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A REMARK ON FLAT TERNARY CYCLOTOMIC POLYNOMIALS

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Abstract. Let $\Phi_n(x)$ be the n-th cyclotomic polynomial. In this paper, for odd primes $p < q < r$ with $q \equiv \pm 1 \pmod p$ and $8r \equiv \pm 1 \pmod{pq}$, we prove that the coefficients of $\Phi_{pqr}(x)$ do not exceed 1 in modulus if and only if (i) $p = 3$, $q \geq 19$ and $q \equiv 1 \pmod 3$ or (ii) $p = 7$, $q \geq 83$ and $q \equiv -1 \pmod 7$.

1. Introduction

Let $\Phi_n(x) = \sum_{m=0}^{\phi(n)} a(n,m)x^m$ be the n-th cyclotomic polynomial and put

$$A(n) = \max\{|a(n,m)| : 0 \leq m \leq \phi(n)\},$$

where ϕ is the Euler totient function. We can deduce that $\Phi_n(x)$ is a monic polynomial over integers by induction on n. It turns out that $A(n) = 1$ when n has no more than two distinct prime factors and this intriguing observation peeked the interest of many mathematicians. In particular, there is a lot of interest in *flat* cyclotomic polynomials (for which $A(n) = 1$, i.e., its nonzero coefficients are 1 or -1). Using basic properties of such polynomials, we have

$$\Phi_{2n}(x) = \pm\Phi_n(-x) \text{ and } \Phi_n(x) = \Phi_{\mathrm{rad}(n)}(x^{n/\mathrm{rad}(n)}),$$

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where $\mathrm{rad}(n)$ denotes the largest square-free factor of n. Therefore, the investigation of $A(n)$ can be reduced to the case when $n = pqr\cdots$, where $p, q, r, \cdots$ are distinct odd primes.

It is trivial to see that $\Phi_p(x) = \sum_{m=0}^{p-1} x^m$ and $A(p) = 1$. In 1883, Migotti ([12]) showed $A(pq) = 1$ and noted that $A(3 \cdot 5 \cdot 7) > 1$ with $a(3 \cdot 5 \cdot 7, 7) = -2$. Approximately one hundred years later, Beiter gave the necessary and sufficient conditions for $A(3qr) = 1$ by established the following result.

PROPOSITION 1.1. *Let $3 < q < r$ be primes such that $r = (wq \pm 1)/h$, $1 < h \leq (q-1)/2$. Then $A(3qr) = 1$ if and only if one of these conditions holds:*

(1) *$w \equiv 0$ and $h + q \equiv 0 \pmod 3$, or*
(2) *$h \equiv 0$ and $w + r \equiv 0 \pmod 3$.*

The proofs are based on the consideration of four types of partitions of m and the contribution of each type to the coefficients of x^m in the polynomial, see [4] for details. So the other case is $n = pqr$ with $5 \leq p < q < r$ primes. Currently, there are several open problems involving ternary cyclotomic polynomials $\Phi_{pqr}(x)$, an interesting and difficult one is to classify all flat ternary cyclotomic polynomials. While it is know that

$$r \equiv \pm 1 \pmod{pq} \Rightarrow A(pqr) = 1,$$

there are examples of flat ternary cyclotomic polynomials not of this form, and no simple general characterization of flatness is known. It has been conjectured by Elder ([6]), however, that if $A(pqr) = 1$ and $r \not\equiv \pm 1 \pmod{pq}$, then necessarily $q \equiv \pm 1 \pmod p$ (the latter condition is not sufficient for flatness in general).

Observing computational data, Broadhurst made the following conjecture about flat ternary cyclotoic polynomials.

CONJECTURE 1.2. *Let $p < q < r$ be odd primes with w the unique integer $0 \leq w \leq \frac{pq-1}{2}$ satisfying $r \equiv \pm w \pmod{pq}$.*

If $w = 1$, then we say that $[p, q, r]$ is of Type 1.

If $w > 1$, $q \equiv 1 \pmod{pw}$ and $p \equiv 1 \pmod w$, then we say that $[p, q, r]$ is of Type 2.

If $w > p$, $q > p(p-1)$, $q \equiv \pm 1 \pmod p$ and $w \equiv \pm 1 \pmod p$, and in the case where $w \equiv 1 \pmod p$ we have $wp \nmid q + 1$ and $wp \nmid q - 1$, then we say that $[p, q, r]$ is of Type 3.

Then $A(pqr) = 1$ if and only if $[p, q, r]$ is of Type 1 or 2, or $[p, q, r]$ is of Type 3 and $\Phi_{pq}(x^s)/\Phi_{pq}(x)$ is flat, where s is the smallest positive integer such that $s \equiv 1 \pmod p$ and $s \equiv \pm r \pmod{pq}$.

In 2007, Kaplan ([9]) proved the following periodicity of $A(pqr)$, which implies that for given p and q, $A(pqr)$ is completed determined by the residue class of $r \bmod pq$.

Proposition 1.3. *Let $3 \le p < q < r$ be primes. Then for any prime $s > q$ such that $s \equiv \pm r \pmod{pq}$, $A(pqr) = A(pqs)$.*

Moreover, if z is the least positive integer such that $zr \equiv \pm 1 \pmod{pq}$, then the smaller the value of z is the simpler analysis of the function $A(pqr)$ appears to be. Consequently, we may try to investigate flatness of $\Phi_{pqr}(x)$ with $q \equiv \pm 1 \pmod{p}$ for small values of z. So far, the analysis has been completed for all $z \le 7$, see [2, 3, 6, 7, 8, 9, 14, 15, 16, 17, 18]. In this paper, we continue the study of the flatness of ternary cyclotomic polynomials $\Phi_{pqr}(x)$ in the case $z = 8$. First note that in this case, by taking $h = 8$, $w \equiv 0 \pmod{3}$ in Proposition 1.1, we have, for odd primes $3 < q < r$ with $q \ge 17$ and $8r \equiv \pm 1 \pmod{3q}$, $A(3qr) = 1$ if and only if $q \ge 19$ and $q \equiv 1 \pmod{3}$. For $q = 5, 7, 11, 13$, by using the PARI/GP system (or consulting literature ([1])) and Proposition 1.3, we obtain $A(3qr) = 2$ when $q = 5, 7, 11, 13$ and $8r \equiv \pm 1 \pmod{3q}$. Therefore, we infer that the following statement holds.

Corollary 1.4. *Let $3 < q < r$ be primes such that $8r \equiv \pm 1 \pmod{3q}$. Then $A(3qr) = 1$ if and only if $q \ge 19$ and $q \equiv 1 \pmod{3}$.*

Our purpose here is to establish the following result.

Theorem 1.5. *Let $3 \le p < q < r$ be primes such that $q \equiv \pm 1 \pmod{p}$ and $8r \equiv \pm 1 \pmod{pq}$. Then $A(pqr) = 1$ if and only if*

(i) *$p = 3$, $q \ge 19$ and $q \equiv 1 \pmod{3}$, or*
(ii) *$p = 7$, $q \ge 83$ and $q \equiv -1 \pmod{7}$.*

We remark that, on invoking Proposition 1.3 and Corollary 1.4, it remains to prove this theorem in the cases

$$p \ge 5,\ q \equiv \pm 1 \pmod{p} \text{ and } 8r \equiv +1 \pmod{pq}.$$

We will present the proof for $p = 5$, $p = 7$, $p > 7$ in Sections 3, 4, 5, respectively.

2. Preliminaries

Recall that the binary cyclotomic polynomial coefficients $a(pq, m)$ have been completely determined in a simple and explicit way, see Lenstra ([11, (2.16)]), Lam and Leung ([10, Theorem]) or Thangadurai ([13, Theorem 2.3]). Considering this in the cases $q \equiv \pm 1 \pmod{p}$, we can obtain the following two useful results.

Lemma 2.1. *Let $3 \le p < q$ be primes such that $q = kp + 1$. Then*

$$a(pq, m) = \begin{cases} 1, & \text{if } m = up \text{ with } 0 \le u \le q - k - 1, \\ -1, & \text{if } m = up + vq + 1 \text{ with } 0 \le u \le k - 1, 0 \le v \le p - 2, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let $3 \le p < q$ be primes such that $q = kp - 1$. Then*

$$a(pq, m) = \begin{cases} 1, & \text{if } m = up + vq \text{ with } 0 \le u \le k-1, 0 \le v \le p-2, \\ -1, & \text{if } m = up + 1 \text{ with } 0 \le u \le q-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

In 2007, by using the fact that

$$\Phi_{pqr}(x) = \frac{1}{1 - x^{pq}} \left(\sum_{i=0}^{p-1} x^i - \sum_{i=0}^{p-1} x^{q+i} \right) \Phi_{pq}(x^r),$$

Kaplan ([9]) proved the following technical lemma, revealing the relationship between coefficients of $\Phi_{pqr}(x)$ and $\Phi_{pq}(x)$.

Lemma 2.3. *Let $3 \le p < q < r$ be primes. Given nonnegative integer l, let $f(i)$ denote the unique value $0 \le f(i) \le pq - 1$ such that*

$$f(i) \equiv \frac{(l-i)}{r} \pmod{pq}. \tag{2.1}$$

(1) *Then*

$$\sum_{i=0}^{p-1} a(pq, f(i)) = \sum_{i=0}^{p-1} a(pq, f(q+i)).$$

(2) *Set*

$$a^*(pq, m) = \begin{cases} a(pq, m), & \text{if } \; m \le \frac{l}{r}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Then

$$a(pqr, l) = \sum_{i=0}^{p-1} a^*(pq, f(i)) - \sum_{i=0}^{p-1} a^*(pq, f(q+i)).$$

3. Proof of Theorem 1.5 when $p = 5$

We will show the non-flatness of $\Phi_{5qr}(x)$ for $q \equiv \pm 1 \pmod 5$ and $8r \equiv 1 \pmod{5q}$ by proving the following two propositions.

Proposition 3.1. *Let $5 < q < r$ be primes such that $q \equiv 1 \pmod 5$ and $8r \equiv 1 \pmod{5q}$.*

(1) *If $q = 11$, then $A(55r) = 3$.*
(2) *If $q > 11$, then $a(5qr, qr + q + 6r + 2) = 2$.*

Proof. (1) By using PARI/GP or consulting literature ([1]), we have $A(5 \cdot 11 \cdot 227) = 3$. Then it follows from $8 \cdot 227 \equiv 1 \pmod{5 \cdot 11}$ and Proposition 1.3 that $A(5 \cdot 11 \cdot r) = 3$ when $8r \equiv 1 \pmod{5 \cdot 11}$.

(2) Let $q > 11$ and $l = qr + q + 6r + 2$. Then by using congruence $f(i) \equiv r^{-1}(l - i) \pmod{5q}$ and $0 \le f(i) \le 5q - 1$, we obtain

$$f(i) = 4q + 22 - 8i \text{ and } f(q+i) = q + 22 - 8i,$$

where $0 \leq i \leq 4$. So

$$f(q+4) < f(q+3) < f(q+2) < \frac{l}{r} < f(q+1) < f(q) < f(4) < \cdots < f(0).$$

By equation (2.2), it follows that

$$a^*(5q, f(i)) = \begin{cases} a(5q, f(i)), & \text{if } i \in \{q+2, q+3, q+4\}, \\ 0, & \text{if } i \in \{0, 1, 2, 3, 4, q, q+1\}. \end{cases}$$

Hence, by Lemma 2.3, we infer that

$$a(5qr, l) = -a(5q, f(q+4)) - a(5q, f(q+3)) - a(5q, f(q+2)). \tag{3.1}$$

On rewriting $f(q+2)$ and $f(q+4)$ as

$$f(q+2) = 1 \cdot 5 + 1 \cdot q + 1 \text{ and } f(q+4) = \tfrac{q-11}{5} \cdot 5 + 1,$$

we obtain from Lemma 2.1 that

$$a(5q, f(q+2)) = a(5q, f(q+4)) = -1.$$

Note that $f(q+3) = q - 2 \equiv 4 \pmod 5$. On invoking Lemma 2.1, we have $a(5q, f(q+3)) \neq 1$. If $a(5q, f(q+3)) = -1$, then, by another application of Lemma 2.1, there must exist integers $0 \leq u \leq \frac{q-1}{p} - 1$ and $0 \leq v \leq 3$ such that $f(q+3) = q - 2 = 5u + vq + 1$. Since $0 < f(q+3) < q$, we have $v = 0$. This yields $q - 2 = 5u + 1$, a contradiction to the fact $q \equiv 1 \pmod 5$. So

$$a(5q, f(q+3)) = 0.$$

Finally, by substituting the values of $a(5q, f(q+i))$ into (3.1), we obtain $a(5qr, l) = 2$. ∎

Proposition 3.2. *Let $5 < q < r$ be primes such that $q \equiv -1 \pmod 5$ and $8r \equiv 1 \pmod{5q}$. Then $a(5qr, 2qr + 10r + 1) = 2$.*

Proof. Let $l = 2qr + 10r + 1$. By using congruence (2.1), we have

$$f(i) = 2q + 18 - 8i \text{ and } f(q+i) = 4q + 18 - 8i,$$

where $0 \leq i \leq 4$. So

$$f(4) < f(3) < f(2) < f(1) < \frac{l}{r} < f(0) < f(q+4) < \cdots < f(q).$$

Then it follows from Lemma 2.3 that

$$a(5qr, l) = a(5q, f(4)) + a(5q, f(3)) + a(5q, f(2)) + a(5q, f(1)).$$

Since $f(1) = 2 \cdot 5 + 2q$ and $f(4) = \frac{q-14}{5} \cdot 5 + q$, we have $a(5q, f(1)) = a(5q, f(4)) = 1$ by Lemma 2.2. Note that $f(2) \equiv 0 \pmod 5$ and $f(3) \equiv 2 \pmod 5$. In view of Lemma 2.2, we infer that $a(5q, f(2)) \neq -1$ and

$a(5q, f(3)) \neq -1$. It is easy to show that neither $f(2)$ nor $f(3)$ can be written in the form $u \cdot 5 + v \cdot q$ for $0 \leq u \leq \frac{q+1}{5}$ and $0 \leq v \leq 3$. Then it follows from Lemma 2.2 that $a(5q, f(2)) = a(5q, f(3)) = 0$, and thus $a(5qr, l) = 2$. ∎

4. Proof of Theorem 1.5 when $p = 7$

In this section, we will give the necessary and sufficient conditions for $\Phi_{7qr}(x)$ to be flat in the cases $q \equiv \pm 1 \pmod 7$ and $8r \equiv 1 \pmod{7q}$ by showing the following two propositions.

Proposition 4.1. *Let $7 < q < r$ be primes such that $q \equiv 1 \pmod 7$ and $8r \equiv 1 \pmod{7q}$.*

(1) *If $q = 29$, then $A(203r) = 2$.*
(2) *If $q > 29$, then $a(7qr, 5qr + q + r + 5) = 2$.*

Proof. (1) If $q = 29$, we obtain $A(7 \cdot 29 \cdot 127) = 2$ by using PARI/GP or [1]. Then it follows from $8 \cdot 127 \equiv 1 \pmod{7 \cdot 29}$ and Lemma 1.3 that $A(7 \cdot 29 \cdot r) = 2$ when $8r \equiv 1 \pmod{7q}$.

(2) Let $l = 5qr + q + r + 5$. By using the congruence (2.1) and $0 \leq f(i) \leq 7q - 1$, we obtain

$$f(i) = 6q + 41 - 8i \text{ and } f(q + i) = 5q + 41 - 8i,$$

where $0 \leq i \leq 6$. Then

$$\begin{aligned} f(q+6) < f(q+5) < \frac{l}{r} \\ < f(q+4) < \cdots < f(q) < f(6) < \cdots < f(0). \end{aligned}$$

Thus, by Lemma 2.3,

$$a(7qr, l) = -a(7q, f(q+6)) - a(7q, f(q+5)). \tag{4.1}$$

Note that $f(q + 5) = 5q + 1$ and $f(q + 6) = \frac{q-8}{7} \cdot 7 + 4q + 1$. It follows from Lemma 2.1 that $a(7q, f(q + 5)) = a(7q, f(q + 6)) = -1$. Hence $a(7qr, l) = 2$. ∎

Proposition 4.2. *Let $7 < q < r$ be primes such that $q \equiv -1 \pmod 7$ and $8r \equiv 1 \pmod{7q}$. Then*

$$A(7qr) = \begin{cases} 2, & \text{if } q = 13, 41, \\ 1, & \text{if } q \geq 83. \end{cases}$$

Proof. The smallest three primes such that $q \equiv -1 \pmod 7$ are 13, 41 and 83. With the help of PARI/GP or [1], we know that $A(7 \cdot 13 \cdot 239) = 2$. On noting that $8 \cdot 239 \equiv 1 \pmod{7 \cdot 13}$, we infer from Proposition 1.3 that $A(7 \cdot 13 \cdot r) = 2$ for r satisfying $8r \equiv 1 \pmod{7 \cdot 13}$. Similarly, we obtain that $A(7 \cdot 41 \cdot r) = 2$ for r with $8r \equiv 1 \pmod{7 \cdot 41}$, since $A(7 \cdot 41 \cdot 1471) = 2$ and $8 \cdot 1471 \equiv 1 \pmod{7 \cdot 41}$.

Next we show that $A(7qr) = 1$ when $q \geq 83$, $q \equiv -1 \pmod 7$ and $8r \equiv 1 \pmod{7q}$. Note that Lemma 2.3 gives

$$a(7qr, l) = \sum_{i=0}^{6} a^*(7q, f(i)) + \sum_{i=0}^{6} \Big(- a^*(7q, f(q+i)) \Big), \tag{4.2}$$

where $f(i) \equiv \frac{l-i}{r} \pmod{7q}$, $0 \leq f(i) \leq 7q - 1$, and

$$a^*(7q, f(i)) = \begin{cases} a(7q, f(i)), & \text{if } f(i) \leq \frac{l}{r}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

As for binary coefficients $a(7q, f(i))$, we can rewrite the results of Lemma 2.2 in the following form

$$a(7q, f(i)) = \begin{cases} 1, & \text{if } f(i) \equiv 0 \pmod 7 \text{ and } 0 \leq f(i) \leq q-6, \\ 1, & \text{if } f(i) \equiv 6 \pmod 7 \text{ and } q \leq f(i) \leq 2q-6, \\ 1, & \text{if } f(i) \equiv 5 \pmod 7 \text{ and } 2q \leq f(i) \leq 3q-6, \\ 1, & \text{if } f(i) \equiv 4 \pmod 7 \text{ and } 3q \leq f(i) \leq 4q-6, \\ 1, & \text{if } f(i) \equiv 3 \pmod 7 \text{ and } 4q \leq f(i) \leq 5q-6, \\ 1, & \text{if } f(i) \equiv 2 \pmod 7 \text{ and } 5q \leq f(i) \leq 6q-6, \\ -1, & \text{if } f(i) \equiv 1 \pmod 7 \text{ and } 1 \leq f(i) \leq 6q-7, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Given $l \in [0, \phi(7qr)]$, the value of $f(i)$ is uniquely defined and we have

$$f(i) \equiv f(0) - 8i \pmod{7q}, \tag{4.5}$$

$$f(q+i) \equiv f(0) - q - 8i \pmod{7q}, \tag{4.6}$$

where $0 \leq i \leq 6$.

For $f(0) = 0$, by using (4.5) and (4.6), we have $f(i) = 7q - 8i$ when $1 \leq i \leq 6$ and $f(q+i) = 6q - 8i$ when $0 \leq i \leq 6$. So

$$f(0) < f(q+6) < \cdots < f(q) < f(6) < \cdots < f(1). \tag{4.7}$$

In the rest of this section, because of space limitation, we set

$$a_i := a(7q, f(i)),$$

and it follows from (4.4) that

Table 1. $f(0) = 0$

	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1
value	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

For any given integer $l \in [0, \phi(7qr)]$, if $f(1) \leq \frac{l}{r}$, then, by (4.2) and (4.3), we infer that

$$a(7qr, l) = \sum_{i=0}^{6} a(7q, f(i)) + \sum_{i=0}^{6} \Big(- a(7q, f(q+i)) \Big) = 0.$$

Otherwise, there must exist two neighboring symbols $f(j_1)$ and $f(j_2)$ in (4.7) such that

$$f(j_1) \le \frac{l}{r} < f(j_2).$$

If $0 \le j_1 \le 6$ (or $q \le j_1 \le q+6$), the value of $a(7qr,l)$ is given by computing the sum of binary coefficients from the start of the third row in Table 1 to $a_{f(j_1)}$ (or $-a_{f(j_1)}$). It is clear to see that the data in Table 1 reveal that the sums in (4.2) are always in the set $\{0,1\}$.

For $f(0)=1$, we have $f(i)=7q-8i+1$, when $1 \le i \le 6$, and $f(q+i) = 6q-8i+1$, when $0 \le i \le 6$. So the inequalities (4.7) still hold in this case. And it follows from (4.4) that

Table 2. $f(0) = 1$

	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1
value	-1	0	0	0	0	0	1	0	0	0	0	0	0	0

Given integer $l \in [0, \phi(7qr)]$, if $f(1) \le \frac{l}{r}$, then

$$a(7qr,l) = \sum_{i=0}^{6} a(7q, f(i)) + \sum_{i=0}^{6} \Big(-a(7q, f(q+i)) \Big) = 0.$$

If $\frac{l}{r} < f(0)$, then

$$a(7qr,l) = \sum_{i=0}^{6} 0 + \sum_{i=0}^{6} 0 = 0.$$

Otherwise, there must exist two neighboring symbols $f(j_1)$ and $f(j_2)$ in (4.7) such that $f(j_1) \le \frac{l}{r} < f(j_2)$. Similarly, the data in Table 2 yield that $a(7qr,l) \in \{-1,0\}$.

Now according to the values of $f(0)$, we give the following tables. The second row of each table is the inequality about $f(i)$ for $i \in \{0,1,2,3,4,5,6,q,q+1,q+2,q+3,q+4,q+5,q+6\}$. In the rest of this section, for the reasons of space, we set

$$a_i := a(7q, f(i))$$

and let $\overline{f(0)}$ be the unique integer such that $0 \le \overline{f(0)} \le 6$ and $\overline{f(0)} \equiv f(0) \pmod 7$. The values of a_i are obtained by using (4.4)–(4.6).

Table 3. $2 \le f(0) \le 7$

	$f(0) < f(q+6) < \cdots < f(q) < f(6) < \cdots < f(1)$													
$f(0)$	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1
2	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
3	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
4	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
5	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
6	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
7	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 4. $8 \leq f(0) \leq 15$

	$f(1) < f(0) < f(q+6) < \cdots < f(q) < f(6) < \cdots < f(2)$													
$f(0)$	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2
8	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
9	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
10	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
11	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
12	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
13	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
14	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
15	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 5. $16 \leq f(0) \leq 23$

	$f(2) < f(1) < f(0) < f(q+6) < \cdots < f(q) < f(6) < \cdots < f(3)$													
$f(0)$	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3
16	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
17	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
18	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
19	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
20	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
21	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
22	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
23	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 6. $24 \leq f(0) \leq 31$

	$f(3) < f(2) < f(1) < f(0) < f(q+6) < \cdots < f(q) < f(6) < f(5) < f(4)$													
$f(0)$	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4
24	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
25	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
26	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
27	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
28	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
29	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
30	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
31	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 7. $32 \leq f(0) \leq 39$

	$f(4) < f(3) < f(2) < f(1) < f(0) < f(q+6) < \cdots < f(q) < f(6) < f(5)$													
$f(0)$	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5
32	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
33	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
34	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
35	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
36	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
37	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
38	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
39	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 8. $40 \leq f(0) \leq 47$

	$f(5) < f(4) < f(3) < f(2) < f(1) < f(0) < f(q+6) < \cdots < f(q) < f(6)$													
$f(0)$	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6
40	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
41	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
42	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
43	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
44	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
45	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
46	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
47	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 9. $48 \leq f(0) \leq q-1$

	$f(6) < \cdots < f(0) < f(q+6) < \cdots < f(q)$													
$\overline{f(0)}$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$
0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
2	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
3	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
4	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
5	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
6	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 10. $q \le f(0) \le q+7$

	$f(q) < f(6) < \cdots < f(0) < f(q+6) < \cdots < f(q+1)$													
$f(0)$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$
q	-1	1	-1	0	0	0	0	1	0	0	0	0	0	0
$q+1$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
$q+2$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$q+3$	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
$q+4$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$q+5$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$q+6$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
$q+7$	-1	1	-1	0	0	0	0	1	0	0	0	0	0	0

Table 11. $q+8 \le f(0) \le q+15$

	$f(q+1) < f(q) < f(6) < \cdots < f(0) < f(q+6) < \cdots < f(q+2)$													
$f(0)$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$
$q+8$	-1	1	-1	0	0	0	0	1	0	0	0	0	0	0
$q+9$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$q+10$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$q+11$	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
$q+12$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$q+13$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$q+14$	0	-1	1	-1	0	0	0	0	1	0	0	0	0	0
$q+15$	-1	1	-1	0	0	0	0	1	0	0	0	0	0	0

Table 12. $q+16 \le f(0) \le q+23$

	$f(q+2) < f(q+1) < f(q) < f(6) < \cdots < f(0) < f(q+6) < \cdots < f(q+3)$													
$f(0)$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$
$q+16$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0
$q+17$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$q+18$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$q+19$	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
$q+20$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$q+21$	0	0	-1	1	-1	0	0	0	0	1	0	0	0	0
$q+22$	0	-1	1	-1	0	0	0	0	1	0	0	0	0	0
$q+23$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0

Table 13. $q+24 \le f(0) \le q+31$

	$f(q+3) < f(q+2) < f(q+1) < f(q) < f(6) < \cdots < f(0) < f(q+6) < f(q+5) < f(q+4)$													
$f(0)$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$
$q+24$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0
$q+25$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$q+26$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$q+27$	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
$q+28$	0	0	0	-1	1	-1	0	0	0	0	1	0	0	0
$q+29$	0	0	-1	1	-1	0	0	0	0	1	0	0	0	0
$q+30$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$q+31$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0

Table 14. $q+32 \le f(0) \le q+39$

	$f(q+4) < f(q+3) < f(q+2) < f(q+1) < f(q) < f(6) < \cdots < f(0) < f(q+6) < f(q+5)$													
$f(0)$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$	$-a_{q+5}$
$q+32$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0
$q+33$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$q+34$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$q+35$	0	0	0	0	-1	1	-1	0	0	0	0	1	0	0
$q+36$	0	0	0	-1	1	-1	0	0	0	0	1	0	0	0
$q+37$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$q+38$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$q+39$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0

Table 15. $q+40 \le f(0) \le q+47$

	$f(q+5) < f(q+4) < f(q+3) < f(q+2) < f(q+1) < f(q) < f(6) < \cdots < f(0) < f(q+6)$													
$f(0)$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0	$-a_{q+6}$
$q+40$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0
$q+41$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$q+42$	0	0	0	0	0	-1	1	-1	0	0	0	0	1	0
$q+43$	0	0	0	0	-1	1	-1	0	0	0	0	1	0	0
$q+44$	0	0	0	-1	1	0	0	0	0	0	1	0	-1	0
$q+45$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$q+46$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$q+47$	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0

Table 16. $q+48 \le f(0) \le 2q-1$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	0	0	0	0	0	-1	1	-1	0	0	0	0	1	0
1	0	0	0	0	-1	1	0	0	0	0	0	1	0	-1
2	0	0	0	-1	1	0	0	0	0	0	1	0	-1	0
3	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
4	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
5	-1	1	0	0	0	0	0	1	0	-1	0	0	0	0
6	1	0	0	0	0	0	-1	0	-1	0	0	0	0	1

Table 17. $2q \le f(0) \le 2q+47$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$2q$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+1$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$2q+2$	0	0	0	0	0	-1	1	-1	0	0	0	0	1	0
$2q+3$	0	0	0	0	-1	1	0	0	0	0	0	1	0	-1
$2q+4$	0	0	0	-1	1	0	0	0	0	0	1	0	-1	0
$2q+5$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$2q+6$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$2q+7$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+8$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+9$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$2q+10$	0	0	0	0	-1	1	0	0	0	0	0	1	0	-1
$2q+11$	0	0	0	-1	1	0	0	0	0	0	1	0	-1	0
$2q+12$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$2q+13$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$2q+14$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+15$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+16$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$2q+17$	0	0	0	0	0	1	0	0	0	0	0	0	0	-1
$2q+18$	0	0	0	-1	1	0	0	0	0	0	1	0	-1	0
$2q+19$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$2q+20$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$2q+21$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+22$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+23$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$2q+24$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$2q+25$	0	0	0	0	1	0	0	0	0	0	0	0	-1	0
$2q+26$	0	0	-1	1	0	0	0	0	0	1	0	-1	0	0
$2q+27$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$2q+28$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+29$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+30$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$2q+31$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$2q+32$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$2q+33$	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
$2q+34$	0	-1	1	0	0	0	0	0	1	0	-1	0	0	0
$2q+35$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+36$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+37$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$2q+38$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$2q+39$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$2q+40$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$2q+41$	0	0	1	0	0	0	0	0	0	0	-1	0	0	0
$2q+42$	-1	1	0	0	0	0	-1	1	0	-1	0	0	0	1
$2q+43$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$2q+44$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$2q+45$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$2q+46$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$2q+47$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0

Table 18. $2q+48 \le f(0) \le 3q-1$														
	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
1	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
2	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
3	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
4	-1	0	1	0	0	0	0	1	0	0	-1	0	0	0
5	0	1	0	0	0	0	-1	0	0	-1	0	0	0	1
6	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0

Table 19. $3q \le f(0) \le 3q+47$														
	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$3q$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+1$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0
$3q+2$	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0
$3q+3$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$3q+4$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$3q+5$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$3q+6$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$3q+7$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+8$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+9$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$3q+10$	0	0	0	0	-1	0	1	-1	0	0	0	1	0	0
$3q+11$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$3q+12$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$3q+13$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$3q+14$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+15$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+16$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$3q+17$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$3q+18$	0	0	0	-1	0	1	0	0	0	0	1	0	0	-1
$3q+19$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$3q+20$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$3q+21$	-1	0	1	0	0	0	-1	1	0	0	0	-1	0	1
$3q+22$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+23$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$3q+24$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$3q+25$	0	0	0	0	1	0	0	0	0	0	0	0	0	-1
$3q+26$	0	0	-1	0	1	0	0	0	0	1	0	0	-1	0
$3q+27$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$3q+28$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+29$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+30$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$3q+31$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$3q+32$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$3q+33$	0	0	0	0	1	0	0	0	0	0	0	0	-1	0
$3q+34$	0	-1	0	1	0	0	0	0	1	0	0	-1	0	0
$3q+35$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+36$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+37$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$3q+38$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$3q+39$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$3q+40$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$3q+41$	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
$3q+42$	-1	0	1	0	0	0	-1	1	0	0	-1	0	0	1
$3q+43$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$3q+44$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$3q+45$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$3q+46$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$3q+47$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0

Table 20. $3q + 48 \le f(0) \le 4q - 1$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
1	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
2	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
3	-1	0	0	1	0	0	0	1	0	0	0	-1	0	0
4	0	0	1	0	0	0	-1	0	0	0	-1	0	0	1
5	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
6	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0

Table 21. $4q \le f(0) \le 4q + 47$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$4q$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+1$	0	0	1	0	0	0	0	0	0	0	-1	0	0	0
$4q+2$	0	1	0	0	0	-1	0	0	0	-1	0	0	1	0
$4q+3$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$4q+4$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$4q+5$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$4q+6$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$4q+7$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+8$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+9$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0
$4q+10$	1	0	0	0	-1	0	0	0	-1	0	0	1	0	0
$4q+11$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$4q+12$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$4q+13$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$4q+14$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+15$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+16$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$4q+17$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$4q+18$	0	0	0	-1	0	0	1	-1	0	0	1	0	0	0
$4q+19$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$4q+20$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$4q+21$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+22$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+23$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$4q+24$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$4q+25$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$4q+26$	0	0	-1	0	0	1	0	0	0	1	0	0	0	-1
$4q+27$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$4q+28$	-1	0	0	1	0	0	-1	1	0	0	-1	0	0	1
$4q+29$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+30$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$4q+31$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$4q+32$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$4q+33$	0	0	0	0	0	1	0	0	0	0	0	0	0	-1
$4q+34$	0	-1	0	0	1	0	0	0	1	0	0	0	-1	0
$4q+35$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+36$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+37$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$4q+38$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$4q+39$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$4q+40$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$4q+41$	0	0	0	0	1	0	0	0	0	0	0	0	-1	0
$4q+42$	-1	0	0	1	0	0	-1	1	0	0	0	-1	0	1
$4q+43$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$4q+44$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$4q+45$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$4q+46$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$4q+47$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1

Table 22. $4q+48 \le f(0) \le 5q-1$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
1	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
2	-1	0	0	0	1	0	0	1	0	0	0	0	-1	0
3	0	0	0	1	0	0	-1	0	0	0	0	-1	0	1
4	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
5	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
6	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0

Table 23. $5q \le f(0) \le 5q+47$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$5q$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+1$	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
$5q+2$	0	0	1	0	0	-1	0	0	0	0	-1	0	1	0
$5q+3$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$5q+4$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$5q+5$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$5q+6$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$5q+7$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+8$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$5q+9$	0	0	1	0	0	0	0	0	0	0	-1	0	0	0
$5q+10$	0	1	0	0	-1	0	0	0	0	-1	0	1	0	0
$5q+11$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$5q+12$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$5q+13$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$5q+14$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+15$	0	0	0	-1	0	1	0	0	0	0	0	-1	1	0
$5q+16$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$5q+17$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0
$5q+18$	1	0	0	-1	0	0	0	0	-1	0	1	0	0	0
$5q+19$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$5q+20$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$5q+21$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+22$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$5q+23$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$5q+24$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$5q+25$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$5q+26$	0	0	-1	0	0	0	1	-1	0	1	0	0	0	0
$5q+27$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$5q+28$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+29$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$5q+30$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$5q+31$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$5q+32$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$5q+33$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$5q+34$	0	-1	0	0	0	1	0	0	1	0	0	0	0	-1
$5q+35$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+36$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$5q+37$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$5q+38$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$5q+39$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$5q+40$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$5q+41$	0	0	0	0	0	1	0	0	0	0	0	0	0	-1
$5q+42$	-1	0	0	0	1	0	-1	1	0	0	0	0	-1	1
$5q+43$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$5q+44$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$5q+45$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$5q+46$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$5q+47$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0

Table 24. $5q+48 \le f(0) \le 6q-1$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$\overline{f(0)}$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
1	-1	0	0	0	0	1	0	1	0	0	0	0	0	-1
2	0	0	0	0	1	0	-1	0	0	0	0	0	-1	1
3	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
4	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
5	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
6	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0

Table 25. $6q \le f(0) \le 6q+47$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$f(0)$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$6q$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+1$	0	0	0	0	1	0	0	0	0	0	0	0	-1	0
$6q+2$	0	0	0	1	0	-1	0	0	0	0	0	-1	1	0
$6q+3$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$6q+4$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$6q+5$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$6q+6$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$6q+7$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+8$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+9$	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
$6q+10$	0	0	1	0	-1	0	0	0	0	0	-1	1	0	0
$6q+11$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$6q+12$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$6q+13$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$6q+14$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+15$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+16$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$6q+17$	0	0	1	0	0	0	0	0	0	0	-1	0	0	0
$6q+18$	0	1	0	-1	0	0	0	0	0	-1	1	0	0	0
$6q+19$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$6q+20$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$6q+21$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+22$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+23$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$6q+24$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
$6q+25$	0	1	0	0	0	0	0	0	0	-1	0	0	0	0
$6q+26$	1	0	-1	0	0	0	0	0	-1	1	0	0	0	0
$6q+27$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$6q+28$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+29$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+30$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$6q+31$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
$6q+32$	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
$6q+33$	1	0	0	0	0	0	0	0	-1	0	0	0	0	0
$6q+34$	0	-1	0	0	0	0	1	-1	1	0	0	0	0	0
$6q+35$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+36$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+37$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$6q+38$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
$6q+39$	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
$6q+40$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
$6q+41$	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
$6q+42$	-1	0	0	0	0	1	-1	1	0	0	0	0	0	0
$6q+43$	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
$6q+44$	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
$6q+45$	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
$6q+46$	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
$6q+47$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

Table 26. $6q + 48 \leq f(0) \leq 7q - 1$

	$f(q+6) < \cdots < f(q) < f(6) < \cdots < f(0)$													
$\overline{f(0)}$	$-a_{q+6}$	$-a_{q+5}$	$-a_{q+4}$	$-a_{q+3}$	$-a_{q+2}$	$-a_{q+1}$	$-a_q$	a_6	a_5	a_4	a_3	a_2	a_1	a_0
0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	1	-1	0	0	0	0	0	0	0
2	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
3	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
4	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
5	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
6	1	-1	0	0	0	0	0	0	0	0	0	0	0	0

It is a routine matter to check that the sum of values about $\pm a_i$, from the start to anywhere of the row in all tables, is equal to -1, 0 or 1. That is to say, the data reveals that the sums in (4.2) are always in the set $\{-1, 0, 1\}$. So $A(7qr) = 1$ in the case $q \geq 83$, $q \equiv -1 \pmod 7$ and $8r \equiv 1 \pmod{7q}$. $\square$

5. Proof of Theorem 1.5 when $p > 7$

The theorem will be completely proved by showing the following two propositions.

Proposition 5.1. *Let $7 < p < q < r$ be primes such that $q = kp - 1$ and $8r \equiv 1 \pmod{pq}$.*

(1) *If $p = 11$, then $a(11qr, qr + 22r + q + 6) \leq -2$.*
(2) *If $p \equiv 1 \pmod 8$, then $a(pqr, pqr - 12qr + q + \frac{7p-7}{8}) \leq -2$.*
(3) *If $p \equiv 3 \pmod 8$ and $p > 11$, then $a(pqr, pqr + pr - 12qr + q + \frac{5p-7}{8}) \leq -2$.*
(4) *If $p \equiv 5 \pmod 8$, then $a(pqr, pqr + 3pr - 11qr + q + \frac{3p-7}{8}) \leq -2$.*
(5) *If $p \equiv 7 \pmod 8$ and $k = 2$, then $a(pqr, 9qr + q + \frac{3p-5}{8}) \leq -2$.*
(6) *If $p \equiv 7 \pmod 8$ and $k = 4$, then $a(pqr, 8qr + q + \frac{p-3}{4}) \leq -2$.*
(7) *If $p \equiv 7 \pmod 8$ and $k \geq 6$, then $a(pqr, 5pr + 7qr + q + \frac{p-7}{8}) \leq -2$.*

Proof. (1) Let $l = qr + 22r + q + 6$. By using congruence (2.1), we have

$$f(i) \equiv 9q + 70 - 8i \pmod{11q}.$$

According to Lemma 2.3, we only consider $f(i)$ for $i \in [0, 10] \cup [q, q+10]$. Since the value of $f(i)$ is in the range $0 \leq f(i) \leq 11q - 1$, we have $f(i) = 9q + 70 - 8i$. Then

$$f(q+10) < \cdots < f(q+6) < \frac{l}{r} < f(q+5) \cdots < f(q) < f(10) < \cdots < f(0).$$

It follows from Lemma 2.3 that

$$a(11qr, l) = -\sum_{i=6}^{10} a(11q, f(q+i)).$$

Since $f(q+6) = 2 \cdot 11 + q$ and $f(q+10) = (k-1)11$, by Lemma 2.2, we have $a(11q, f(q+6)) = a(11q, f(q+10)) = 1$. Thus

$$a(11qr, l) = -2 - a(11q, f(q+7)) - a(11q, f(q+8)) - a(11q, f(q+9)).$$

It is easy to see that $f(q+7) \equiv 2 \pmod{11}$, $f(q+8) \equiv 5 \pmod{11}$ and $f(q+9) \equiv 8 \pmod{11}$. In view of Lemma 2.2, we infer $a(11q, f(q+i)) \in \{0,1\}$ when $i = 7, 8, 9$. Therefore, $a(11qr, l) \leq -2$.

(2) Let $l = pqr - 12qr + q + \frac{7p-7}{8}$. By using congruence $f(i) \equiv \frac{(l-i)}{r} \pmod{pq}$, we have $f(i) \equiv pq + 7p - 4q - 8i - 7 \pmod{pq}$. According to Lemma 2.3, we only consider $f(i)$ for $i \in [0, p-1] \cup [q, q+p-1]$. Since $0 \leq f(i) \leq pq - 1$, we obtain

$$f(i) = pq + 7p - 4q - 8i - 7. \tag{5.1}$$

Then we have

$$f(q+p-1) < \cdots < f(q + \frac{7p-7}{8}) < \frac{l}{r},$$

$$\frac{l}{r} < f(q + \frac{7p+1}{8}) < \cdots < f(q) < f(p-1) < \cdots < f(0).$$

So, by Lemma 2.3,

$$a(pqr, l) = -\sum_{i=\frac{7p-7}{8}}^{p-1} a(pq, f(q+i)). \tag{5.2}$$

Note that $f(q + \frac{7p-7}{8}) = (p-12)q$ and $f(q+p-1) = (k-1)p + (p-13)q$. It follows from Lemma 2.2 that $a(pq, f(q + \frac{7p-7}{8})) = a(pq, f(q+p-1)) = 1$. Substituting this into (5.2) yields

$$a(pqr, l) = -2 - \sum_{i=\frac{7p+1}{8}}^{p-2} a(pq, f(q+i)).$$

As is known to all, the binary coefficient $a(pq, f(q+i))$ takes on one of three values: -1, 0 or 1. For the purpose of proving $a(pqr, l) \leq -2$, it suffices to show that

$$a(pq, f(q+i)) \neq -1 \text{ when } \tfrac{7p+1}{8} \leq i \leq p-2.$$

If the statement was not true, then, by Lemma 2.2, we certainly have

$$f(q+i) \equiv 1 \pmod{p}.$$

Applying (5.1) to the above congruence gives

$$8i - 4 \equiv 0 \pmod{p}.$$

Combing this and $7p - 3 \leq 8i - 4 \leq 8p - 20$, we obtain $8i - 4 = 7p$, a contradiction to $p \equiv 1 \pmod 8$. Hence $a(pqr, l) \leq -2$. (3) Let $l = pqr + pr - 12qr + q + \frac{5p-7}{8}$. By using congruence (2.1) and $p > 11$, we have $f(i) = pq - 4q + 6p - 7 - 8i$, where $i \in [0, p-1] \cup [q, q+p-1]$. Then $\frac{l}{r} > f(i)$ whenever $i \in \{q + \frac{5p-7}{8}, q + \frac{5p+1}{8}, \cdots, q+p-1\}$ and $\frac{l}{r} < f(i)$ whenever $i \in$

$\{0, 1 \cdots, p-1\} \cup \{q, q+1, \cdots, q+\frac{5p-15}{8}\}$. Note that $f(q+\frac{5p-7}{8}) = p+(p-12)q$ and $f(q+p-1) = (k-2)p+(p-13)q$. So, by Lemmas 2.2 and 2.3,

$$a(pqr, l) = -\sum_{i=\frac{5p-7}{8}}^{p-1} a(pq, f(q+i)) = -2 - \sum_{i=\frac{5p+1}{8}}^{p-2} a(pq, f(q+i)).$$

It is clear that $a(pq, f(q+i)) \in \{-1, 0, 1\}$. In order to show $a(pqr, l) \leq -2$, we only need to prove that $a(pq, f(q+i)) \neq -1$ for $\frac{5p+1}{8} \leq i \leq p-2$. If $a(pq, f(q+i)) = -1$, then, by Lemma 2.2, we infer

$$f(q+i) \equiv 5 - 8i \equiv 1 \pmod{p}.$$

Since $5p-3 \leq 8i-4 \leq 8p-20$, we obtain $8i-4 = 5p, 6p, 7p$. This contradicts the fact $p \equiv 3 \pmod 8$. Hence $a(pqr, l) \leq -2$.

(4) Let $l = pqr+3pr-11qr+q+\frac{3p-7}{8}$. By substituting l into congruence $rf(i) \equiv l-i \pmod{pq}$, we have $f(i) = pq-3q+6p-7-8i$, where $i \in [0, p-1] \cup [q, q+p-1]$. On invoking Lemma 2.3, we can obtain

$$a^*(pq, f(i)) = \begin{cases} a(pq, f(i)), & \text{if } i \in [q+\frac{3p-7}{8}, q+p-1], \\ 0, & \text{if } i \in [0, p-1] \cup [q, q+\frac{3p-15}{8}]. \end{cases}$$

Then

$$a(pqr, l) = -\sum_{i=\frac{3p-7}{8}}^{p-1} a(pq, f(q+i)). \tag{5.3}$$

Since $f(q+\frac{3p-7}{8}) = 3p+(p-11)q$ and $f(q+p-1) = (k-2)p+(p-12)q$, we have $a(pq, f(q+\frac{3p-7}{8})) = a(pq, f(q+p-1)) = 1$ by Lemma 2.2. Applying this to (5.3) gives

$$a(pqr, l) = -2 - \sum_{i=\frac{3p+1}{8}}^{p-2} a(pq, f(q+i)).$$

Next we use Lemma 2.2 to show that

$$a(pq, f(q+i)) \neq -1 \text{ for } \tfrac{3p+1}{8} \leq i \leq p-2.$$

If the statement would not hold, then

$$f(q+i) \equiv 4 - 8i \equiv 1 \pmod{p}.$$

It follows from $\frac{3p+1}{8} \leq i \leq p-2$ that

$$8i - 3 = 3p, 4p, 5p, 6p, 7p.$$

This is contrary to $p \equiv 5 \pmod 8$. Then in the range $\frac{3p+1}{8} \leq i \leq p-2$, the quantity $a(pq, f(q+i))$ takes on one of two values: 0 or 1, and thus $a(pqr, l) \leq -2$.

(5) Let $l = 9qr + q + \frac{3p-5}{8}$. Proceeding as before, we have $f(i) = 3p + 17q - 5 - 8i$, where $i \in [0, p-1] \cup [q, q+p-1]$. According to Lemma 2.3, we deduce that

$$a(pqr, l) = -\sum_{i=\frac{3p-5}{8}}^{p-1} a(pq, f(q+i)).$$

On noting that $q = 2p-1$, we have $f(q + \frac{3p-5}{8}) = 9q$ and $f(q+p-1) = p+6q$. It follows from Lemma 2.2 that

$$a(pq, f(q + \frac{3p-5}{8})) = a(pq, f(q+p-1)) = 1,$$

and then

$$a(pqr, l) = -2 - \sum_{i=\frac{3p+3}{8}}^{p-2} a(pq, f(q+i)).$$

Our task now is to show

$$f(q+i) \not\equiv 1 \pmod{p} \text{ when } \tfrac{3p+3}{8} \le i \le p-2.$$

If the assertion was false, then $f(q+i) \equiv -8i - 14 \equiv 1 \pmod{p}$. Since $\frac{3p+3}{8} \le i \le p-2$, we obtain $8i + 15 = 4p, 5p, 6p, 7p$, a contradiction to $p \equiv 7 \pmod{8}$. On invoking Lemma 2.2, we infer that $a(pq, f(q+i)) \in \{0, 1\}$ for $\frac{3p+3}{8} \le i \le p-2$. Therefore, $a(pqr, l) \le -2$.

(6) Let $l = 8qr + q + \frac{p-3}{4}$, where $q = 4p - 1$. By using the congruence (2.1), we have $f(i) = 2p+16q-6-8i$ when $0 \le i \le p-1$ and $q \le i \le q+p-1$. It follows from Lemma 2.3 that

$$a(pqr, l) = -\sum_{i=\frac{p-3}{4}}^{p-1} a(pq, f(q+i)).$$

Note that $f(q + \frac{p-3}{4}) = 8q$ and $f(q+p-1) = 2p+6q$. In view of Lemma 2.2, we have $a(pq, f(q + \frac{p-3}{4})) = a(pq, f(q+p-1)) = 1$, and then

$$a(pqr, l) = -2 - \sum_{i=\frac{p+1}{4}}^{p-2} a(pq, f(q+i)).$$

Let $\frac{p+1}{4} \le i \le p-2$. We claim that $f(q+i) \not\equiv 1 \pmod{p}$. If otherwise, then

$$f(q+i) \equiv -14 - 8i \equiv 1 \pmod{p}.$$

Since $2p+17 \le 8i+15 \le 8p-1$, we obtain $8i+15 = 3p, 4p, 5p, 6p, 7p$. This leads to a contradiction to $p \equiv 7 \pmod{8}$. So, by Lemma 2.2, $a(pq, f(q+i)) = 0$ or 1. Hence $a(pqr, l) \le -2$.

(7) Our argument here proceeds along the same lines. Taking $l = 5pr + 7qr + q + \frac{p-7}{8}$ in congruence (2.1), we have $f(i) = 6p + 15q - 7 - 8i$, where $i \in [0, p-1] \cup [q, q+p-1]$. According to Lemma 2.3, we deduce that

$$a(pqr, l) = -\sum_{i=\frac{p-7}{8}}^{p-1} a(pq, f(q+i)).$$

On noting that $f(q + \frac{p-7}{8}) = 5p + 7q$ and $f(q+p-1) = (k-2)p + 6q$, we have, in light of $k \geq 6$ and Lemma 2.2,

$$a(pq, f(q + \frac{p-7}{8})) = a(pq, f(q+p-1)) = 1,$$

and then

$$a(pqr, l) = -2 - \sum_{i=\frac{p+1}{8}}^{p-2} a(pq, f(q+i)).$$

Let $\frac{p+1}{8} \leq i \leq p-2$. Our goal now is to show

$$f(q+i) \not\equiv 1 \pmod{p}.$$

If the assertion was false, then $f(q+i) \equiv -8i - 14 \equiv 1 \pmod{p}$. Since $\frac{p+1}{8} \leq i \leq p-2$, we obtain $8i + 15 = 2p, 3p, 4p, 5p, 6p, 7p$, a contradiction to $p \equiv 7 \pmod{8}$. On invoking Lemma 2.2, we infer that $a(pq, f(q+i)) \in \{0, 1\}$.

Finally, we obtain $a(pqr, l) \leq -2$. This completes the proof. ∎

Proposition 5.2. *Let $7 < p < q < r$ be odd primes such that $q = kp + 1$ and $8r \equiv 1 \pmod{pq}$.*

(1) *If $p \equiv 1 \pmod{8}$, then*

$$2 \leq \begin{cases} a(pqr, 6pr + 5qr + q + 4r + \frac{3p-11}{8}), & \text{if } k = 2, \\ a(pqr, pqr - 9qr + q + r + \frac{p-5}{4}), & \text{if } k = 4, \\ a(pqr, pqr + 5pr - 9qr + q + r + \frac{p-9}{8}), & \text{if } k \geq 6. \end{cases}$$

(2) *If $p \equiv 3 \pmod{8}$, then*

$$2 \begin{cases} = A(pqr), & \text{if } k = 2 \text{ and } p = 11, \\ \leq a(pqr, pqr - pr - 8qr + q + \frac{p-11}{8}), & \text{if } k = 2 \text{ and } p > 11, \\ \leq a(pqr, pqr - pr - 10qr + q + \frac{3p-9}{8}), & \text{if } k = 4, \\ \leq a(pqr, pqr + 3pr - 9qr + q + r + \frac{3p-9}{8}), & \text{if } k \geq 6. \end{cases}$$

(3) *If $p \equiv 5 \pmod{8}$, then*

$$2 \leq \begin{cases} a(pqr, pqr + 3pr - 13qr + q + 2r + \frac{5p-9}{8}), & \text{if } k = 2, \\ a(pqr, pqr + pr - 10qr + q + r + \frac{5p-9}{8}), & \text{if } k \geq 4. \end{cases}$$

(4) *If $p \equiv 7 \pmod{8}$, then $2 \leq a(pqr, pqr - 10qr + q + r + \frac{7p-9}{8})$.*

PROOF. The proof of this proposition follows in a similar manner and so is omitted. ∎

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