&\quad+\frac{5+4}{2}\left(1-\operatorname{sgn}\left(\frac{5+4}{6}-\left\lfloor\frac{5+4}{6}\right\rfloor\right)\right)\\
&\quad+\frac{3(5+1)}{2}\left(1-\operatorname{sgn}\left(\frac{5+3}{6}-\left\lfloor\frac{5+3}{6}\right\rfloor\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{2}\left(1-\operatorname{sgn}\left(\frac{5+2}{6}-\left\lfloor\frac{5+2}{6}\right\rfloor\right)\right)\\
&-\frac{5+1}{2}\left(1-\operatorname{sgn}\left(\frac{5+1}{6}-\left\lfloor\frac{5+1}{6}\right\rfloor\right)\right)+\frac{3\cdot 5}{2}\left(1-\operatorname{sgn}\left(\frac{5}{6}-\left\lfloor\frac{5}{6}\right\rfloor\right)\right)\\
=&\ldots=-3.
\end{aligned}
$$

Zadatak 5. Dokažimo

$$
\overline{S}_{2,2}(4n)=\sum_{m=1}^{4n} f_2(m)\, m^2=-16n^2-4n. \tag{53}
$$

Rješenje. Navedimo poznatu formulu

$$
\overline{S}_2(n)=1^2-2^2+3^2-\ldots+(-1)^{n+1}n^2=\frac{1}{2}(-1)^{n+1}n(n+1). \tag{54}
$$

Iz (34) i (54), promatrajmo sumu

$$
\begin{aligned}
\overline{S}_{2,2}(4n)&=\sum_{m=1}^{4n} m^2 \operatorname{sgn}\left(\sin\frac{2m-1}{4}\pi\right)\\
&=1^2+2^2-3^2-4^2+5^2+6^2-\ldots-(4n-1)^2-(4n)^2\\
&=(1^2-3^2+5^2-\ldots-(4n-1)^2)+4(1^2-2^2+3^2-\ldots-(2n)^2)\\
&=(2\cdot 1-1)^2-(2\cdot 2-1)^2+(2\cdot 3-1)^2-\ldots-(2\cdot(2n)-1)^2+4\overline{S}_2(2n)\\
&=8\overline{S}_2(2n)-4\overline{S}_1(2n)=8\left(-\frac{1}{2}\right)2n\,(2n+1)-4\frac{1-(4n+1)}{1}=-16n^2-4n
\end{aligned}
$$

dakle (53) je točno.

Provjera. Ako $n=1$, izravno je $1^2+2^2-3^2-4^2=-20$, a pomoću (53): $\overline{S}_{2,2}(4\cdot 1)=-16\cdot 1^2-4\cdot 1=-20$.

Zadatak 6. (poopćenje Z5) Dokažimo

$$
\begin{aligned}
\overline{S}_{2,2}(n)&=1^2+2^2-3^2-4^2+5^2+6^2-\ldots+f_2(m)\,n^2=\sum_{m=1}^{n} f_2(m)\, m^2\\
&=n\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right)\\
&\quad+(n^2+n-1)\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right)\\
&\quad-(n+1)\left(1-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right)-(n^2+n)\left(1-\operatorname{sgn}\left(\frac{n}{4}-\left\lfloor\frac{n}{4}\right\rfloor\right)\right).
\end{aligned} \tag{55}
$$

Rješenje. Iz (53) slijedi:

$$
1^2+2^2-3^2-4^2+\ldots-(4n-4)^2+(4n-3)^2=(4n-2)-2=(4n-3), \tag{56}
$$

$$
1^2+2^2-3^2-4^2+\ldots+(4n-3)^2+(4n-2)^2=(4n-1)^2-4n=(4n-2)^2+(4n-2)-1, \tag{57}
$$

$$
1^2+2^2-3^2-4^2+\ldots+(4n-2)^2-(4n-1)^2=-4n=-(4n-1)-1, \tag{58}
$$

$$1^2+2^2-3^2-4^2+\ldots-(4n-1)^2-(4n)^2=-16n^2-4n=-(4n)^2-(4n). \quad (59)$$

Napomenimo, da je ovdje zgodno ići od (59) do (56). Iz ovih relacija slijedi (55), jer smo: u (54) izraz $4n-3$ zamijenili s n, ako je $(n+3)\equiv 0 \pmod 4$; u (55) izraz $4n-2$ zamijenili s n, ako je $(n+2)\equiv 0 \pmod 4$; u (58) izraz $4n-1$ zamijenili s n, ako je $(n+1)\equiv 0 \pmod 4$; u (56) izraz $4n$ zamijenili s n, ako je $n\equiv 0 \pmod 4$.

Provjera. Provjerimo (55) npr. za $n=7$: $1^2+2^2-3^2-4^2+5^2+6^2-7^2=-8$. Ako sada uzmemo sumand $-(n+1)\left(1-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right)$ na desnoj strani (53), jer su ostali 0, dobivamo $-(7+1)\left(1-\operatorname{sgn}\left(\frac{8}{4}-\left\lfloor\frac{8}{4}\right\rfloor\right)\right)=-8$, i to je točno.

Zadatak 7. Dokažimo

$$\begin{aligned}\overline{S}_{3,2}(n)&=1^2+2^2+3^2-4^2-5^2-6^2+\ldots+f_3(n)n^2=\sum_{m=1}^{n} f_3(m)\,m^2\\
&=\frac{-n^2+3n}{2}\left(1-\operatorname{sgn}\left(\frac{n+5}{6}-\left\lfloor\frac{n+5}{6}\right\rfloor\right)\right)\\
&\quad+\frac{n^2+5n-4}{2}\left(1-\operatorname{sgn}\left(\frac{n+4}{6}-\left\lfloor\frac{n+4}{6}\right\rfloor\right)\right)\\
&\quad+\frac{3n^2+3n-8}{2}\left(1-\operatorname{sgn}\left(\frac{n+3}{6}-\left\lfloor\frac{n+3}{6}\right\rfloor\right)\right)\\
&\quad+\frac{n^2-3n-8}{2}\left(1-\operatorname{sgn}\left(\frac{n+2}{6}-\left\lfloor\frac{n+2}{6}\right\rfloor\right)\right)\\
&\quad+\frac{-n^2-5n-4}{2}\left(1-\operatorname{sgn}\left(\frac{n+1}{6}-\left\lfloor\frac{n+1}{6}\right\rfloor\right)\right)\\
&\quad+\frac{-3n^2-3n}{2}\left(1-\operatorname{sgn}\left(\frac{n}{6}-\left\lfloor\frac{n}{6}\right\rfloor\right)\right).\end{aligned} \quad (60)$$

Rješenje. Pokažimo najprije

$$\overline{S}_{3,2}(6n)=\sum_{m=1}^{n} f_3(6n)\,(6n)^2=-54n^2-9n. \quad (61)$$

Redom imamo:

$$\begin{aligned}&1^2+2^2+3^2-4^2-5^2-6^2+7^2+8^2+9^2-\ldots-(6n-2)^2-(6n-1)^2-(6n)^2\\
&=(1^2-4^2+7^2-\ldots-(6n-2)^2)+(2^2-5^2+8^2-\ldots-(6n-1)^2)\\
&\quad+(3^2-6^2+9^2-\ldots-(6n)^2)\\
&=((3\cdot1-2)^2-(3\cdot2-2)^2+(3\cdot3-2)^2-\ldots-(3(2n)-2)^2)\\
&\quad+((3\cdot1-1)^2-(3\cdot2-1)^2+(3\cdot3-1)^2-\ldots-(3(2n)-1)^2)\\
&\quad+9(1^2-2^2+3^2-\ldots-(2n)^2)\end{aligned}$$

$$= 9\overline{S}_2(2n) - 12\overline{S}_1(2n) + 9\overline{S}_2(2n) - 6\overline{S}_1(2n) + 9\overline{S}_2(2n) = 27\overline{S}_2(2n) - 18\overline{S}_1(2n)$$
$$= 27\frac{1}{2}(-1)2n\,(2n+1) - \frac{9}{2}\,(1-4n-1) = -54n^2 - 9n.$$

Sada ćemo lijevu stranu od (61) "skraćivati", tako da zadnji član lijeve strane te jednakosti prebacujemo na desnu stranu i transformiramo, a odatle slijede jednakosti:

$$1^2+2^2+3^2-4^2-5^2-6^2+\ldots-(6n)^2 = -54n^2-9n = -\frac{3}{2}(6n)^2 - \frac{3}{2}(6n), \quad (62)$$

$$1^2+2^2+3^2-\ldots-(6n-1)^2 = -\frac{1}{2}(6n)^2 - \frac{3}{2}\,(6n) = -\frac{1}{2}(6n-1)^2 - \frac{5}{2}\,(6n-1) - 2, \quad (63)$$

$$1^2+2^2+3^2-\ldots-(6n-2)^2 = \frac{1}{2}(6n-1)^2 - \frac{5}{2}\,(6n-1) - 2 = \frac{1}{2}(6n-2)^2 - \frac{3}{2}\,(6n-2) - 4, \quad (64)$$

$$1^2+2^2+3^2-\ldots+(6n-3)^2 = \frac{3}{2}(6n-2)^2 - \frac{3}{2}\,(6n-2) - 4 = \frac{3}{2}(6n-3)^2 + \frac{3}{2}\,(6n-3) - 4, \quad (65)$$

$$1^2+2^2+3^2-\ldots+(6n-4)^2 = \frac{1}{2}(6n-3)^2 + \frac{3}{2}\,(6n-3) - 4 = \frac{1}{2}(6n-4)^2 + \frac{5}{2}\,(6n-4) - 2, \quad (66)$$

$$1^2+2^2+3^2-\ldots+(6n-5)^2 = -\frac{1}{2}(6n-4)^2 + \frac{5}{2}\,(6n-4) - 2 = -\frac{1}{2}(6n-5)^2 + \frac{3}{2}\,(6n-5). \quad (67)$$

Kao u Z6 zaključujemo da vrijedi (60).

Provjera. Ako je $n = 1$, izravno je $\overline{S}_{3,2}(1) = 1^2 = 1$, a pomoću (60) je $\overline{S}_{3,2}(1) = \dfrac{-1^2+3\cdot 1}{2}\left(1-\operatorname{sgn}\left(\dfrac{1+5}{6} - \left\lfloor \dfrac{1+5}{6} \right\rfloor\right)\right) + 0+0+0+0+0 = 1.$

Ako je $n = 2$, izravno je $\overline{S}_{3,2}(2) = 1^2+2^2 = 5$, a pomoću (60) je $\overline{S}_{3,2}(2) = 0 + \dfrac{2^2+5\cdot 2-4}{2}\left(1-\operatorname{sgn}\left(\dfrac{2+4}{6} - \left\lfloor \dfrac{2+4}{6} \right\rfloor\right)\right) + 0+0+0+0 = 5.$

Ako je $n = 3$, izravno je $\overline{S}_{3,2}(3) = 1^2+2^2+3^2 = 14$, a pomoću (60) je $\overline{S}_{3,2}(3) = 0+0+\dfrac{3\cdot 3^2+3\cdot 3-8}{2}\left(1-\operatorname{sgn}\left(\dfrac{3+3}{6} - \left\lfloor \dfrac{3+3}{6} \right\rfloor\right)\right) + 0+0+0 = \ldots = 14.$

Ako je $n = 4$, izravno je $\overline{S}_{3,2}(4) = 1^2+2^2+3^2-4^2 = -2$, a pomoću (60) je $\overline{S}_{3,2}(4) = 0+0+0+\dfrac{4^2-3\cdot 4-8}{2}\left(1-\operatorname{sgn}\left(\dfrac{4+2}{6} - \left\lfloor \dfrac{4+2}{6} \right\rfloor\right)\right) + 0+0 = \ldots = -2.$

Ako je $n = 5$, izravno je $\overline{S}_{3,2}(5) = 1^2+2^2+3^2-4^2-5^2 = -27$, a pomoću (60) je $\overline{S}_{3,2}(5) = 0+0+0+0+\dfrac{-5^2-5\cdot 5-4}{2}\left(1-\operatorname{sgn}\left(\dfrac{5+1}{6} - \left\lfloor \dfrac{5+1}{6} \right\rfloor\right)\right) + 0 = \ldots = -27.$

Ako je $n = 6$, izravno je $\overline{S}_{3,2}(6) = 1^2+2^2+3^2-4^2-5^2-6^2 = -63$, a pomoću (60) je $\overline{S}_{3,2}(6) = 0+0+0+0+0+\dfrac{-3\cdot 6^2-3\cdot 6}{2}\left(1-\operatorname{sgn}\left(\dfrac{n}{6} - \left\lfloor \dfrac{n}{6} \right\rfloor\right)\right) = \ldots = -63.$

Odredimo jednu "jubilarnu" sumu:

$$\begin{aligned}\overline{S}_{3,2}(2021) &= 1^2+2^2+3^2-4^2-5^2-6^2+7^2+\ldots+2019^2-2020^2-2021^2\\ &= \frac{-2021^2-5\cdot 2021-4}{2}\left(1-\operatorname{sgn}\left(\frac{2021+1}{6}-\left\lfloor\frac{2021+1}{6}\right\rfloor\right)\right)\\ &= -4\,094\,550.\end{aligned}$$

Dakle, zadnja dva člana sume su negativna jer je $2021 \equiv 5 \pmod 6$.

Napomena 4. Sada ćemo ove rezultate primijeniti na neke složenije sume.

Zadatak 8. Dokažimo

$$\begin{aligned}S(n) &= 1\cdot 2+2\cdot 3-3\cdot 4-4\cdot 5+\ldots+f_2(n)n\,(n+1)\\ &= (n+1)\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right)\\ &\quad+\left(n^2+2n\right)\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right)\\ &\quad-(n+1)\left(1-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right)-\left(n^2+2n\right)\left(1-\operatorname{sgn}\left(\frac{n}{4}-\left\lfloor\frac{n}{4}\right\rfloor\right)\right).\end{aligned} \tag{68}$$

Uputa. Očito je $S(n)=\overline{S}_{2,2}(n)+\overline{S}_{2,1}(n)$, pa ako tu uvrstimo (55) i (35), dobivamo (68).

Provjera. Ako je $n=7$, izravna vrijednost je $S(7)=1\cdot 2+2\cdot 3-3\cdot 4-4\cdot 5+5\cdot 6+6\cdot 7-7\cdot 8=\ldots=-8$, a iz (68) slijedi $S(7)=0+0-(7+1)\left(1-\operatorname{sgn}\left(\frac{7+1}{4}-\left\lfloor\frac{7+1}{4}\right\rfloor\right)\right)-0=-8$, što je istinito.

Zadatak 9. Dokažimo

$$\begin{aligned}\overline{S}_{2,0}(n) &= f_2(1)+f_2(2)+f_2(3)+\ldots+f_2(n)\\ &= 4-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)-2\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\\ &\quad-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\end{aligned} \tag{69}$$

Rješenje. Budući je

$$\begin{aligned}\overline{S}_{2,0}(1)&=\overline{S}_{2,0}(5)=\ldots=\overline{S}_{2,0}\,(4m+1)=1\\ \overline{S}_{2,0}(2)&=\overline{S}_{2,0}(6)=\ldots=\overline{S}_{2,0}\,(4m+2)=2\\ \overline{S}_{2,0}(3)&=\overline{S}_{2,0}(7)=\ldots=\overline{S}_{2,0}\,(4m+3)=1\\ \overline{S}_{2,0}(4)&=\overline{S}_{2,0}(8)=\ldots=\overline{S}_{2,0}\,(4m)=0,\end{aligned}$$

gdje je $m=0,1,2,\ldots,$ zaključujemo

$$\begin{aligned}\overline{S}_{2,0}(n) &= 1\cdot\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right)+2\cdot\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right)\\ &\quad+1\cdot\left(1-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right),\end{aligned} \tag{70}$$

a nakon sređivanja dobivamo (69).

Provjera. Ako je $n=5$, izravno je $\overline{S}_{2,0}(5)=1+1-1-1+1=1$ i pomoću (69) $\overline{S}_{2,0}(5)=4-0-2-1=1$, pa je (69) u tom slučaju istinito.

Zadatak 10. Dokažimo

$$\begin{aligned}\overline{S}_{3,0}(n)&=f_3(1)+f_3(2)+f_3(3)+\ldots+f_3(n)\\&=9-\operatorname{sgn}\left(\frac{n+5}{6}-\left\lfloor\frac{n+5}{6}\right\rfloor\right)-2\operatorname{sgn}\left(\frac{n+4}{6}-\left\lfloor\frac{n+4}{6}\right\rfloor\right)\\&\quad-3\operatorname{sgn}\left(\frac{n+3}{6}-\left\lfloor\frac{n+3}{6}\right\rfloor\right)\\&\quad-2\operatorname{sgn}\left(\frac{n+2}{6}-\left\lfloor\frac{n+2}{6}\right\rfloor\right)-\operatorname{sgn}\left(\frac{n+1}{6}-\left\lfloor\frac{n+1}{6}\right\rfloor\right).\end{aligned}\tag{71}$$

Uputa. Analogni postupak kao u Z9.

Provjera. Ako je $n=5$, izravno je $\overline{S}_{3,0}(5)=1+1+1-1-1=1$, a pomoću (71) je također $\overline{S}_{3,0}(5)=9-1-2-3-2-0=1$, što smo i očekivali.

Zadatak 11. (Generalizacija)

$$\begin{aligned}\overline{S}_{k,0}(n)=k^2-\sum_{i=1}^{k}i\cdot\operatorname{sgn}\left(\frac{n+2k-i}{2k}-\left\lfloor\frac{n+2k-i}{2k}\right\rfloor\right)\\+\sum_{i=1}^{k-1}i\cdot\operatorname{sgn}\left(\frac{n+i}{2k}-\left\lfloor\frac{n+i}{2k}\right\rfloor\right).\end{aligned}\tag{72}$$

Uputa. Dokaz se bazira na sumi

$$\begin{aligned}1+2+\ldots+(k-1)+k+(k-1)+\ldots2+1&=2\,(1+2+\ldots(k-1))+k\\&=2\cdot\frac{(k-1)\,k}{2}+k=k^2.\end{aligned}$$

Zadatak 12. Dokažimo

$$\begin{aligned}S(n)&=1\cdot3+3\cdot5-5\cdot7-7\cdot9+\ldots+f_2(n)\,(2n-1)\,(2n+1)\\&=(4n-1)\left(1-\operatorname{sgn}\left(\frac{n+3}{4}-\left\lfloor\frac{n+3}{4}\right\rfloor\right)\right)\\&\quad+\left(4n^2+4n-6\right)\left(1-\operatorname{sgn}\left(\frac{n+2}{4}-\left\lfloor\frac{n+2}{4}\right\rfloor\right)\right)\\&\quad-(4n+5)\left(1-\operatorname{sgn}\left(\frac{n+1}{4}-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right)\\&\quad-\left(4n^2+4n\right)\left(1-\operatorname{sgn}\left(\frac{n}{4}-\left\lfloor\frac{n}{4}\right\rfloor\right)\right).\end{aligned}\tag{73}$$

Uputa. Budući je $S(n)=4\overline{S}_{2,2}(n)-\overline{S}_{2,0}(n)$, ako tu uvrstimo (55) i (70), dobivamo (73).

Provjera. Ako je $n=3$, izravno dobijemo $S(3)=1\cdot3+3\cdot5-5\cdot7=-17$.

Iz (73) je $S(3)=0+0-17+0=-17$, a time smo provjerili formulu za $n=3$.

(nastavit će se u sljedećem broju)