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LIMIT THEOREMS FOR NUMBERS SATISFYING A CLASS OF TRIANGULAR ARRAYS

Igoris Belovas
Vilnius University, Lithuania

Abstract. The paper extends the investigations of limit theorems for numbers satisfying a class of triangular arrays, defined by a bivariate linear recurrence with bivariate linear coefficients. We obtain the partial differential equation and special analytical expressions for the numbers using a semi-exponential generating function. We apply the results to prove the asymptotic normality of special classes of the numbers and specify the convergence rate to the limiting distribution. We demonstrate that the limiting distribution is not always Gaussian.

1. Introduction

Let us consider the numbers $a_{n,k}$, satisfying a class of triangular arrays, defined by a bivariate linear recurrence with bivariate linear coefficients.

Definition 1.1. *Let Ψ be a real non-zero matrix,*

$$\Psi = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \tag{1.1}$$

then

$$a_{n,k} = \begin{cases} 1, & \text{for } n=0 \text{ and } k=0, \\ 0, & \text{for } n<k \text{ or } n<0 \text{ or } k<0, \\ (\psi_{1,1}n+\psi_{1,2}k+\psi_{1,3})a_{n-1,k-1} & \\ +(\psi_{2,1}n+\psi_{2,2}k+\psi_{2,3})a_{n-1,k}\,, & \text{otherwise.} \end{cases} \tag{1.2}$$

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Numbers a_{nk} include many combinatorial numbers, for instance, binomial coefficients, k-permutations of n without repetition, Morgan numbers, Stirling numbers of the first kind and the second kind, non-central Stirling numbers, Eulerian numbers, Lah numbers, numbers of the tribonacci triangle, see [4], as well as some generalizations of the numbers mentioned above (cf., e.g., [26, 29] and the references therein).

In this research, we establish limit theorems for numbers satisfying a class of triangular arrays, extending, particularly, the investigations of Canfield, Kyriakoussis and Vamvakari, see [11, 21, 22, 23, 24, 25]. The paper is organized as follows. The first part is the introduction. Section 2 shows how the underlying recurrence relation translates into a partial differential equation for the corresponding bivariate semi-exponential generating function. We receive special analytical expressions of the numbers a_{nk} as well. In Section 3, central limit theorems for some special cases of the numbers a_{nk} are proved. The rates of convergence to the limiting distribution are specified. In Section 4, we discuss the findings of Kyriakoussis on asymptotic normality of the numbers, defined by a bivariate linear recurrence with bivariate linear coefficients (see [21, Corollary 2.1]), and present a counterexample to his result. We prove that the limiting distribution for the class is not always Gaussian.

Throughout this paper, we denote by C_n^k the binomial coefficients, by $W(x)$ - the Lambert W function, by $\Phi(x)$ - the cumulative distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt, \qquad x \in \mathbb{R}.$$

By $S(n,k)$ we denote the Stirling numbers of the second kind, counting the number of partitions of a set of size n into k disjoint non-empty subset, and by $A(n,k)$ - the Eulerian numbers, counting the number of permutations of the numbers 1 to n in which precisely k elements are greater than the previous element. Next, $T_n(x)$, $A_n(x)$ and $\omega_n(x)$ stand for the Touchard, the Eulerian and the geometric polynomials respectively,

$$T_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \quad A_n(x) = \sum_{k=0}^{n} A(n,k)x^k, \quad \omega_n(x) = \sum_{k=0}^{n} k!S(n,k)x^k.$$

All limits, unless specified otherwise, are taken as $n \to \infty$.

2. Generating functions and analytic expressions of the numbers, satisfying a class of triangular arrays

We may view the recurrent expression for the numbers a_{nk} (1.2) as a partial difference equation with linear coefficients. First, let us introduce the semi-exponential generating function of the numbers,

$$F(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{n,k}\frac{x^n}{n!}y^k=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_{n,k}\frac{x^n}{n!}y^k. \tag{2.1}$$

Next we establish the following theorem.

Theorem 2.1. *Let*

$$\xi_1=\psi_{1,1}+\psi_{1,2}+\psi_{1,3},\qquad \xi_2=\psi_{2,1}+\psi_{2,3},$$

then the generating function $F(x,y)$ satisfies the linear first-order partial differential equation

$$(1-\psi_{1,1}xy-\psi_{2,1}x)F_x-(\psi_{1,2}y^2+\psi_{2,2}y)F_y=(\xi_1 y+\xi_2)F, \tag{2.2}$$

with the initial condition $F(0,y)=1$.

Proof. By the definition of the numbers $a_{n,k}$ (1.2), we have

$$\begin{aligned} a_{n,k}&=(\psi_{1,1}n+\psi_{1,2}k+\psi_{1,3})a_{n-1,k-1}\\ &\quad+(\psi_{2,1}n+\psi_{2,2}(k+1)+(\psi_{2,3}-\psi_{2,2}))a_{n-1,k}. \end{aligned} \tag{2.3}$$

Substituting the expression into the generating function (2.1), we get

$$\begin{aligned} F(x,y)&=\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_{n,k}\frac{x^n}{n!}y^k+\sum_{n=0}^{\infty}a_{n,0}\frac{x^n}{n!}\\ &=\psi_{1,1}\sum_{n=1}^{\infty}\sum_{k=1}^{n}na_{n-1,k-1}\frac{x^n}{n!}y^k+\psi_{1,2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}ka_{n-1,k-1}\frac{x^n}{n!}y^k\\ &\quad+\psi_{1,3}\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_{n-1,k-1}\frac{x^n}{n!}y^k+\psi_{2,1}\sum_{n=1}^{\infty}\sum_{k=1}^{n}na_{n-1,k}\frac{x^n}{n!}y^k\\ &\quad+\psi_{2,2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}(k+1)a_{n-1,k}\frac{x^n}{n!}y^k\\ &\quad+(\psi_{2,3}-\psi_{2,2})\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_{n-1,k}\frac{x^n}{n!}y^k+\sum_{n=0}^{\infty}a_{n,0}\frac{x^n}{n!}. \end{aligned} \tag{2.4}$$

Next,

$$
\begin{aligned}
F(x,y) &= \psi_{1,1}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} n a_{n-1,k}\frac{x^n}{n!}y^{k+1} + \psi_{1,2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(k+1)a_{n-1,k}\frac{x^n}{n!}y^{k+1} \\
&\quad + \psi_{1,3}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} a_{n-1,k}\frac{x^n}{n!}y^{k+1} + \psi_{2,1}\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}(n+1)a_{n,k}\frac{x^{n+1}}{(n+1)!}y^k \\
&\quad + \psi_{2,2}\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}(k+1)a_{n,k}\frac{x^{n+1}}{(n+1)!}y^k \\
&\quad + (\psi_{2,3}-\psi_{2,2})\sum_{n=0}^{\infty}\sum_{k=1}^{n+1} a_{n,k}\frac{x^{n+1}}{(n+1)!}y^k + \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!}.
\end{aligned}
\tag{2.5}
$$

Thus we obtain

$$
\begin{aligned}
F(x,y) &= \psi_{1,1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1)a_{n,k}\frac{x^{n+1}}{(n+1)!}y^{k+1} \\
&\quad + \psi_{1,2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)a_{n,k}\frac{x^{n+1}}{(n+1)!}y^{k+1} \\
&\quad + \psi_{1,3}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k}\frac{x^{n+1}}{(n+1)!}y^{k+1} + \psi_{2,1}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k}\frac{x^{n+1}}{n!}y^k \\
&\quad - \psi_{2,1}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{n!} + \psi_{2,2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)a_{n,k}\frac{x^{n+1}}{(n+1)!}y^k \\
&\quad - \psi_{2,2}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{(n+1)!} + (\psi_{2,3}-\psi_{2,2})\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k}\frac{x^{n+1}}{(n+1)!}y^k \\
&\quad - (\psi_{2,3}-\psi_{2,2})\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!}.
\end{aligned}
\tag{2.6}
$$

Hence, by the definition of the generating function $F(x,y)$ in (2.1), we receive

$$
\begin{aligned}
F &= \psi_{1,1}xyF + \psi_{1,2}y\int_0^x (yF)_y dt + \psi_{1,3}y\int_0^x F dt + \psi_{2,1}xF \\
&\quad + \psi_{2,2}\int_0^x (yF)_y dt + (\psi_{2,3}-\psi_{2,2})\int_0^x F dt + g(x),
\end{aligned}
\tag{2.7}
$$

with

$$
\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!} - \psi_{2,1}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{n!} \\
&\quad - \psi_{2,2}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{(n+1)!} - (\psi_{2,3}-\psi_{2,2})\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!} - \psi_{2,1}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{n!} - \psi_{2,3}\sum_{n=0}^{\infty} a_{n,0}\frac{x^{n+1}}{(n+1)!}.
\end{aligned} \tag{2.8}
$$

Next, we prove that $g(x) \equiv 1$. Indeed,

$$
\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!} - \sum_{n=0}^{\infty} a_{n,0}(\psi_{21}(n+1)+\psi_{2,3})\frac{x^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!} - \sum_{n=0}^{\infty} a_{n+1,0}\frac{x^{n+1}}{(n+1)!} = a_{0,0} = 1.
\end{aligned} \tag{2.9}
$$

Thus, using (2.7), we obtain

$$
\begin{aligned}
(1-\psi_{1,1}xy-\psi_{2,1}x)F &= (\psi_{1,2}y+\psi_{2,2})\int_0^x (yF)_y\,dt \\
&\quad + (\psi_{1,3}y+\psi_{2,3}-\psi_{2,2})\int_0^x F\,dt + 1.
\end{aligned} \tag{2.10}
$$

Calculating the derivative of the expression with respect to x, we receive

$$
\begin{aligned}
&(-\psi_{1,1}y-\psi_{2,1})F + (1-\psi_{1,1}xy-\psi_{2,1}x)F_x \\
&\quad = (\psi_{1,2}y+\psi_{2,2})(yF_y+F) + (\psi_{1,3}y+\psi_{2,3}-\psi_{2,2})F,
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-\psi_{1,1}xy-\psi_{2,1}x)F_x - (\psi_{1,2}y^2+\psi_{2,2}y)F_y \\
&\quad = ((\psi_{1,1}+\psi_{1,2}+\psi_{1,3})y + (\psi_{2,1}+\psi_{2,3}))F,
\end{aligned}
$$

with $F(0,y)=1$ yielding us the statement of the theorem. ∎

REMARK 2.2. Using a substitution $F(x,y)=\Theta(x,y)A(y)$, we can reduce the linear partial differential equation (2.2) into its homogeneous form.

REMARK 2.3. The dual numbers $\tilde{a}_{n,k} := a_{n,n-k}$ are generated by the matrix

$$
\begin{pmatrix} \psi_{2,1}+\psi_{2,2} & -\psi_{2,2} & \psi_{2,3} \\ \psi_{1,1}+\psi_{1,2} & -\psi_{1,2} & \psi_{1,3} \end{pmatrix}. \tag{2.11}
$$

The double semi-exponential generating function $\tilde{F}(x,y)$ of the dual numbers (2.11) equals $\tilde{F}(x,y) = F(xy, y^{-1})$.

Solving the linear first-order partial differential equation (2.2), we obtain the generating function $F(x, y)$. The formal Taylor series in two variables for the generating function equals

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\partial^{n+k}}{\partial x^n \partial y^k}F(x,y)\bigg|_{(0,0)}\right)\frac{x^n y^k}{n!k!}.$$

Hence, the partial differentiation of the double semi-exponential generating function $F(x, y)$ at $(0, 0)$, yields the analytic expressions of the numbers

$$a_{n,k} = \frac{1}{k!}\frac{\partial^{n+k}}{\partial x^n \partial y^k}F(x,y)\bigg|_{(0,0)}. \tag{2.12}$$

Note that, for $\psi_{1,1} = 0$, we can separate arguments while solving the linear partial differential equation (2.2). Hence the next theorem follows.

THEOREM 2.4. *For $\psi_{1,2}, \psi_{2,1}, \psi_{2,2} \neq 0$, numbers generated by the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}$$

(i) *have the generating function*

$$F(x,y) = (1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{-\frac{\xi_1}{\psi_{1,2}}}, \tag{2.13}$$

(ii) *and the analytic expression*

$$a_{n,k} = \frac{\prod_{j=1}^{k}(\psi_{1,2}j+\psi_{1,3})}{k!(\psi_{2,2})^k}\sum_{m=0}^{k}(-1)^m C_k^m \prod_{s=1}^{n}(\psi_{2,2}(k-m)+\psi_{2,1}s+\psi_{2,3}). \tag{2.14}$$

PROOF. By Theorem 2.1, the numbers correspond the differential equation

$$(1-\psi_{2,1}x)F_x - (\psi_{1,2}y^2+\psi_{2,2}y)F_y = (\xi_1 y + \xi_2)F, \qquad F(0,y) = 1. \tag{2.15}$$

We can solve the linear first-order partial differential equation using the method of characteristics (cf. [13, 16, 20]).

I. Along the line Γ: $x = \eta_1(t) = 0$, $y = \eta_2(t) = t$, $F = 1$ we have

$$\Delta = (1-\psi_{2,1}\eta_1(t))y_t - (-(\psi_{1,2}\eta_2^2(t)+\psi_{2,2}\eta_2(t)))x_t = 1 \neq 0. \tag{2.16}$$

Thus, there exists a solution to the Cauchy problem.

II. Characteristic equations, corresponding (2.15) linear first-order partial differential equation, are

$$\frac{dx}{1-\psi_{2,1}x} = \frac{dy}{-(\psi_{1,2}y^2+\psi_{2,2}y)} = \frac{dF}{(\xi_1 y+\xi_2)F}. \tag{2.17}$$

Thus we have characteristic curves

$$\begin{cases} (1-\psi_{2,1}x)^{-\frac{1}{\psi_{2,1}}}(\psi_{1,2}y+\psi_{2,2})^{-\frac{1}{\psi_{2,2}}}y^{\frac{1}{\psi_{2,2}}} = C_1, \\ (\psi_{1,2}y+\psi_{2,2})^{\frac{\xi_1}{\psi_{1,2}}-\frac{\xi_2}{\psi_{2,2}}}y^{\frac{\xi_2}{\psi_{2,2}}}F = C_2. \end{cases} \tag{2.18}$$

III. Taking into account the equations of the line Γ, we calculate

$$\begin{cases} t = \frac{\psi_{2,2}}{C_1^{-\psi_{2,2}}-\psi_{1,2}}, \\ (\psi_{1,2}t+\psi_{2,2})^{\frac{\xi_1}{\psi_{1,2}}-\frac{\xi_2}{\psi_{2,2}}}t^{\frac{\xi_2}{\psi_{2,2}}} = C_2. \end{cases}$$

Eliminating t, we obtain the expression for parameters C_1 and C_2,

$$\psi_{2,2}^{\frac{\xi_1}{\psi_{1,2}}}\left(C_1^{-\psi_{2,2}}\right)^{\frac{\xi_1}{\psi_{1,2}}-\frac{\xi_2}{\psi_{2,2}}}(C_1^{-\psi_{2,2}}-\psi_{1,2})^{-\frac{\xi_1}{\psi_{1,2}}} = C_2. \tag{2.19}$$

IV. Substituting (2.18) curves into (2.19), we get

$$F(x,y) = \psi_{2,2}^{\frac{\xi_1}{\psi_{1,2}}}\frac{(1-\psi_{2,1}x)^{\frac{\psi_{2,2}}{\psi_{2,1}}\left(\frac{\xi_1}{\psi_{1,2}}-\frac{\xi_2}{\psi_{2,2}}\right)}}{\left((1-\psi_{2,1}x)^{\frac{\psi_{2,2}}{\psi_{2,1}}}(\psi_{1,2}y+\psi_{2,2})-\psi_{1,2}y\right)^{\frac{\xi_1}{\psi_{1,2}}}},$$

yielding us the first statement of the theorem.

To obtain the analytic expression we apply the formula

$$\frac{d^m}{dt^m}(at+b)^c = m!\binom{c}{m}a^m(at+b)^{c-m} = \frac{\prod_{j=0}^{m-1}(ac-aj)}{(at+b)^{m-c}}. \tag{2.20}$$

By formula (2.12), we have

$$\begin{aligned} a_{n,k} &= \frac{1}{k!}\frac{\partial^n}{\partial x^n}(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}} \\ &\qquad \cdot \frac{\partial^k}{\partial y^k}\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{-\frac{\xi_1}{\psi_{1,2}}}\Bigg|_{(0,0)} \\ &= \binom{-\frac{\xi_1}{\psi_{1,2}}}{k}\left(\frac{\psi_{1,2}}{\psi_{2,2}}\right)^k \\ &\qquad \cdot \frac{\partial^n}{\partial x^n}\frac{(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}\left(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^k}{\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{\frac{\xi_1}{\psi_{1,2}}+k}}\Bigg|_{(0,0)} \\ &= \binom{-\frac{\xi_1}{\psi_{1,2}}}{k}\left(\frac{\psi_{1,2}}{\psi_{2,2}}\right)^k \end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{\partial^n}{\partial x^n}\left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}(k-j)-\frac{\xi_2}{\psi_{2,1}}} \right.\\
&\qquad \left. \cdot \left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y-\frac{\psi_{1,2}}{\psi_{2,2}}y(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}}\right)^{-\frac{\xi_1}{\psi_{1,2}}-k} \right)_{(0,0)}\\
&= \binom{-\frac{\xi_1}{\psi_{1,2}}}{k} \left(\frac{\psi_{1,2}}{\psi_{2,2}}\right)^k\\
&\quad \cdot \frac{\partial^n}{\partial x^n}\left(\sum_{j=0}^{k}\sum_{s=0}^{\infty} (-1)^{k-j} \binom{k}{j} (1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}(k-j)-\frac{\xi_2}{\psi_{2,1}}} \right.\\
&\qquad \cdot \binom{-\frac{\xi_1}{\psi_{1,2}}-k}{s} (-1)^s \left(\frac{\psi_{1,2}}{\psi_{2,2}}\right)^s\\
&\qquad \left. \cdot y^s (1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}s} \left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y\right)^{-\frac{\xi_1}{\psi_{1,2}}-k-s} \right)_{(0,0)}\\
&= \binom{-\frac{\xi_1}{\psi_{1,2}}}{k} \frac{(\psi_{1,2})^k}{(\psi_{2,2})^k}\\
&\quad \cdot \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\partial^n}{\partial x^n} \left((1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}(k-j)-\frac{\xi_2}{\psi_{2,1}}} \right)_{x=0}\\
&= \frac{\prod_{s=0}^{k-1}\left(-\frac{\xi_1}{\psi_{1,2}}-s\right)}{k!} \frac{(\psi_{1,2})^k}{(\psi_{2,2})^k}\\
&\quad \cdot \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} n! \binom{-\frac{\psi_{2,2}}{\psi_{2,1}}(k-j)-\frac{\xi_2}{\psi_{2,1}}}{n} (-\psi_{2,1})^n\\
&= \frac{\prod_{s=0}^{k-1}\left((\psi_{1,2}+\psi_{1,3})+\psi_{1,2}s\right)}{k!(\psi_{2,2})^k}\\
&\quad \cdot \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \prod_{m=0}^{n-1} \left(\psi_{2,2}(k-j)+(\psi_{2,1}+\psi_{2,3})+\psi_{2,1}s\right),
\end{aligned}
$$

yielding us the second statement of the theorem. ∎

Let us consider special cases of the numbers of Theorem 2.4.

COROLLARY 2.5. *Numbers generated by the matrix* $\begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ 0 & \psi_{2,2} & \psi_{2,3} \end{pmatrix}$

(i) *have the generating function*

$$(2.21)\qquad F(x,y)=\begin{cases} e^{\psi_{2,3}x}\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y(1-e^{\psi_{2,2}x})\right)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}, & \psi_{1,2}\neq 0,\ \psi_{2,2}\neq 0;\\ \exp\left(\psi_{2,3}x+\frac{\psi_{1,3}}{\psi_{2,2}}y(e^{\psi_{2,2}x}-1)\right), & \psi_{1,2}=0,\ \psi_{2,2}\neq 0;\\ e^{\psi_{2,3}x}\left(1-\psi_{1,2}xy\right)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}, & \psi_{1,2}\neq 0,\ \psi_{2,2}=0 \end{cases}$$

(ii) *and the analytic expression*

$$(2.22)\qquad a_{n,k}=\begin{cases} (k!)^{-1}(\psi_{2,2})^{-k}\prod_{j=1}^{k}(\psi_{1,2}j+\psi_{1,3}) & \psi_{1,2}\neq 0,\\ \cdot\sum_{m=0}^{k}(-1)^m C_k^m(\psi_{2,2}(k-m)+\psi_{2,3})^n, & \psi_{2,2}\neq 0;\\ (k!)^{-1}\left(\psi_{1,3}/\psi_{2,2}\right)^k & \psi_{1,2}=0,\\ \cdot\sum_{m=0}^{k}(-1)^m C_k^m(\psi_{2,2}(k-m)+\psi_{2,3})^n, & \psi_{2,2}\neq 0;\\ C_n^k(\psi_{2,3})^{n-k}\prod_{j=1}^{k}(\psi_{1,2}j+\psi_{1,3}), & \psi_{1,2}\neq 0,\ \psi_{2,2}=0. \end{cases}$$

PROOF. I.1. The first part of the first statement of the corollary can be proved by the method of characteristics, analogically to the proof of the first statement of Theorem 2.4, however it is enough to notice that in the formula (2.13) we have

$$\lim_{\psi_{2,1}\to 0}(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}=e^{\psi_{2,3}x},\qquad \lim_{\psi_{2,1}\to 0}(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}}=e^{\psi_{2,2}x},$$

yielding us the statement.

II.1. The first part of the second statement of the corollary we can obtain by differentiating the generating function (cf. (2.12)), or by substituting $\psi_{2,1}=0$ into (2.14).

I.2. The second part of the first statement of the corollary can be proved by the method of characteristics, analogically to the proof of the first statement of Theorem 2.4. However, calculating the limit of the generating function in (2.13) while $\psi_{1,2}\to 0$,

$$\lim_{\psi_{1,2}\to 0}(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}$$
$$=(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}\lim_{\psi_{1,2}\to 0}\left(1-\frac{\psi_{1,2}}{\psi_{2,2}}y((1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}}-1)\right)^{-\frac{\psi_{1,3}}{\psi_{1,2}}},$$

and applying the formula

$$\lim_{t\to 0}(1-at)^{-\frac{b}{t}}=e^{ab},$$

we receive the statement.

II.2. The second part of the second statement of the corollary we obtain by differentiating the generating function (cf. (2.12)), or by substituting $\psi_{1,2} = 0$ into (2.14).

I.3. The third part of the first statement of the corollary can be proved by the method of characteristics, analogically to the proof of the first statement of Theorem 2.4. However, calculating the limit of the first generating function in (2.21), while $\psi_{2,2} \to 0$, and noticing that

$$\lim_{t\to 0} \frac{1-e^{xt}}{t} = -x,$$

we obtain the corresponding statement of the corollary.

II.3. The third part of the second statement of the corollary we obtain by differentiating the third generating function in (2.21). Using (2.20), we obtain

$$\begin{aligned}
a_{n,k} &= \frac{1}{k!}\frac{\partial^n}{\partial x^n}\, e^{\psi_{2,3}x}\frac{\partial^k}{\partial y^k}(1-\psi_{1,2}xy)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}\Bigg|_{(0,0)} \\
&= \frac{1}{k!}\frac{\partial^n}{\partial x^n}\, e^{\psi_{2,3}x}k!\binom{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}{k}(-\psi_{1,2}x)^k(1-\psi_{1,2}xy)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}-k}\Bigg|_{(0,0)} \\
&= \frac{\prod_{j=0}^{k}(\psi_{1,2}k+\psi_{1,3})}{k!}\frac{\partial^n}{\partial x^n} \\
&\qquad \cdot \sum_{s=0}^{\infty}\binom{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}-k}{s}(-\psi_{1,2}x)^s y^s e^{\psi_{2,3}x}x^k\Bigg|_{(0,0)} \\
&= \frac{\prod_{j=0}^{k}(\psi_{1,2}k+\psi_{1,3})}{k!}\left(x^k e^{\psi_{2,3}x}\right)^{(n)}_{x=0} \\
&= \frac{\prod_{j=0}^{k}(\psi_{1,2}k+\psi_{1,3})}{k!}C_n^k k!(\psi_{2,3})^{n-k},
\end{aligned}$$

thus completing the proof. ∎

Theorem 2.6. *Numbers generated by the matrix* $\begin{pmatrix} \psi_{1,1} & 0 & \psi_{1,3} \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}$

(i) *have the generating function*

$$F(x,y) = (1-(\psi_{1,1}y+\psi_{2,1})x)^{-\frac{\xi_1 y+\xi_2}{\psi_{1,1}y+\psi_{2,1}}}, \tag{2.23}$$

(ii) *and the analytic expression*

$$a_{n,k} = \sum_{k_1+\ldots+k_n=k,\ k_j\in\{0,1\}} \prod_{j=1}^{n} (b(k_j) + (j-1)c(k_j)) \tag{2.24}$$
$$= \sum_{k_1+\cdots+k_n=k,\ k_j\in\{0,1\}} \prod_{j=1}^{n} (\psi_{2-k_j,1} j + \psi_{2-k_j,3}),$$

where

$$b(k_j) = \begin{cases} \psi_{2,1} + \psi_{2,3}, & \text{for } k_j = 0, \\ \psi_{1,1} + \psi_{1,3}, & \text{for } k_j = 1, \end{cases} \qquad c(k_j) = \begin{cases} \psi_{2,1}, & \text{for } k_j = 0, \\ \psi_{1,1}, & \text{for } k_j = 1. \end{cases}$$

PROOF. If $\psi_{1,2} = \psi_{2,2} = 0$, then we can solve (2.2) equation as an ordinary differential equation

$$(1 - \psi_{1,1}xy - \psi_{2,1}x)F_x = (\xi_1 y + \xi_2)F, \qquad F(0, y) = 1.$$

Solving the Cauchy problem we obtain

$$\log F = \int_0^x \frac{\xi_1 y + \xi_2}{1 - (\psi_{11} y + \psi_{21})t} dt = -\frac{\xi_1 y + \xi_2}{\psi_{1,1} y + \psi_{2,1}} \log(1 - (\psi_{1,1} y + \psi_{2,1})x),$$

which yields us the first statement of the lemma.

Next, we calculate the analytic expression. Let us denote

$$B = \xi_1 y + \xi_2, \qquad C = \psi_{1,1} y + \psi_{2,1}.$$

Then, applying (2.20), we get

$$a_{n,k} = \frac{1}{k!} \frac{\partial^{n+k}}{\partial x^n \partial y^k} \left((1 - Cx)^{-\frac{B}{C}}\right)_{(0,0)} \frac{1}{k!} \frac{\partial^k}{\partial y^k} \left(\frac{\prod_{j=0}^{n-1}(Cj + B)}{(1 - Cx)^{\frac{B}{C}+n}}\right)_{(0,0)}$$
$$= \frac{1}{k!} \frac{\partial^k}{\partial y^k} \left(\sum_{s=0}^{\infty} \binom{-\frac{B}{C} - n}{s} (-C)^s x^s \prod_{j=0}^{n-1} (Cj + B)\right)_{(0,0)}$$
$$= \sum_{\sum k_j = k,\ k_j\in\{0,1\}} \prod_{j=1}^{n} (B + C(j-1))^{(k_j)}_{y=0},$$

where

$$\begin{cases} B'_{y=0} = B(1) = \xi_1 = \psi_{1,1} + \psi_{1,3}, \\ B_{y=0} = B(0) = \xi_2 = \psi_{2,1} + \psi_{2,3}, \end{cases} \quad \text{and} \quad \begin{cases} C'_{y=0} = C(1) = \psi_{1,1}, \\ C_{y=0} = C(0) = \psi_{2,1}. \end{cases}$$

We can rewrite the analytic expression in the following way,

$$a_{n,k} = \sum_{k_1+\cdots+k_n=k,\ k_j\in\{0,1\}} \prod_{j=1}^{n} (\psi_{2-k_j,1} j + \psi_{2-k_j,3}),$$

thus completing the proof. ∎

COROLLARY 2.7. *Numbers generated by the matrix* $\begin{pmatrix} \psi_{1,1} & 0 & 0 \\ \psi_{2,1} & 0 & 0 \end{pmatrix}$

(i) *have the generating function*

$$F(x,y) = (1 - (\psi_{1,1}y + \psi_{2,1})x)^{-1}, \tag{2.25}$$

(ii) *and the analytic expression*

$$a_{n,k} = n!C_n^k(\psi_{1,1})^k(\psi_{2,1})^{n-k}. \tag{2.26}$$

PROOF. The proof of the corollary follows the outline of the proof of Theorem 2.6 (with coefficients of the generating matrix $\psi_{1,3} = \psi_{2,3} = 0$). ∎

In the next section, we use the results of Corollary 2.5 and Theorem 2.6 to establish central limit theorems for particular numbers satisfying a class of triangular arrays.

3. LIMIT THEOREMS FOR NUMBERS SATISFYING A CLASS OF TRIANGULAR ARRAYS

Limit theorems for numbers satisfying a class of triangular arrays can be established using ordinary or semi-exponential generating functions (cf. [2, 3]), moment generating functions (cf. [1, 4, 6, 8, 28]) and probability generating functions (see [7, 18]).

Let Ω_n be an integral random variable with probability mass function

$$P(\Omega_n = k) = \frac{a_{n,k}}{\sum_{j=0}^n a_{n,j}}. \tag{3.1}$$

The moment generating function of the random variable Ω_n equals

$$M_n(s) = E(e^{\Omega_n s}) = \sum_{k=0}^n P(\Omega_n = k)e^{ks} = \left(\sum_{k=0}^n a_{n,k}\right)^{-1} \sum_{k=0}^n a_{n,k}e^{ks}. \tag{3.2}$$

Let us denote

$$S_n = \sum_{k=0}^n a_{n,k}.$$

Combining the definition of the semi-exponent generating function (2.1) with (3.2), we obtain

$$F(x,e^s) = \sum_{n=0}^\infty \frac{x^n}{n!} \sum_{k=0}^n a_{n,k}e^{sk} = \sum_{n=0}^\infty \frac{x^n}{n!} S_n M_n(s).$$

Thus, the partial differentiation of the double semi-exponential generating function $F(x,y)$ at $x = 0$, yields us the moment generating function

$$M_n(s) = S_n^{-1} \left.\frac{\partial^n}{\partial x^n} F(x,e^s)\right|_{x=0}. \tag{3.3}$$

Since $M_n(0) = 1$, the formula for the sum S_n follows,

$$S_n = \frac{\partial^n}{\partial x^n} F(x, e^s)\Big|_{(0,0)}. \tag{3.4}$$

We use Hwang's result on the convergence rate in the central limit theorem for combinatorial structures (see Corollary 2 from Section 4 in [19]) to establish the limit theorem for a special case of the numbers a_{nk} and specify the rate of convergence to the limiting distribution.

LEMMA 3.1 (Hwang [19]). *Let $P_n(z)$ be a probability generating function of the random variable Ω_n, taking only non-negative integral values, with expectation μ_n and variance σ_n^2. Suppose that, for each fixed $n \geqslant 1$, $P_n(z)$ is a Hurwitz polynomial (a polynomial whose zeros are located in the left halfplane of the complex plane or on the imaginary axis). If $\sigma_n \to \infty$, then, Ω_n satisfies*

$$P\left(\frac{\Omega_n - \mu_n}{\sigma_n} < x\right) = \Phi(x) + O\left(\frac{1}{\sigma_n}\right), \qquad x \in \mathbb{R}. \tag{3.5}$$

Let us formulate central limit theorems (Theorem 3.2 and Theorem 3.3).

THEOREM 3.2. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with probability mass function* (3.1)*. Let the nonzero elements of the matrix*

$$\begin{pmatrix} \psi_{1,1} & 0 & \psi_{1,3} \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}, \tag{3.6}$$

generating the numbers $a_{n,k}$, satisfy the inequalities

$$\frac{\psi_{2,1} j + \psi_{2,3}}{\psi_{1,1} j + \psi_{1,3}} > 0, \tag{3.7}$$

for $1 \leqslant j \leqslant n$, then

$$F_n(\sigma_n x + \mu_n) = \begin{cases} \Phi(x) + O\left(\frac{1}{\sqrt{n}}\right), & \psi_{1,1} \neq 0,\ \psi_{2,1} \neq 0, \\ \Phi(x) + O\left(\frac{1}{\sqrt{n}}\right), & \psi_{1,1} = \psi_{2,1} = 0, \\ \Phi(x) + O\left(\frac{1}{\sqrt{\log n}}\right), & \psi_{1,1} = 0 \text{ or } \psi_{2,1} = 0, \end{cases} \qquad x \in \mathbb{R}. \tag{3.8}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\begin{aligned} \mu_n &= \sum_{j=1}^{n} \frac{\psi_{1,1} j + \psi_{1,3}}{(\psi_{1,1} j + \psi_{1,3}) + (\psi_{2,1} j + \psi_{2,3})}, \\ \sigma_n^2 &= \sum_{j=1}^{n} \frac{(\psi_{1,1} j + \psi_{1,3})(\psi_{2,1} j + \psi_{2,3})}{((\psi_{1,1} j + \psi_{1,3}) + (\psi_{2,1} j + \psi_{2,3}))^2}. \end{aligned} \tag{3.9}$$

Proof. First we derive the moment generating function. By Theorem 2.6, the semi-exponential generating function of the numbers (3.6) equals

$$F(x,y) = (1 - (\psi_{1,1}y + \psi_{2,1})x)^{-\frac{\xi_1 y + \xi_2}{\psi_{1,1}y+\psi_{2,1}}}. \tag{3.10}$$

Calculating the partial derivative by x and taking into account the formula (2.20), we obtain that

$$\begin{aligned} \frac{\partial^n}{\partial x^n} F(x,y)\Big|_{x=0} &= \prod_{j=0}^{n-1} ((\xi_1 y + \xi_2) + (\psi_{1,1}y + \psi_{2,1})j) \\ &= \prod_{j=0}^{n-1} ((\psi_{1,1}y + \psi_{2,1})(j+1) + (\psi_{1,3}y + \psi_{2,3})). \end{aligned} \tag{3.11}$$

Combining (3.3) and (3.4), we have that the moment generating function equals

$$\begin{aligned} M_n(s) &= S_n^{-1} \prod_{j=1}^{n} ((\psi_{1,1}e^s + \psi_{2,1})j + (\psi_{1,3}e^s + \psi_{2,3})) \\ &= \frac{\prod_{j=1}^{n}((\psi_{1,1}e^s + \psi_{2,1})j + (\psi_{1,3}e^s + \psi_{2,3}))}{\prod_{j=1}^{n}((\psi_{1,1} + \psi_{2,1})j + (\psi_{1,3} + \psi_{2,3}))}, \end{aligned} \tag{3.12}$$

and

$$\log M_n(s) = \sum_{j=1}^{n} \log((\psi_{1,1}e^s + \psi_{2,1})j + (\psi_{1,3}e^s + \psi_{2,3})) - \log S_n. \tag{3.13}$$

Thus,

$$(\log M_n(s))' = \sum_{j=1}^{n} \frac{(\psi_{1,1}j + \psi_{1,3})e^s}{(\psi_{1,1}e^s + \psi_{2,,1})j + (\psi_{1,3}e^s + \psi_{2,3})}, \tag{3.14}$$

and

$$(\log M_n(s))'' = \sum_{j=1}^{n} \frac{e^s(\psi_{1,1}j + \psi_{1,3})(\psi_{2,1}j + \psi_{2,3})}{((\psi_{1,1}e^s + \psi_{2,1})j + (\psi_{1,3}e^s + \psi_{2,3}))^2}. \tag{3.15}$$

Next we calculate the expectation

$$\begin{aligned} \mu_n = M_n'(0) &= \underbrace{M_n(0)}_{=1} (\log M_n(s))'|_{s=0} \\ &= \sum_{j=1}^{n} \frac{\psi_{1,1}j + \psi_{1,3}}{(\psi_{1,1} + \psi_{2,1})j + (\psi_{1,3} + \psi_{2,3})}, \end{aligned} \tag{3.16}$$

and the variance

$$\begin{aligned}\sigma_n^2 &= M_n''(0) - M_n'^2(0)\\ &= \underbrace{M_n'(0)}_{\mu_n}\underbrace{(\log M_n(s))'|_{s=0}}_{=\mu_n} + \underbrace{M_n(0)}_{=1}(\log M_n(s))''|_{s=0} - \underbrace{M_n'^2(0)}_{=\mu_n^2}\\ &= \sum_{j=1}^{n} \frac{(\psi_{1,1}j+\psi_{1,3})(\psi_{2,1}j+\psi_{2,3})}{((\psi_{1,1}+\psi_{2,1})j+(\psi_{1,3}+\psi_{2,3}))^2}.\end{aligned} \tag{3.17}$$

The probability generating functon is

$$P_n(z) = M_n(\log z) = \frac{\prod_{j=1}^{n}((\psi_{1,1}j+\psi_{1,3})z+(\psi_{2,1}j+\psi_{2,3}))}{\prod_{j=1}^{n}((\psi_{1,1}j+\psi_{1,3})+(\psi_{2,1}j+\psi_{2,3}))}. \tag{3.18}$$

The roots of the polynomial (3.18) are negative. Indeed, by the condition (3.7) of the theorem, we have

$$z_j = -\frac{\psi_{2,1}j+\psi_{2,3}}{\psi_{1,1}j+\psi_{1,3}} < 0. \tag{3.19}$$

Now let us consider the variance (cf. (3.17))

$$\begin{aligned}\sigma_n^2 &= \sum_{j=1}^{n} \frac{(\psi_{1,1}j+\psi_{1,3})(\psi_{2,1}j+\psi_{2,3})}{((\psi_{1,1}j+\psi_{1,3})+(\psi_{2,1}j+\psi_{2,3}))^2}\\ &= \sum_{j=1}^{n}\left(\frac{(\psi_{1,1}j+\psi_{1,3})+(\psi_{2,1}j+\psi_{2,3})}{\sqrt{\psi_{1,1}j+\psi_{1,3}}\sqrt{\psi_{2,1}j+\psi_{2,3}}}\right)^{-2}\\ &= \sum_{j=1}^{n}\left(\underbrace{\sqrt{\frac{\psi_{1,1}j+\psi_{1,3}}{\psi_{2,1}j+\psi_{2,3}}}}_{=:\delta_j}+\sqrt{\frac{\psi_{2,1}j+\psi_{2,3}}{\psi_{1,1}j+\psi_{1,3}}}\right)^{-2} = \sum_{j=1}^{n}\left(\delta_j+\frac{1}{\delta_j}\right)^{-2}.\end{aligned} \tag{3.20}$$

I. Let $\psi_{1,1} \neq 0$ and $\psi_{2,1} \neq 0$. Consequently,

$$\lim_{j\to\infty}\delta_j = \sqrt{\frac{\psi_{1,1}}{\psi_{2,1}}} = const > 0,$$

yielding us $\sigma_n^2 \to \infty$. On the other hand, $\delta_j + 1/\delta_j \geqslant 2$. Hence,

$$\sigma_n^2 = \sum_{j=1}^{n}\underbrace{\left(\delta_j+\frac{1}{\delta_j}\right)^{-2}}_{\leqslant 1/4} \leqslant n/4, \tag{3.21}$$

Thus, we receive $\sigma_n = O(\sqrt{n})$. The first statement of the theorem follows.

II. Let $\psi_{1,1} = \psi_{2,1} = 0$. By (3.17), we obtain

$$\sigma_n^2 = \sum_{j=1}^{n} \frac{\psi_{1,3}\psi_{2,3}}{(\psi_{1,3} + \psi_{2,3})^2} = Cn.$$

Hence, $\sigma_n^2 \to \infty$ and $\sigma_n = O(\sqrt{n})$, yielding us the second statement of the theorem.

III. Let $\psi_{2,1} = 0$ (note that the case of $\psi_{1,1} = 0$ can be addressed in a similar way) and

$$\alpha = \frac{\psi_{2,3}}{\psi_{1,1}}, \ \beta = \frac{\psi_{1,3} + \psi_{2,3}}{\psi_{1,1}}.$$

Now,

$$\sigma_n^2 = \psi_{2,3} \sum_{j=1}^{n} \frac{\psi_{1,1} j + \psi_{1,3} + \psi_{2,3} - \psi_{2,3}}{(\psi_{1,1} j + \psi_{1,3} + \psi_{2,3})^2} = \underbrace{\alpha \sum_{j=1}^{n} \frac{1}{j+\beta}}_{=\alpha \log n + O(1)} - \underbrace{\alpha^2 \sum_{j=1}^{n} \frac{1}{(j+\beta)^2}}_{=:D_n}.$$

Since

$$\lim_{n\to\infty} D_n = const > 0,$$

we obtain $\sigma_n^2 \to \infty$ and $\sigma_n^2 = \alpha \log n + O(1)$, yielding us the third statement of the theorem. ∎

Theorem 3.3. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with probability mass function* (3.1). *Let the non-zero elements of the matrix*

$$\begin{pmatrix} 0 & 0 & \psi_{1,3} \\ 0 & \psi_{2,2} & 0 \end{pmatrix}, \tag{3.22}$$

generating the numbers $a_{n,k}$, be positive (negative) and $\alpha = \psi_{1,3}/\psi_{2,2}$, then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right), \qquad x \in \mathbb{R}. \tag{3.23}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\begin{aligned} \mu_n &= \frac{T_{n+1}(\alpha)}{T_n(\alpha)} - \alpha, \\ \sigma_n^2 &= \frac{T_{n+2}(\alpha)}{T_n(\alpha)} - \left(\frac{T_{n+1}(\alpha)}{T_n(\alpha)}\right)^2 - \alpha. \end{aligned} \tag{3.24}$$

Here $T_n(x)$ stand for Touchard polynomials.

Proof. The proof of the theorem follows the outline of the proof of Theorem 3.2. It is important to note that the probability generating function of the random variable Ω_n (cf. (3.18)) is given in terms of Touchard polynomials (A.5). Details for the proof are presented in Appendix A. ∎

Theorem 3.3 allows us to receive the symmetric result for the dual numbers (2.11). We can formulate the subsequent corollary.

Corollary 3.4. *Let the coefficients $\psi_{1,3}$ and $\psi_{2,2}$ of the numbers generated by the matrix*

$$\begin{pmatrix} \psi_{2,2} & -\psi_{2,2} & 0 \\ 0 & 0 & \psi_{1,3} \end{pmatrix}, \tag{3.25}$$

be positive (negative) and $\alpha = \psi_{1,3}/\psi_{2,2}$, then the cumulative distribution function of the corresponding random variable (3.1)

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right), \qquad x \in \mathbb{R}. \tag{3.26}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\begin{aligned} \mu_n &= n - \frac{T_{n+1}(\alpha)}{T_n(\alpha)} + \alpha, \\ \sigma_n^2 &= \frac{T_{n+2}(\alpha)}{T_n(\alpha)} - \left(\frac{T_{n+1}(\alpha)}{T_n(\alpha)}\right)^2 - \alpha. \end{aligned} \tag{3.27}$$

To prove the next central limit theorem (Theorem 3.6) we use the following result for the geometric polynomials.

Theorem 3.5 (Belovas [5]). *Let $x > 0$ be fixed, then*

$$\omega_n(x) = \frac{n!}{(1+x)\log^{n+1}\left(1+\frac{1}{x}\right)}\left(1 + O\left(\left(1 + \frac{4\pi^2}{\log^2\left(1+\frac{1}{x}\right)}\right)^{-\frac{n+1}{2}}\right)\right). \tag{3.28}$$

Theorem 3.6. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with the probability mass function* (3.1). *Let the non-zero elements of the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & 0 \\ 0 & \psi_{2,2} & 0 \end{pmatrix}, \tag{3.29}$$

generating the numbers $a_{n,k}$, be positive (negative) and $\beta = \psi_{1,2}/\psi_{2,2}$, then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O\left(\frac{1}{\sqrt{n}}\right), \qquad x \in \mathbb{R}. \tag{3.30}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\begin{aligned} \mu_n &= \frac{1}{\beta+1}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} - \frac{\beta}{\beta+1}, \\ \sigma_n^2 &= \frac{1}{(\beta+1)^2}\left(\frac{\omega_{n+2}(\beta)}{\omega_n(\beta)} - \frac{\omega_{n+1}^2(\beta)}{\omega_n^2(\beta)}\right) - \frac{\beta}{(\beta+1)^2}\left(\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} + 1\right), \end{aligned} \tag{3.31}$$

respectively.

Proof. The proof of the theorem follows the outline of the proof of Theorem 3.3. It is important to note that the probability generating function of the random variable Ω_n (cf. (A.5)) is given in terms of geometric polynomials (B.5). Details for the proof are presented in Appendix B.

It is interesting to note the similarity in expressions of the variance σ_n^2 in Theorem 3.3 and Theorem 3.6,

$$\sigma_n^2 = \alpha(\alpha(\log T_n(\alpha))')', \qquad \sigma_n^2 = \beta(\beta(\log \omega_n(\beta))')',$$

respectively (cf. (A.13)). $\square$

Theorem 3.6 allows us to receive the symmetric result for the dual numbers. We can formulate the subsequent corollary.

Corollary 3.7. *Let the coefficients $\psi_{1,3}$ and $\psi_{2,2}$ of the numbers generated by the matrix*

$$\begin{pmatrix} \psi_{2,2} & -\psi_{2,2} & 0 \\ \psi_{1,2} & -\psi_{1,2} & 0 \end{pmatrix}, \tag{3.32}$$

be positive (negative) and $\beta = \psi_{1,2}/\psi_{2,2}$, then the cumulative distribution function of the corresponding random variable (3.1) *is*

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O\left(\frac{1}{\sqrt{n}}\right), \qquad x \in \mathbb{R}. \tag{3.33}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\begin{aligned} \mu_n &= n - \frac{1}{\beta+1}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} + \frac{\beta}{\beta+1}, \\ \sigma_n^2 &= \frac{1}{(\beta+1)^2}\left(\frac{\omega_{n+2}(\beta)}{\omega_n(\beta)} - \frac{\omega_{n+1}^2(\beta)}{\omega_n^2(\beta)}\right) - \frac{\beta}{(\beta+1)^2}\left(\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} + 1\right), \end{aligned} \tag{3.34}$$

respectively.

As we can see, all limiting distributions, received in Theorems 3.2, 3.3, 3.6 and Corollaries 3.4, 3.7 are Gaussian. Is the normal distribution a limiting law in a general central limit theorem for the numbers $a_{n,k}$, satisfying a class of triangular arrays, defined by a bivariate linear recurrence with bivariate linear coefficients (see (1.1)-(1.2))? We will address the problem in the next section.

4. Limit theorems with non-Gaussian limiting distributions

Kyriakoussis (cf. [21, Corollary 2.1]) claimed a general result for nonnegative numbers satisfying a class of triangular arrays.

Theorem 4.1 (Kyriakoussis [21]). *Nonnegative numbers $a_{n,k}$ generated by the matrix*

$$\begin{pmatrix} c_6 & c_5 & c_4 \\ c_3 & c_2 & c_1 \end{pmatrix} \tag{4.1}$$

are asymptotically normal (i.e., satisfy a central limit theorem) with the expectation μ_n and the variance σ_n^2,

$$\begin{aligned} \mu_n &= -n\frac{r'(0)}{r(0)} + \frac{A'(0)}{A(0)}, \\ \sigma_n^2 &= n\left(\left(\frac{r'(0)}{r(0)}\right)^2 - \frac{r''(0)}{r(0)}\right) + \frac{A''(0)}{A(0)} - \left(\frac{A'(0)}{A(0)}\right)^2, \end{aligned} \tag{4.2}$$

respectively. Here $r(s)$ and $A(s)$ are the solutions of the differential equations

$$\begin{aligned} &(c_2 + c_5e^s)r'(s) - (c_3 + c_6e^s)r(s) = c_0, \qquad c_0 \text{ constant}, \\ &(c_2 + c_5e^s)A'(s) + ((c_4 + c_5)e^s + c_1)A(s) = 0, \\ &\left(\frac{r'(0)}{r(0)}\right)^2 - \frac{r''(0)}{r(0)} \neq 0, \qquad A(0) \neq 0. \end{aligned} \tag{4.3}$$

However the proposition is flawed. Let us give a counterexample ehxibiting not asymptotically normal numbers $a_{n,k}$. First let us provide an auxiliary lemma on the convergence of the moment generating functions. Let $\{F_n(x)\}$ be a sequence of distribution functions and $\{M_n(x)\}$ be the sequence of corresponding moment generating functions, which exist in some neighborhood of 0. Pointwise convergence of $M_n(x)$ to $M(x)$ in some neighborhood of 0 implies weak convergence of $F_n(x)$ to $F(x)$.

Lemma 4.2 (Mukherjea et al. [28]). *Let a and b be positive and $a < b$. If*

$$\lim_{n\to\infty} M_n(t) = M(t),$$

whenever $a < t < b$. Then

$$\lim_{n\to\infty} F_n(t) = F(t),$$

for every number $x \in \mathbb{R}$, at which $F(x)$ is continuous.

Note that the result is correct if a positive interval is replaced by a negative one, see [31]. Now let us consider a central limit theorem for generalized k-permutations of n without repetition.

Theorem 4.3. *Let the non-zero elements of the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & 0 \\ 0 & 0 & \psi_{2,3} \end{pmatrix}, \tag{4.4}$$

generating the numbers $a_{n,k}$, be positive (negative). Then the numbers $a_{n,k}$ and their dual numbers $\tilde{a}_{n,k} = a_{n,n-k}$, generated by the matrix

$$\begin{pmatrix} 0 & 0 & \psi_{2,3} \\ \psi_{1,2} & -\psi_{1,2} & 0 \end{pmatrix} \tag{4.5}$$

are asymptotically Poissonian.

Proof. First we derive the moment generating function. By Table 2 (entry 1) and Table 1 (entry 3) of Appendix C, semi-exponential generating functions of the numbers (4.4) and (4.5) are equal to

$$F(x,y) = e^{\psi_{2,3}x}(1-\psi_{1,2}xy)^{-1}, \; \tilde{F}(x,y) = e^{\psi_{2,3}xy}(1-\psi_{1,2}x)^{-1}, \tag{4.6}$$

respectively. Calculating the partial derivative by x using the general Leibniz rule and taking into account the formula (2.20), we obtain that

$$\begin{aligned} \frac{\partial^n}{\partial x^n} F(x,y)\bigg|_{x=0} &= \sum_{m=0}^{n} C_n^m \psi_{2,3}^{n-m} \prod_{j=0}^{m-1} (\psi_{1,2}y + \psi_{1,2}yj) \\ &= \psi_{2,3}^n \sum_{m=0}^{n} C_n^m m! \left(\frac{\psi_{1,2}}{\psi_{2,3}} y\right)^m, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \frac{\partial^n}{\partial x^n} \tilde{F}(x,y)\bigg|_{x=0} &= \sum_{m=0}^{n} C_n^m (\psi_{2,3}y)^m \prod_{j=0}^{n-m-1} (\psi_{1,2} + \psi_{1,2}j) \\ &= \psi_{1,2}^n \sum_{m=0}^{n} C_n^m (n-m)! \left(\frac{\psi_{2,3}}{\psi_{1,2}} y\right)^m. \end{aligned} \tag{4.8}$$

Let us denote $\lambda = \psi_{2,3}/\psi_{1,2}$. Combining (3.3) and (3.4), we have that the moment generating functions are equal to

$$\begin{aligned} M_n(s) &= S_n^{-1} \psi_{2,3}^n \sum_{m=0}^{n} C_n^m m! \lambda^{-m} e^{ms} = \frac{\sum_{m=0}^{n} C_n^m m! \lambda^{-m} e^{ms}}{\sum_{m=0}^{n} C_n^m m! \lambda^{-m}} \\ &= \Bigg(\underbrace{\sum_{m=0}^{n} \frac{\lambda^m}{m!}}_{=:e_n(\lambda)}\Bigg)^{-1} \sum_{m=0}^{n} \frac{\lambda^m e^{(n-m)s}}{m!}, \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
\tilde{M}_n(s) &= S_n^{-1}\psi_{1,2}^n \sum_{m=0}^{n} C_n^m (n-m)!\lambda^m e^{ms} \\
&= \frac{\sum_{m=0}^{n} C_n^m (n-m)!\lambda^m e^{ms}}{\sum_{m=0}^{n} C_n^m (n-m)!\lambda^m} = \Bigg(\underbrace{\sum_{m=0}^{n} \frac{\lambda^m}{m!}}_{=:e_n(\lambda)}\Bigg)^{-1} \sum_{m=0}^{n} \frac{\lambda^m e^{ms}}{m!}.
\end{aligned} \tag{4.10}
$$

Let us consider the convergence of moment generating functions (4.9) and (4.10).

Note that

$$
e_n(\lambda) = e^{\lambda}\left(1 + O\left(\frac{\lambda^{n+1}}{(n+1)!}\right)\right).
$$

The moment generating function of the numbers (4.5) equals

$$
\tilde{M}_n(s) = e_n^{-1}(\lambda)\sum_{m=0}^{n} \frac{\lambda^m e^{ms}}{m!} = e^{\lambda(e^s-1)}\left(1 + O\left(\frac{\lambda^n \max(1, e^{ns})}{(n+1)!}\right)\right). \tag{4.11}
$$

Thus, by Lemma 4.2, the numbers $\tilde{a}_{nk}$ are asymptotically Poissonian, i.e. $\tilde{\Omega}_n \sim \mathrm{Pois}(\lambda)$. The symmetric statement for the dual numbers a_{nk} follows.

$\square$

Remark 4.4. Let $\psi_{1,2} = \psi_{2,3} = 1$ (k-permutations of n without repetition). The probability generating functions of the numbers (4.4) and dual numbers (4.5) are equal to

$$
\begin{aligned}
P_n(z) &= M_n(\log z) = e_n^{-1}(1) \sum_{m=0}^{n} \frac{z^m}{(n-m)!}, \\
\tilde{P}_n(z) &= \tilde{M}_n(\log z) = e_n^{-1}(1) \sum_{m=0}^{n} \frac{z^m}{m!},
\end{aligned} \tag{4.12}
$$

respectively. Calculating the roots of $P_5(z)$ and $\tilde{P}_5(z)$ numerically, we receive

$$
\begin{aligned}
&z_1 = -0.459..., \; z_{2,3} = -0.295... \pm 0.303...i, \; z_{4,5} = 0.024... \pm 0.318i, \\
&\tilde{z}_1 = -2.181..., \; \tilde{z}_{2,3} = -1.650... \pm 1.694...i, \; \tilde{z}_{4,5} = 0.240... \pm 3.128i.
\end{aligned}
$$

Indeed, $\Re(z_{4,5}) > 0$ and $\Re(\tilde{z}_{4,5}) > 0$. Thus, the probability generating functons (4.12) do not satisfy the conditions of Lemma 3.1 (because the polynomials are not Hurwitz polynomials).

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Appendix A. Proof of Theorem 3.3

Proof. First we derive the moment generating function. By Corollary 2.5 (see (2.21)), the semi-exponential generating function of the numbers (3.22) equals

$$F(x,y) = \exp\left(\alpha y(e^{\psi_{2,2}x} - 1)\right). \tag{A.1}$$

By the properties of the Touchard polynomials $T_n(x)$, we have that they can be defined by the exponential generating function, see [32],

$$\exp\left(x(e^t - 1)\right) = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}. \tag{A.2}$$

Combining (A.1) and (A.2), we get

$$\frac{\partial^n}{\partial x^n}F(x,y)\bigg|_{x=0} = (\psi_{2,2})^n T_n\left(\alpha y\right). \tag{A.3}$$

Next, combining (3.3) and (3.4), we have that the moment generating function equals

$$M_n(s) = S_n^{-1}(\psi_{2,2})^n T_n\left(\alpha e^s\right) = \frac{T_n\left(\alpha e^s\right)}{T_n\left(\alpha\right)}. \tag{A.4}$$

Hence, the probability generating functon is

$$P_n(z) = M_n(\log z) = \frac{T_n\left(\alpha z\right)}{T_n\left(\alpha\right)}. \tag{A.5}$$

Harper showed that the roots of the Touchard polynomials are real, distinct, and non-positive (see [17, Lemma 1]). Since $\alpha > 0$, the polynomial (A.5) is a Hurwitz polynomial.

Next we calculate the expectation μ_n and the variance σ_n^2. The Touchard polynomials satisfy the following recurrence relation (cf. [12, 14, 27]),

$$(x + x\partial_x)T_n(x) = T_{n+1}(x). \tag{A.6}$$

Hence,

$$\begin{aligned} T_n'(x) &= x^{-1}T_{n+1}(x) - T_n(x),\\ T_n''(x) &= x^{-2}T_{n+2}(x) - x^{-2}(1+2x)T_{n+1}(x) + T_n(x). \end{aligned} \tag{A.7}$$

By (A.4), we obtain that the derivative of the moment generating function equals

$$\begin{aligned} M_n'(s) &= \alpha e^s \left(T_n\left(\alpha\right)\right)^{-1} T_n'\left(\alpha e^s\right),\\ M_n''(s) &= \alpha e^s \left(T_n\left(\alpha\right)\right)^{-1} \left(T_n'\left(\alpha e^s\right) + \alpha e^s T_n''\left(\alpha e^s\right)\right). \end{aligned} \tag{A.8}$$

Combining (A.7) and (A.8), we obtain the expectation

$$\begin{aligned} \mu_n = M_n'(0) &= \alpha\left(T_n\left(\alpha\right)\right)^{-1} T_n'\left(\alpha\right)\\ &= \alpha\left(T_n\left(\alpha\right)\right)^{-1}\left(\alpha^{-1}T_{n+1}(\alpha) - T_n(\alpha)\right)\\ &= \left(T_n\left(\alpha\right)\right)^{-1} T_{n+1}(\alpha) - \alpha \end{aligned} \tag{A.9}$$

and the variance

$$\begin{aligned} \sigma_n^2 &= M_n''(0) - M_n'^2(0)\\ &= \frac{\alpha T_n'(\alpha) + \alpha^2 T_n''(\alpha)}{T_n(\alpha)} - \left(\frac{T_{n+1}(\alpha)}{T_n(\alpha)} - \alpha\right)^2\\ &= \frac{T_{n+1}(\alpha) - \alpha T_n(\alpha) + T_{n+2}(\alpha) - (1+2\alpha)T_{n+1}(\alpha) + \alpha^2 T_n(\alpha)}{T_n(\alpha)}\\ &\quad - \frac{T_{n+1}^2(\alpha)}{T_n^2(\alpha)} + 2\alpha\frac{T_{n+1}(\alpha)}{T_n(\alpha)} - \alpha^2 = \frac{T_{n+2}(\alpha)}{T_n(\alpha)} - \frac{T_{n+1}^2(\alpha)}{T_n^2(\alpha)} - \alpha. \end{aligned} \tag{A.10}$$

Using (A.7), we get

$$\begin{aligned} T_{n+1}(x) &= xT_n'(x) + xT_n(x),\\ T_{n+2}(x) &= x^2T_n''(x) + (2x^2+x)T_n'(x) + (x^2+x)T_n(x). \end{aligned} \tag{A.11}$$

Hence,

$$\begin{aligned} \frac{T_{n+1}(x)}{T_n(x)} &= x\frac{T_n'(x)}{T_n(x)} + x = x(\log T_n(x))' + x,\\ \frac{T_{n+2}(x)}{T_n(x)} &= x^2\frac{T_n''(x)}{T_n(x)} + (2x^2+x)\frac{T_n'(x)}{T_n(x)} + (x^2+x)\\ &= x^2(\log T_n(x))'' + x^2((\log T_n(x))')^2\\ &\quad + (2x^2+x)(\log T_n(x))' + (x^2+x). \end{aligned} \tag{A.12}$$

Combining (A.10) and (A.12), we receive the variance

$$\begin{aligned}\sigma_n^2 &= \alpha^2(\log T_n(\alpha))'' + \alpha^2((\log T_n(\alpha))')^2 + (2\alpha^2+\alpha)(\log T_n(\alpha))' \\ &\quad + (\alpha^2+\alpha) - (\alpha(\log T_n(\alpha))' + \alpha)^2 - \alpha \\ &= \alpha^2(\log T_n(\alpha))'' + \alpha(\log T_n(\alpha))' = \alpha(\alpha(\log T_n(\alpha))')'.\end{aligned} \tag{A.13}$$

The asymptotic formula for the Touchard polynomials (see [30, Theorem 1]), for the fixed positive x, is

$$T_n(x) \sim \frac{\Gamma(n+1)\exp(-x+(n+1)/W\left(\frac{n+1}{x}\right))}{\sqrt{2\pi(n+1)\left(1+W\left(\frac{n+1}{x}\right)\right)}W^n\left(\frac{n+1}{x}\right)}. \tag{A.14}$$

Let

$$V_n = W\left(\frac{n+1}{\alpha}\right), \qquad V_n' = W'\left(\frac{n+1}{\alpha}\right), \qquad V_n'' = W''\left(\frac{n+1}{\alpha}\right),$$

then,

$$(\log T_n(\alpha))' \sim -1 + (n+1)\frac{V_n'}{\alpha^2}\left(\frac{n+1}{V_n^2} + \frac{1/2}{1+V_n} + \frac{n}{V_n}\right), \tag{A.15}$$

and, by (A.13),

$$\begin{aligned}\frac{\sigma_n^2}{\alpha} &\sim \left(-\alpha + (n+1)\frac{V_n'}{\alpha}\left(\frac{n+1}{V_n^2} + \frac{1/2}{1+V_n} + \frac{n}{V_n}\right)\right)' \\ &= -1 + (n+1)\frac{-(n+1)V_n''/\alpha - V_n'}{\alpha^2}\left(\frac{n+1}{V_n^2} + \frac{1/2}{1+V_n} + \frac{n}{V_n}\right) \\ &\quad + (n+1)\frac{V_n'}{\alpha}\left(\frac{2(n+1)^2V_n'}{\alpha^2V_n^3} + \frac{(n+1)V_n'}{2\alpha^2(1+V_n)^2} + \frac{n(n+1)V_n'}{\alpha^2V_n^2}\right).\end{aligned} \tag{A.16}$$

Note that the asymptotic of the principal branch of the Lambert W function is

$$W(x) = \log x - \log\log x + O\left(\frac{\log\log x}{\log x}\right), \qquad x \to \infty. \tag{A.17}$$

Hence,

$$\begin{aligned}\frac{n+1}{\alpha}V_n' &= 1 + O\left(\frac{1}{\log n}\right), \\ \frac{(n+1)^2}{\alpha^2}V_n'' &= -1 + O\left(\frac{1}{\log n}\right).\end{aligned} \tag{A.18}$$

Combining (A.16) and (A.18), we obtain

$$\sigma_n^2 \sim -\alpha + \frac{2(n+1)}{V_n^3} + \frac{1}{2(1+V_n)^2} + \frac{n}{V_n^2} + O\left(\frac{n}{\log^3 n}\right). \tag{A.19}$$

Hence,

$$\sigma_n^2 \sim \frac{n}{V_n^2} \qquad \text{or} \qquad \sigma_n \sim \frac{\sqrt{n}}{\log n}. \tag{A.20}$$

Thus $\sigma_n \to \infty$, yielding us, by Lemma 3.1, the statement of the theorem. □

Appendix B. Proof of Theorem 3.6

Proof. First we derive the moment generating function. By Corollary 2.5 (see (2.21)), the bivariate semi-exponential generating function of the numbers (3.29) equals

$$F(x,y) = \left(1 + \beta y (1 - e^{\psi_{2,2} x})\right)^{-1}. \tag{B.1}$$

The geometric polynomials $\omega(x)$ can be defined by the exponential generating function, see [10],

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} \omega_n(x) \frac{t^n}{n!}. \tag{B.2}$$

Combining (B.1) and (B.2)) we get that

$$\left.\frac{\partial^n}{\partial x^n} F(x,y)\right|_{x=0} = (\psi_{2,2})^n \omega_n(\beta y). \tag{B.3}$$

Next, combining (3.3), (3.4) and (B.3), we obtain that the moment generating function equals

$$M_n(s) = S_n^{-1} (\psi_{2,2})^n \omega_n(\beta e^s) = \frac{\omega_n(\beta e^s)}{\omega_n(\beta)}. \tag{B.4}$$

The probability generating functon is

$$P_n(z) = M_n(\log z) = \frac{\omega_n(\beta z)}{\omega_n(\beta)}. \tag{B.5}$$

By the relation between the Eulerian and the geometric polynomials, see [9],

$$A_n(x) = (1-x)^n \omega_n\left(\frac{x}{1-x}\right). \tag{B.6}$$

Frobenius showed that the roots of the Eulerian polynomials $A_n(x)$ are real, distinct and negative, see [15]. Since $\beta > 0$, then, combining (B.5) and (B.6), we obtain the same result for the roots of the probability generating function. Thus $P_n(z)$ is a Hurwitz polynomial.

Next we calculate the expectation μ_n and the variance σ_n^2. The geometric polynomials satisfy the following recurrence relation (see [10, Proposition 13]),

$$\omega_{n+1}(x) = (x^2 + x)\omega_n'(x) + x\omega_n(x). \tag{B.7}$$

Hence,

$$\begin{aligned} \omega_n'(x) &= \frac{1}{x^2+x}\omega_{n+1}(x) - \frac{1}{x+1}\omega_n(x), \\ \omega_n''(x) &= \frac{1}{(x^2+x)^2}\omega_{n+2}(x) - \frac{4x+1}{(x^2+x)^2}\omega_{n+1}(x) + \frac{2}{(x+1)^2}\omega_n(x). \end{aligned} \tag{B.8}$$

By (B.4), we obtain that the derivatives of the moment generating function are equal to

$$\begin{aligned} M_n'(s) &= \beta e^s \left(\omega_n(\beta)\right)^{-1} \omega_n'(\beta e^s), \\ M_n''(s) &= \beta e^s \left(\omega_n(\beta)\right)^{-1} \left(\omega_n'(\beta e^s) + \beta e^s \omega_n''(\beta e^s)\right). \end{aligned} \tag{B.9}$$

Combining (B.8) and (B.9), we obtain the expectation

$$\begin{aligned} \mu_n = M_n'(0) &= \beta \left(\omega_n(\beta)\right)^{-1} \omega_n'(\beta) \\ &= \frac{1}{\beta+1}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} - \frac{\beta}{\beta+1} \end{aligned} \tag{B.10}$$

and the variance

$$\begin{aligned} \sigma_n^2 &= M_n''(0) - M_n'^2(0) \\ &= \frac{\beta\omega_n'(\beta) + \beta^2\omega_n''(\beta)}{\omega_n(\beta)} - \left(\frac{1}{\beta+1}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} - \frac{\beta}{\beta+1}\right)^2 \\ &= \beta^2\left(\frac{1}{(\beta^2+\beta)^2}\frac{\omega_{n+2}(\beta)}{\omega_n(\beta)} - \frac{4\beta+1}{(\beta^2+\beta)^2}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} + \frac{2}{(\beta+1)^2}\right) \\ &\quad + \beta\left(\frac{1}{\beta^2+\beta}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} - \frac{1}{\beta+1}\right) \\ &\quad - \frac{1}{(\beta+1)^2}\frac{\omega_{n+1}^2(\beta)}{\omega_n^2(\beta)} + \frac{2\beta}{(\beta+1)^2}\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} - \frac{\beta^2}{(\beta+1)^2} \\ &= \frac{1}{(\beta+1)^2}\left(\frac{\omega_{n+2}(\beta)}{\omega_n(\beta)} - \frac{\omega_{n+1}^2(\beta)}{\omega_n^2(\beta)}\right) - \frac{\beta}{(\beta+1)^2}\left(\frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} + 1\right). \end{aligned} \tag{B.11}$$

By (3.28), we have that

$$\frac{\omega_{n+2}(\beta)}{\omega_n(\beta)} \sim \frac{(n+1)(n+2)}{\log^2\left(1+\beta^{-1}\right)}, \qquad \frac{\omega_{n+1}(\beta)}{\omega_n(\beta)} \sim \frac{n+1}{\log\left(1+\beta^{-1}\right)}. \tag{B.12}$$

Hence, by (3.31),

$$\sigma_n^2 \sim \frac{1}{(\beta+1)^2}\left(\frac{n+1}{\log^2\left(1+\beta^{-1}\right)}\right) - \frac{\beta}{(\beta+1)^2}\left(\frac{n+1}{\log\left(1+\beta^{-1}\right)} + 1\right), \tag{B.13}$$

and

$$\sigma_n^2 \sim \underbrace{\frac{1-\beta\log\left(1+\beta^{-1}\right)}{(\beta+1)^2\log^2\left(1+\beta^{-1}\right)}}_{=:C_\beta>0} n. \tag{B.14}$$

Thus $\sigma_n \to \infty$ yields the statement of the theorem by Lemma 3.1. ∎

Appendix C. Synoptic table of the results for special cases of the numbers $a_{n,k}$

Table 1. Generating functions and analytical expressions

Generating matrices Ψ Generating functions $F(x,y)$ and analytical expressions of $a_{n,k}$
$\Psi = \begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \qquad \psi_{1,2}, \psi_{2,1}, \psi_{2,2} \neq 0,$ $F(x,y) = (1-\psi_{2,1}x)^{-\frac{\psi_2}{\psi_{2,1}}} \left(1 + \frac{\psi_{1,2}}{\psi_{2,2}} y (1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{-\frac{\psi_1}{\psi_{1,2}}},$ $a_{n,k} = \frac{\prod_{j=1}^k (\psi_{1,2}j+\psi_{1,3})}{k!(\psi_{2,2})^k} \sum_{m=0}^k (-1)^m C_k^m \prod_{s=1}^n (\psi_{2,2}(k-m) + \psi_{2,1}s + \psi_{2,3}).$
$\Psi = \begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ 0 & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \qquad \psi_{1,2}, \psi_{2,2} \neq 0,$ $F(x,y) = e^{\psi_{2,3}x} \left(1 + \frac{\psi_{1,2}}{\psi_{2,2}} y (1 - e^{\psi_{2,2}x})\right)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}},$ $a_{n,k} = \frac{\prod_{j=1}^k (\psi_{1,2}j+\psi_{1,3})}{k!(\psi_{2,2})^k} \sum_{m=0}^k (-1)^m C_k^m (\psi_{2,2}(k-m) + \psi_{2,3})^n.$
$\Psi = \begin{pmatrix} 0 & 0 & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \qquad \psi_{2,1}, \psi_{2,2} \neq 0\ ,$ $F(x,y) = (1-\psi_{2,1}x)^{-\frac{\psi_2}{\psi_{2,1}}} \exp\left(\frac{\psi_{1,3}}{\psi_{2,2}} y ((1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}} - 1)\right),$ $a_{n,k} = \frac{(\psi_{1,3})^k}{k!(\psi_{2,2})^k} \sum_{m=0}^k (-1)^m C_k^m \prod_{s=1}^n (\psi_{2,2}(k-m) + \psi_{2,1}s + \psi_{2,3})\ .$
$\Psi = \begin{pmatrix} 0 & 0 & \psi_{13} \\ 0 & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \qquad \psi_{2,2} \neq 0,$ $F(x,y) = \exp\left(\psi_{2,3}x + \frac{\psi_{1,3}}{\psi_{2,2}} y (e^{\psi_{2,2}x} - 1)\right),$ $a_{n,k} = \frac{1}{k!} \left(\frac{\psi_{1,3}}{\psi_{2,2}}\right)^k \sum_{m=0}^k (-1)^m C_k^m (\psi_{2,2}(k-m) + \psi_{2,3})^n.$

TABLE 2. Generating functions and analytical expressions

Generating matrices Ψ Generating functions $F(x,y)$ and analytical expressions of $a_{n,k}$
$\Psi = \begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ 0 & 0 & \psi_{2,3} \end{pmatrix}$, $F(x,y) = e^{\psi_{2,3}x}\,(1-\psi_{1,2}xy)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}$, $a_{n,k} = C_n^k(\psi_{2,3})^{n-k}\prod_{j=1}^{k}(\psi_{1,2}j+\psi_{1,3})$.
$\Psi = \begin{pmatrix} \psi_{1,1} & 0 & \psi_{1,3} \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}$, $F(x,y) = (1-(\psi_{1,1}y+\psi_{2,1})x)^{-\frac{\xi_1 y+\xi_2}{\psi_{1,1}y+\psi_{2,1}}}$, $a_{n,k} = \sum_{k_1+\cdots+k_n=k,\ k_j\in\{0,1\}}\prod_{j=1}^{n}(\psi_{2-k_j,1}j+\psi_{2-k_j,3})$.
$\Psi = \begin{pmatrix} 0 & 0 & \psi_{1,3} \\ 0 & 0 & \psi_{2,3} \end{pmatrix}$, $F(x,y) = e^{\psi_{2,3}x+\psi_{1,3}xy}$, $a_{n,k} = C_n^k(\psi_{1,3})^k(\psi_{2,3})^{n-k}$.
$\Psi = \begin{pmatrix} \psi_{1,1} & 0 & 0 \\ \psi_{2,1} & 0 & 0 \end{pmatrix}$, $F(x,y) = (1-(\psi_{1,1}y+\psi_{2,1})x)^{-1}$, $a_{n,k} = n!C_n^k(\psi_{1,1})^k(\psi_{2,1})^{n-k}$.

I. Belovas
Institute of Data Science and Digital Technologies
Vilnius University
04812 Vilnius
Lithuania
E-mail: Igoris.Belovas@mif.vu.lt

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