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CONTINUITY OF GENERALIZED RIESZ POTENTIALS FOR DOUBLE PHASE FUNCTIONALS WITH VARIABLE EXPONENTS

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Abstract. In this note, we discuss the continuity of generalized Riesz potentials $I_\rho f$ of functions in Morrey spaces $L^{\Phi,\nu(\cdot)}(G)$ of double phase functionals with variable exponents.

1. Introduction

The double phase functional introduced by Zhikov ([30]) in the 1980s has been studied intensively by many mathematichans. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione in [1, 4, 5] studied a double phase functional

$$\tilde{\Phi}(x,t) = t^p + a(x)t^q,\ x \in \mathbf{R}^N,\ t \geq 0,$$

where $N \geq 2$, $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$. We refer to [10, 16] for Sobolev's inequality, [11] for Trudinger's inequality and e.g. [2, 7, 8] for other double phase problems.

For $0 < \alpha < N$ and a locally integrable function f on $\mathbf{R}^N$ the Riesz potential $I_\alpha f$ of order α is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\,dy.$$

In [12] we discussed the continuity of Riesz potentials $I_\alpha f$ of functions in Morrey spaces $L^{\tilde{\Phi},\nu}(\mathbf{R}^N)$ of the double phase functionals $\tilde{\Phi}(x,t)$. We refer to [16, Section 5] for the $L^{\tilde{\Phi}}$ case and [14] for the $L^{p(\cdot),\nu(\cdot)}$ case.

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In the present note, we consider the case $\Phi(x,t)$ is a double phase functional given by

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(x) < q(x)$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ ([10], cf. [3, 25]).

To obtain general results, we consider the family (ρ) of all functions ρ satisfying the following conditions: $\rho : (0,\infty) \to (0,\infty)$ is a measurable function such that

$$\int_0^r \rho(s)\,\frac{ds}{s} < \infty$$

for all sufficiently small $r > 0$ and there exists constants $0 < k < 1$, $0 < k_1 < k_2$ and $C_\rho > 0$ such that

$$\sup_{kr \le s \le r} \rho(s) \le C_\rho \int_{k_1 r}^{k_2 r} \rho(s)\,\frac{ds}{s} \tag{1.1}$$

for all $r > 0$ (e.g. [6, 26]). We do not postulate the doubling condition on ρ.

Example 1.1. If ρ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that $C^{-1} \le \rho(r)/\rho(s) \le C$ for $1/2 \le r/s \le 2$, then ρ satisfies (1.1) whenever $k = 1/2$ and $2k_1 = k_2$. If ρ is increasing, then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$. If $\alpha \in \mathbf{R}$ such that $\rho(r) = r^\alpha e^{-1/r}$, then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1/4$ and $k_2 = 1/2$. See also [20, Lemma 2.5], [23, 26] and [27, Remark 2.2].

Let G be an open bounded set in $\mathbf{R}^N$. For a function $\rho \in (\rho)$, we define the generalized Riesz potential $I_\rho f$ of f by

$$I_\rho f(x) = \int_G \frac{\rho(|x-y|) f(y)}{|x-y|^N}\,dy,$$

where $f \in L^1(G)$. We write $I_\rho f = I_\alpha f$ when $\rho(r) = r^\alpha$, $0 < \alpha < N$. We refer to [15, 21, 22, 24, 29] etc. for the study of $I_\rho f$.

Our aim in this note is to study the continuity of generalized Riesz potential $I_\rho f$ of functions f in Morrey spaces $L^{\Phi,\nu(\cdot)}(G)$ of the double phase functionals with variable exponents (Theorem 2.2), as an extension of [12, Theorem 4.1].

2. Definitions and the main Theorem

Throughout this paper, let C denote various constants independent of the variables in question and log be a natural logarithm.

Let $p(\cdot)$ be a measurable functions on G such that

(P1) $1 \le p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$, (P2) $p(\cdot)$ is log-Hölder continuous on G, namely

$$|p(x) - p(y)| \le \frac{C_p}{\log(e + 1/|x-y|)} \quad (x,\, y \in G),$$

with a constant $C_p \ge 0$.

Let $\nu(\cdot)$ be a measurable functions on G such that

$$0 < \nu^- := \inf_{x \in G} \nu(x) \le \sup_{x \in G} \nu(x) =: \nu^+ < \infty.$$

Let $B(x,r)$ denote the open ball centered at $x \in \mathbf{R}^N$ with radius $r > 0$. For a set $E \subset \mathbf{R}^N$, $|E|$ denotes the Lebesgue measure of E. Set $d_G = \sup\{|x-y| : x, y \in G\}$. Morrey space with variable exponents $L^{p(\cdot),\nu(\cdot)}(G)$ is the family of measurable functions f on G satisfying

$$L^{p(\cdot),\nu(\cdot)}(G) = \Bigg\{ f \in L^1_{loc}(G)\,; \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)}}{|B(x,r)|} \int_{G \cap B(x,r)} |f(y)|^{p(y)}\, dy < \infty \Bigg\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot),\nu(\cdot)}(G)} = \inf \Bigg\{ \lambda > 0\,; \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)}}{|B(x,r)|} \int_{G \cap B(x,r)} \left(\frac{|f(y)|}{\lambda} \right)^{p(y)} dy \le 1 \Bigg\}$$

(cf. see [18]).

We consider a function $\Phi(x,t) : G \times [0,\infty) \to [0,\infty)$ satisfying the following conditions (Φ1) and (Φ2):

(Φ1) $\Phi(\,\cdot\,,t)$ is measurable on G for each $t \ge 0$ and $\Phi(x,\,\cdot\,)$ is convex on $[0,\infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \ge 1$ such that $A_1^{-1} \le \Phi(x,1) \le A_1$ for all $x \in G$.

The Musielak-Orlicz-Morrey space $L^{\Phi,\nu(\cdot)}(G)$ is defined by

$$L^{\Phi,\nu(\cdot)}(G) = \Bigg\{ f \in L^1_{\rm loc}(G)\ : \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)}}{|B(x,r)|} \int_{G \cap B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda} \right) dy < \infty \text{ for some } \lambda > 0 \Bigg\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi,\nu(\cdot)}(G)} = \inf \Bigg\{ \lambda > 0\ : \sup_{x \in G, 0<r<d_G} \frac{r^{\nu(x)}}{|B(x,r)|} \int_{G \cap B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda} \right) dy \le 1 \Bigg\}$$

(see [9, 19]).

Let $q(\cdot)$ be a measurable function on G such that

(Q1) $1 \le q^- := \inf_{x\in G} q(x) \le \sup_{x\in G} q(x) =: q^+ < \infty$,

(Q2) $q(\cdot)$ is log-Hölder continuous on G, namely

$$|q(x) - q(y)| \le \frac{C_q}{\log(e + 1/|x-y|)} \quad (x,\, y \in G),$$

with a constant $C_q \ge 0$.

In what follows, set

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(x) < q(x)$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ ([10], cf. [3, 25]).

Remark 2.1. Let $f \in L^{\Phi,\nu(\cdot)}(G)$ be a measurable function on G. Then note that $f \in L^{p(\cdot),\nu(\cdot)}(G)$ and $bf \in L^{q(\cdot),\nu(\cdot)}(G)$.

We state the following, as an extension of [12, Theorem 4.1].

Theorem 2.2. *Let $\rho \in (\rho)$. Assume that there are constants $\eta_1 > 0, \eta_2 > 0, \tau > 0$ and $C_0 > 0$ such that*

$$\left| \frac{\rho(|x-y|)}{|x-y|^N} - \frac{\rho(|z-y|)}{|z-y|^N} \right| \le C_0 \frac{|x-z|^{\eta_1}}{|x-y|^{\eta_2}} \frac{\rho(\tau|x-y|)}{|x-y|^N} \tag{2.1}$$

whenever $x, y, z \in G$ and $|x-z| \le |x-y|/2$. Abbreviate

$$\begin{aligned} \psi(x,z,r) \equiv &\int_0^{4k_2 r} s^{-\nu(x)/p(x)+\theta} \rho(s)\, \frac{ds}{s} + \int_0^{4k_2 r} s^{-\nu(x)/q(x)} \rho(s)\, \frac{ds}{s} \\ &+ \int_0^{6k_2 r} s^{-\nu(z)/p(z)+\theta} \rho(s)\, \frac{ds}{s} + \int_0^{6k_2 r} s^{-\nu(z)/q(z)} \rho(s)\, \frac{ds}{s} \\ &+ r^{\theta} \int_{k_1 r}^{4k_2 d_G} s^{-\nu(z)/p(z)} \rho(s)\, \frac{ds}{s} \\ &+ r^{\eta_1} \int_{2k_1 \tau r}^{4k_2 \tau d_G} s^{-\nu(x)/p(x)-\eta_2+\theta} \rho(s)\, \frac{ds}{s} \\ &+ r^{\eta_1} \int_{2k_1 \tau r}^{4k_2 \tau d_G} s^{-\nu(x)/q(x)-\eta_2} \rho(s)\, \frac{ds}{s} \end{aligned}$$

for $x, z \in G$ and $0 < r \le d_G$, where k_1 and k_2 are constants in (ρ). Assume that $\psi(x,z,r) < \infty$ for all $x, z \in G$ and $0 < r \le d_G$. Then there exists a constant $C > 0$ such that

$$|b(x) I_\rho f(x) - b(z) I_\rho f(z)| \le C \psi(x,z,|x-z|)$$

for all $x, z \in G$ and measurable functions f on G with $\|f\|_{L^{\Phi,\nu(\cdot)}(G)} \le 1$.

REMARK 2.3. Let $\rho(r) = r^{\alpha} e^{-1/r}$ be as in Example 1.1. Then the mean value property implies that (2.1) holds for $\eta_1 = 1, \eta_2 = 2$ and $\tau = 3/2$. Note here that there exists a constant $C \geq 1$ such that

$$C^{-1}\frac{\rho(r)}{r^{N+2}} \leq \frac{d}{dr}\left(\frac{\rho(r)}{r^N}\right) \leq C\frac{\rho(r)}{r^{N+2}}$$

for all $0 < r \leq d_G$ and $|x-y|/2 \leq |x-y+t(z-x)| \leq 3|x-y|/2$ for $0 \leq t \leq 1$ and $|x-z| \leq |x-y|/2$.

REMARK 2.4. If $\rho(r)r^a$ is increasing for some $a \geq 0$ and $\rho(r)/r^b$ is decreasing for some $b \geq 0$, then ρ satisfies the doubling condition and

$$\left|\frac{\rho(r)}{r^N} - \frac{\rho(s)}{s^N}\right| \leq C_0|r-s|\frac{\rho(r)}{r^{N+1}}, \quad \text{for } \frac{1}{2} \leq \frac{s}{r} \leq 2.$$

See [21].

3. Corollaries

In this section, we give consequences of our theorem.

COROLLARY 3.1. *Let $\rho(r) = r^{\alpha}(\log(e+1/r))^{\beta}$ for $\alpha > 0$ and $\beta \in \mathbf{R}$. Suppose $\inf_{x\in G}(\nu(x)-\alpha p(x)) > 0$ and $\inf_{x\in G}((\alpha+\theta)p(x)-\nu(x)) > 0$. Further suppose $\inf_{x\in G}(\nu(x)-(\alpha-1)q(x)) > 0$ and $\inf_{x\in G}(\alpha q(x)-\nu(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}|b(x)&I_\rho f(x) - b(z)I_\rho f(z)| \\ &\leq C\big\{|x-z|^{\alpha-\nu(x)/p(x)+\theta} + |x-z|^{\alpha-\nu(x)/q(x)} \\ &\quad + |x-z|^{\alpha-\nu(z)/p(z)+\theta} + |x-z|^{\alpha-\nu(z)/q(z)}\big\}(\log(e+1/|x-z|))^{\beta}\end{aligned}$$

for all $x, z \in G$ and measurable functions f on G with $\|f\|_{L^{\Phi,\nu(\cdot)}(G)} \leq 1$.

PROOF. Since $\inf_{x\in G}((\alpha+\theta)p(x)-\nu(x)) > 0$, taking ε_1 such that $0 < \varepsilon_1 < \alpha - \nu(x)/p(x) + \theta$, there exists a constant $c_1 > 0$ such that

$$s_1^{\alpha-\nu(x)/p(x)+\theta-\varepsilon_1}(\log(e+1/s_1))^{\beta} \leq c_1 s_2^{\alpha-\nu(x)/p(x)+\theta-\varepsilon_1}(\log(e+1/s_2))^{\beta}$$

whenever $0 < s_1 \leq s_2$ (see e.g. [17, 28]). Therefore we have

$$\begin{aligned}\int_0^{4k_2 r} &s^{\alpha-\nu(x)/p(x)+\theta}(\log(e+1/s))^{\beta}\,\frac{ds}{s} \\ &\leq c_1(4k_2 r)^{\alpha-\nu(x)/p(x)+\theta-\varepsilon_1}(\log(e+1/(4k_2 r)))^{\beta}\int_0^{4k_2 r} s^{\varepsilon_1}\,\frac{ds}{s} \\ &\leq Cr^{\alpha-\nu(x)/p(x)+\theta}(\log(e+1/r))^{\beta}\end{aligned}$$

and, similarly we obtain

$$\int_0^{6k_2 r} s^{\alpha-\nu(z)/p(z)+\theta}(\log(e+1/s))^{\beta}\,\frac{ds}{s} \leq Cr^{\alpha-\nu(z)/p(z)+\theta}(\log(e+1/r))^{\beta}.$$

Since $\inf_{x\in G}(\alpha q(x)-\nu(x))>0$, we obtain

$$\int_0^{4k_2 r} s^{\alpha-\nu(x)/q(x)}(\log(e+1/s))^\beta \frac{ds}{s} \le C r^{\alpha-\nu(x)/q(x)}(\log(e+1/r))^\beta$$

and

$$\int_0^{6k_2 r} s^{\alpha-\nu(z)/q(z)}(\log(e+1/s))^\beta \frac{ds}{s} \le C r^{\alpha-\nu(z)/q(z)}(\log(e+1/r))^\beta.$$

Since $\inf_{x\in G}(\nu(x)-\alpha p(x))>0$, taking ε_2 such that $0<\varepsilon_2<\nu(z)/p(z)-\alpha$, there exists a constant $c_2>0$ such that

$$s_2^{\alpha-\nu(z)/p(z)+\varepsilon_2}(\log(e+1/s_2))^\beta \le c_2 s_1^{\alpha-\nu(z)/p(z)+\varepsilon_2}(\log(e+1/s_1))^\beta$$

whenever $0<s_1\le s_2$, so that

$$\begin{aligned} r^\theta &\int_{k_1 r}^{4k_2 d_G} s^{\alpha-\nu(z)/p(z)}(\log(e+1/s))^\beta \frac{ds}{s} \\ &\le c_2 r^\theta (k_1 r)^{\alpha-\nu(z)/p(z)+\varepsilon_2}(\log(e+1/(k_1 r)))^\beta \int_{k_1 r}^{4k_2 d_G} s^{-\varepsilon_2}\frac{ds}{s} \\ &\le C r^{\alpha-\nu(z)/p(z)+\theta}(\log(e+1/r))^\beta. \end{aligned}$$

We also have

$$\begin{aligned} r\int_{2k_1\tau r}^{4k_2\tau d_G} & s^{\alpha-\nu(x)/p(x)-1+\theta}(\log(e+1/s))^\beta \frac{ds}{s} \\ &\le C r^{\alpha-\nu(x)/p(x)+\theta}(\log(e+1/r))^\beta, \end{aligned}$$

since $\inf_{x\in G}(\nu(x)-(\alpha+\theta-1)p(x)) \ge \inf_{x\in G}(\nu(x)-\alpha p(x))>0$, and

$$r\int_{2k_1\tau r}^{4k_2\tau d_G} s^{\alpha-\nu(x)/q(x)-1}(\log(e+1/s))^\beta \frac{ds}{s} \le C r^{\alpha-\nu(x)/q(x)}(\log(e+1/r))^\beta,$$

since $\inf_{x\in G}(\nu(x)-(\alpha-1)q(x))>0$.

Collecting these facts, we obtain by our assumptions

$$\begin{aligned} \psi(x,z,r) \le C\big\{ & r^{\alpha-\nu(x)/p(x)+\theta} + r^{\alpha-\nu(x)/q(x)} \\ & + r^{\alpha-\nu(z)/p(z)+\theta} + r^{\alpha-\nu(z)/q(z)}\big\}(\log(e+1/r))^\beta < \infty \end{aligned}$$

for $x,z\in G$ and $0<r\le d_G$. By Theorem 2.2, we obtain the required result. $\square$

Corollary 3.2. *Suppose* $\inf_{x\in G}(\nu(x)-\alpha p(x))>0$ *and* $\inf_{x\in G}((\alpha+\theta)p(x)-\nu(x))>0$. *Further suppose* $\inf_{x\in G}(\nu(x)-(\alpha-1)q(x))>0$ *and* $\inf_{x\in G}(\alpha q(x)-\nu(x))>0$. *Then there exists a constant* $C>0$ *such that*

$$\begin{aligned} |b(x)I_\alpha f(x)-b(z)I_\alpha f(z)| \le C\big\{ & |x-z|^{\alpha-\nu(x)/p(x)+\theta} + |x-z|^{\alpha-\nu(x)/q(x)} \\ & + |x-z|^{\alpha-\nu(z)/p(z)+\theta} + |x-z|^{\alpha-\nu(z)/q(z)}\big\} \end{aligned}$$

for all $x,z\in G$ *and measurable functions* f *on* G *with* $\|f\|_{L^{\Phi,\nu(\cdot)}(G)}\le 1$.

PROOF. This is the case $\beta = 0$ in Corollary 3.1. ∎

Compare Corollaries 3.1 and 3.2 with [12, Theorem 4.1].

COROLLARY 3.3. *Let $\rho(r) = r^{\alpha} e^{-1/r}$ be as in Example 1.1. Then there exists a constant $C > 0$ such that*

$$|b(x) I_\rho f(x) - b(z) I_\rho f(z)| \le C|x - z|^{\theta}$$

for all $x, z \in G$ and measurable functions f on G with $\|f\|_{L^{\Phi,\nu(\cdot)}(G)} \le 1$.

PROOF. Since, for $a \in \mathbf{R}$, there exists a constant $c > 0$ such that

$$\int_0^r s^a e^{-1/s} \frac{ds}{s} \le c r^{\theta}$$

for all $0 < r \le d_G$, it follows from Remark 2.3 that

$$\psi(x, z, r) \le C(r + r^{\theta}) \le C r^{\theta}$$

for all $x, z \in G$ and $0 < r \le d_G$, since $\theta \in (0, 1]$. Hence, we obtain the required inequality. ∎

COROLLARY 3.4. *Let $\rho(r) = r^{\alpha}(\log(e + 1/r))^{\beta}$ for $\alpha > 0$ and $\beta \in \mathbf{R}$. Suppose $\inf_{x\in G}(\nu(x) - (\alpha - 1)p(x)) > 0$ and $\inf_{x\in G}(\alpha p(x) - \nu(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &|I_\rho f(x) - I_\rho f(z)| \\ &\quad \le C \left\{ |x - z|^{\alpha - \nu(x)/p(x)} + |x - z|^{\alpha - \nu(z)/p(z)} \right\} (\log(e + 1/|x - z|))^{\beta} \end{aligned}$$

for all $x, z \in G$ and measurable functions f on G with $\|f\|_{L^{p(\cdot),\nu(\cdot)}(G)} \le 1$.

PROOF. To show this, we take $b(\cdot) \equiv 1$ and $q(\cdot) = p(\cdot)$ in the proof of Theorem 2.2. As in the proof of Corollary 3.1, we obtain the result. ∎

4. LEMMAS

Before giving a proof of Theorem 2.2, we prepare two lemmas. To prove the following lemma, (P2) and (Q2) were used.

LEMMA 4.1 ([13, Lemma 2.1], cf. [14, Lemma 2.7]). *There exists a constant $C > 0$ such that*

$$\frac{r^{\nu(x)/p(x)}}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|\, dy \le C$$

for all $x \in G$, $0 < r < d_G$ and measurable functions f on G with $\|f\|_{L^{p(\cdot),\nu(\cdot)}(G)} \le 1$.

LEMMA 4.2. *Let $\tau > 0, \beta \in \mathbf{R}$ and $\rho \in (\rho)$. Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot),\nu(\cdot)}(G)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{G\cap B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \le C\int_0^{2k_2\tau r} s^{-\nu(x)/p(x)-\beta}\rho(s)\,\frac{ds}{s} \tag{4.1}$$

and

$$\int_{G\setminus B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \le C\int_{k_1\tau r}^{4k_2\tau d_G} s^{-\nu(x)/p(x)-\beta}\rho(s)\,\frac{ds}{s} \tag{4.2}$$

for all $x \in G$ and $0 < r \le d_G$, where k_1 and k_2 are constants in (ρ).

PROOF. Let f be a nonnegative function on G such that $\|f\|_{L^{p(\cdot),\nu(\cdot)}(G)} \le 1$. Take $\gamma \in \mathbf{R}$ such that $1 < \gamma \le \min\{1/k, 2\}$. If $y \in G \cap (B(x,\gamma^j r) \setminus B(x,\gamma^{j-1}r))$ for $j \in \mathbf{Z}$, then a geometric observation and (1.1) show

$$\begin{aligned}\frac{\rho(\tau|x-y|)}{|x-y|^{N+\beta}} &\le \frac{\max\{1,\gamma^{N+\beta}\}}{(\gamma^j r)^{N+\beta}} \sup_{\gamma^{j-1}\tau r\le s\le \gamma^j\tau r}\rho(s)\\ &\le \frac{\max\{1,\gamma^{N+\beta}\}}{(\gamma^j r)^{N+\beta}} \sup_{k\gamma^{j}\tau r\le s\le \gamma^j\tau r}\rho(s)\\ &\le \frac{C_\rho\max\{1,\gamma^{N+\beta}\}}{(\gamma^j r)^{N+\beta}} \int_{\gamma^j k_1\tau r}^{\gamma^j k_2\tau r}\rho(s)\,\frac{ds}{s},\end{aligned}$$

by $\gamma \le 1/k$. By Lemma 4.1, we have

$$\frac{1}{|B(x,\gamma^j r)|}\int_{G\cap B(x,\gamma^j r)} f(y)\,dy \le C_1(\gamma^j r)^{-\nu(x)/p(x)}$$

for some constant $C_1 > 0$, so that

$$\begin{aligned}&\int_{G\cap(B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r))} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy\\ &\le \frac{C_\rho\sigma_N\max\{1,\gamma^{N+\beta}\}}{(\gamma^j r)^{\beta}}\int_{\gamma^j k_1\tau r}^{\gamma^j k_2\tau r}\rho(s)\,\frac{ds}{s}\cdot\frac{1}{|B(x,\gamma^j r)|}\int_{G\cap B(x,\gamma^j r)} f(y)\,dy\\ &\le \frac{C_\rho\sigma_N\max\{1,\gamma^{N+\beta}\}}{(\gamma^j r)^{\beta}}\int_{\gamma^j k_1\tau r}^{\gamma^j k_2\tau r}\rho(s)\,\frac{ds}{s}\cdot C_1\left(\gamma^j r\right)^{-\nu(x)/p(x)}\\ &\le C_1C_\rho\sigma_N\max\{1,2^{N+\beta}\}\left(\gamma^j r\right)^{-\nu(x)/p(x)-\beta}\int_{\gamma^j k_1\tau r}^{\gamma^j k_2\tau r}\rho(s)\,\frac{ds}{s}\\ &\le C_1C_\rho\sigma_N\max\{1,2^{N+\beta}\}\max\left\{(\tau k_1)^{\nu(x)/p(x)+\beta},(\tau k_2)^{\nu(x)/p(x)+\beta}\right\}\\ &\quad\times\int_{\gamma^j k_1\tau r}^{\gamma^j k_2\tau r} s^{-\nu(x)/p(x)-\beta}\rho(s)\,\frac{ds}{s}\end{aligned}$$

$$\leq C_2 \int_{\gamma^j k_1 \tau r}^{\gamma^j k_2 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s},$$

where σ_N denotes the volume of the unit ball $B(0,1)$ and

$$C_2 = C_1 C_\rho \sigma_N \max\left\{1, 2^{N+\beta}\right\} \times \max\left\{(\tau k_1)^{\nu^+/p^-+\beta}, (\tau k_1)^{\nu^-/p^++\beta}, (\tau k_2)^{\nu^+/p^-+\beta}, (\tau k_2)^{\nu^-/p^++\beta}\right\}.$$

Therefore we have

(4.3)

$$\int_{G\cap(B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r))} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \leq C_2 \int_{\gamma^j k_1 \tau r}^{\gamma^j k_2 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s}. \tag{4.3}$$

Let j_0 be the smallest integer such that $k_2/k_1 \leq \gamma^{j_0}$. Using (4.3), we obtain

$$\begin{aligned}
&\int_{G\cap B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \\
&= \sum_{j=0}^{\infty} \int_{G\cap(B(x,\gamma^{-j}r)\setminus B(x,\gamma^{-j-1}r))} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \\
&\leq C_2 \sum_{j=0}^{\infty} \int_{\gamma^{-j} k_1 \tau r}^{\gamma^{-j} k_2 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s} \\
&\leq C_2 \sum_{j=0}^{\infty} \int_{\gamma^{-j} k_1 \tau r}^{\gamma^{-j+j_0} k_1 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s} \\
&\leq j_0 C_2 \int_0^{2k_2\tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s},
\end{aligned}$$

which proves (4.1).

Let j_1 be the smallest integer such that $d_G \leq \gamma^{j_1} r$. If we use (4.3),

$$\begin{aligned}
&\int_{G\setminus B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \\
&\leq \sum_{j=1}^{j_1} \int_{G\cap(B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r))} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}}\,dy \\
&\leq C_2 \sum_{j=1}^{j_1} \int_{\gamma^j k_1 \tau r}^{\gamma^j k_2 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s} \\
&\leq C_2 \sum_{j=1}^{j_1} \int_{\gamma^j k_1 \tau r}^{\gamma^{j+j_0} k_1 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s} \\
&\leq C_2 j_0 \int_{k_1 \tau r}^{4k_2\tau d_G} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s}.
\end{aligned}$$

Thus, (4.2) follows. □

5. Proof of Theorem 2.2

Without loss of generality, we can assume that f is a nonnegative function on G such that $\|f\|_{L^{\Phi,\nu(\cdot)}(G)} \le 1$. First note from (2.1) that for $x, z \in G$ and $r = |x-z|$

$$
\begin{aligned}
&|b(x)I_\rho f(x) - b(z)I_\rho f(z)| \\
&\quad \le b(x) \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)f(y)}{|x-y|^N}\,dy \\
&\quad\quad + b(z) \int_{G\cap B(x,2r)} \frac{\rho(|z-y|)f(y)}{|z-y|^N}\,dy \\
&\quad\quad + |b(x)-b(z)| \int_{G\setminus B(x,2r)} \frac{\rho(|z-y|)f(y)}{|z-y|^N}\,dy \\
&\quad\quad + b(x) \int_{G\setminus B(x,2r)} \left| \frac{\rho(|x-y|)}{|x-y|^N} - \frac{\rho(|z-y|)}{|z-y|^N} \right| f(y)\,dy \\
&\quad \le C\Bigg\{ b(x) \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)f(y)}{|x-y|^N}\,dy \\
&\quad\quad + b(z) \int_{G\cap B(z,3r)} \frac{\rho(|z-y|)f(y)}{|z-y|^N}\,dy + r^\theta \int_{G\setminus B(z,r)} \frac{\rho(|z-y|)f(y)}{|z-y|^N}\,dy \\
&\quad\quad + r^{\eta_1} b(x) \int_{G\setminus B(x,2r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\eta_2}}\,dy \Bigg\} \\
&\quad = C\left\{ I_1(x) + \tilde{I}_1(z) + I_2(z) + I_3(x) \right\}.
\end{aligned}
$$

For $I_1(x)$, we have

$$
\begin{aligned}
I_1(x) &\le \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)}{|x-y|^N} |b(x)-b(y)| f(y)\,dy \\
&\quad + \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)}{|x-y|^N} b(y) f(y)\,dy \\
&\le C \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)f(y)}{|x-y|^{N-\theta}}\,dy + \int_{G\cap B(x,2r)} \frac{\rho(|x-y|)\{b(y)f(y)\}}{|x-y|^N}\,dy \\
&= C I_{11}(x) + I_{12}(x).
\end{aligned}
$$

By (4.1), we obtain

$$
I_{11}(x) \le C \int_0^{4k_2 r} s^{-\nu(x)/p(x)+\theta} \rho(s)\,\frac{ds}{s},
$$

and

$$I_{12}(x) \le C \int_0^{4k_2 r} s^{-\nu(x)/q(x)} \rho(s) \, \frac{ds}{s}.$$

For $\tilde{I}_1(z)$, we have by (4.1)

$$\tilde{I}_1(z) \le C \left\{ \int_0^{6k_2 r} s^{-\nu(z)/p(z)+\theta} \rho(s) \, \frac{ds}{s} + \int_0^{6k_2 r} s^{-\nu(z)/q(z)} \rho(s) \, \frac{ds}{s} \right\},$$

as in the estimate of $I_{11}(x)$ and $I_{12}(x)$.

For $I_2(z)$, we have by (4.2)

$$I_2(z) \le C r^{\theta} \int_{k_1 r}^{4k_2 d_G} s^{-\nu(z)/p(z)} \rho(s) \, \frac{ds}{s}.$$

Finally, for $I_3(x)$ we have

$$\begin{aligned} I_3(x) &\le r^{\eta_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|)}{|x-y|^{N+\eta_2}} |b(x) - b(y)| f(y) \, dy \\ &\quad + r^{\eta_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|)}{|x-y|^{N+\eta_2}} b(y) f(y) \, dy \\ &\le C r^{\eta_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|) f(y)}{|x-y|^{N-\theta+\eta_2}} \, dy \\ &\quad + r^{\eta_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|)\{b(y) f(y)\}}{|x-y|^{N+\eta_2}} \, dy \\ &= C I_{31}(x) + I_{32}(x). \end{aligned}$$

Note from (4.2) that

$$I_{31}(x) \le C r^{\eta_1} \int_{2k_1 \tau r}^{4k_2 \tau d_G} s^{-\nu(x)/p(x) - \eta_2 + \theta} \rho(s) \, \frac{ds}{s}$$

and

$$I_{32}(x) \le C r^{\eta_1} \int_{2k_1 \tau r}^{4k_2 \tau d_G} s^{-\nu(x)/q(x) - \eta_2} \rho(s) \, \frac{ds}{s}.$$

Collecting these facts, we obtain

$$|b(x) I_\rho f(x) - b(z) I_\rho f(z)| \le C \psi(x, z, r).$$

Thus this theorem is proved.

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