CONTINUITY OF GENERALIZED RIESZ POTENTIALS FOR DOUBLE PHASE FUNCTIONALS WITH VARIABLE EXPONENTS

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Abstract. In this note, we discuss the continuity of generalized Riesz potentials $I_\alpha f$ of functions in Morrey spaces $L^{\Phi,\nu}(G)$ of double phase functionals with variable exponents.

1. Introduction

The double phase functional introduced by Zhikov ([30]) in the 1980s has been studied intensively by many mathematicians. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione in [1, 4, 5] studied a double phase functional 

$$\tilde{\Phi}(x,t) = t^p + a(x)t^q, \quad x \in \mathbb{R}^N, \quad t \geq 0,$$

where $N \geq 2$, $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0, 1]$. We refer to [10, 16] for Sobolev’s inequality, [11] for Trudinger’s inequality and e.g. [2, 7, 8] for other double phase problems.

For $0 < \alpha < N$ and a locally integrable function $f$ on $\mathbb{R}^N$ the Riesz potential $I_\alpha f$ of order $\alpha$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^N} |x-y|^{\alpha-N} f(y) \, dy.$$

In [12] we discussed the continuity of Riesz potentials $I_\alpha f$ of functions in Morrey spaces $L^{\tilde{\Phi},\nu}(\mathbb{R}^N)$ of the double phase functionals $\tilde{\Phi}(x,t)$. We refer to [16, Section 5] for the $L^\Phi$ case and [14] for the $L^{p(\cdot),\nu(\cdot)}$ case.

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In the present note, we consider the case \( \Phi(x,t) \) is a double phase functional given by
\[
\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},
\]
where \( p(x) < q(x) \) and \( b(\cdot) \) is non-negative, bounded and Hölder continuous of order \( \theta \in (0,1] \) ([10], cf. [3, 25]).

To obtain general results, we consider the family \( (\rho) \) of all functions \( \rho \) satisfying the following conditions:
\[
\rho : (0, \infty) \to (0, \infty) \text{ is a measurable function such that }
\int_0^r \rho(s) \frac{ds}{s} < \infty
\]
for all sufficiently small \( r > 0 \) and there exists constants \( 0 < k < 1 \), \( 0 < k_1 < k_2 \) and \( C_\rho > 0 \) such that
\[
\sup_{k_2 \leq s \leq r} \rho(s) \leq C_\rho \int_{k_1 r}^{k_2 r} \rho(s) \frac{ds}{s}
\]
for all \( r > 0 \) (e.g. [6, 26]). We do not postulate the doubling condition on \( \rho \).

**Example 1.1.** If \( \rho \) satisfies the doubling condition, that is, there exists a constant \( C > 0 \) such that \( C^{-1} \leq \rho(r)/\rho(s) \leq C \) for \( 1/2 \leq r/s \leq 2 \), then \( \rho \) satisfies (1.1) whenever \( k = 1/2 \) and \( 2k_1 = k_2 \). If \( \rho \) is increasing, then \( \rho \) satisfies (1.1) with \( k = 1/2, k_1 = 1/4 \) and \( k_2 = 1/2 \). See also [20, Lemma 2.5], [23, 26] and [27, Remark 2.2].

Let \( G \) be an open bounded set in \( \mathbb{R}^N \). For a function \( \rho \in (\rho) \), we define the generalized Riesz potential \( I_\rho f \) of \( f \) by
\[
I_\rho f(x) = \int_G \rho(|x-y|) f(y) \frac{dy}{|x-y|^N},
\]
where \( f \in L^1(G) \). We write \( I_\rho f = I_\rho f \) when \( \rho(r) = r^\alpha \), \( 0 < \alpha < N \). We refer to [15, 21, 22, 24, 29] etc. for the study of \( I_\rho f \).

Our aim in this note is to study the continuity of generalized Riesz potential \( I_\rho f \) of functions \( f \) in Morrey spaces \( L^{\Phi, \nu(\cdot)}(G) \) of the double phase functionals with variable exponents (Theorem 2.2), as an extension of [12, Theorem 4.1].

### 2. Definitions and the main Theorem

Throughout this paper, let \( C \) denote various constants independent of the variables in question and \( \log \) be a natural logarithm.

Let \( p(\cdot) \) be a measurable functions on \( G \) such that
\[
(P1) \ 1 \leq p^- := \inf_{x \in G} p(x) \leq \sup_{x \in G} p(x) =: p^+ < \infty,
\]
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(P2) $p(\cdot)$ is log-Hölder continuous on $G$, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(e + 1/|x - y|)} \quad (x, y \in G),$$

with a constant $C_p \geq 0$.

Let $\nu(\cdot)$ be a measurable functions on $G$ such that

$$0 < \nu^- := \inf_{x \in G} \nu(x) \leq \sup_{x \in G} \nu(x) =: \nu^+ < \infty.$$

Let $B(x, r)$ denote the open ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$.

For a set $E \subset \mathbb{R}^N$, $|E|$ denotes the Lebesgue measure of $E$. Set $d_G = \sup\{|x - y| : x, y \in G\}$. Morrey space with variable exponents $L^{p(\cdot), \nu(\cdot)}(G)$ is the family of measurable functions $f$ on $G$ satisfying

$$L^{p(\cdot), \nu(\cdot)}(G) = \left\{ f \in L^1_{\text{loc}}(G) : \sup_{x \in G, 0 < r < d_G} \frac{r^{p(x)}}{|B(x, r)|} \int_{G \cap B(x, r)} |f(y)|^p \nu(y) \, dy < \infty \right\}.$$ 

It is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot), \nu(\cdot)}(G)} = \inf \left\{ \lambda > 0 : \sup_{x \in G, 0 < r < d_G} \frac{r^{p(x)}}{|B(x, r)|} \int_{G \cap B(x, r)} \left( \frac{|f(y)|}{\lambda} \right)^p \nu(y) \, dy \leq 1 \right\}.$$

(cf. see [18]).

We consider a function $\Phi(x, t) : G \times [0, \infty) \to [0, \infty)$ satisfying the following conditions (Φ1) and (Φ2):

(Φ1) $\Phi(\cdot, t)$ is measurable on $G$ for each $t \geq 0$ and $\Phi(x, \cdot)$ is convex on $[0, \infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \geq 1$ such that $A_1^{-1} \leq \Phi(x, 1) \leq A_1$ for all $x \in G$.

The Musielak-Orlicz-Morrey space $L^{\Phi, \nu(\cdot)}(G)$ is defined by

$$L^{\Phi, \nu(\cdot)}(G) = \left\{ f \in L^1_{\text{loc}}(G) : \sup_{x \in G, 0 < r < d_G} \frac{r^{\nu(x)}}{|B(x, r)|} \int_{G \cap B(x, r)} \Phi \left( y, \frac{|f(y)|}{\lambda} \right) \, dy < \infty \right\}.$$ 

for some $\lambda > 0$.

It is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi, \nu(\cdot)}(G)} = \inf \left\{ \lambda > 0 : \sup_{x \in G, 0 < r < d_G} \frac{r^{\nu(x)}}{|B(x, r)|} \int_{G \cap B(x, r)} \Phi \left( y, \frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}.$$
(see [9, 19]).

Let \( q(\cdot) \) be a measurable function on \( G \) such that

(Q1) \( 1 \leq q^- := \inf_{x \in G} q(x) \leq \sup_{x \in G} q(x) =: q^+ < \infty \),

(Q2) \( q(\cdot) \) is \( p \)-log-Hölder continuous on \( G \), namely

\[
|q(x) - q(y)| \leq \frac{C_q}{\log(e + 1/|x - y|)} \quad (x, y \in G),
\]

with a constant \( C_q \geq 0 \).

In what follows, let

\[
\Phi(x, t) = t^{p(x)} + (b(x)t)^{q(x)},
\]

where \( p(x) < q(x) \) and \( b(\cdot) \) is non-negative, bounded and Hölder continuous of order \( \theta \in (0, 1) \) ([10], cf. [3, 25]).

**Remark 2.1.** Let \( f \in L^{\Phi, \nu}(G) \) be a measurable function on \( G \). Then note that \( f \in L^{p(\cdot), \nu(\cdot)}(G) \) and \( bf \in L^{\Phi(\cdot), \nu(\cdot)}(G) \).

We state the following, as an extension of [12, Theorem 4.1].

**Theorem 2.2.** Let \( \rho \in (\rho) \). Assume that there are constants \( \eta_1 > 0, \eta_2 > 0, \tau > 0 \) and \( C_0 > 0 \) such that

\[
(2.1) \quad \left| \frac{\rho(|x - y|)}{|x - y|^N} - \frac{\rho(|z - y|)}{|z - y|^N} \right| \leq C_0 \frac{|x - z|^{\eta_1}}{|x - y|^{\eta_2}} \rho(\tau|x - y|) \quad \text{whenever } x, y, z \in G \text{ and } |x - z| \leq |x - y|/2.
\]

Abbreviate

\[
\psi(x, z, r) = \int_0^{4k_2r} s^{-\nu(x)/p(x) + \theta} \rho(s) \frac{ds}{s} + \int_0^{4k_2r} s^{-\nu(x)/q(x)} \rho(s) \frac{ds}{s}
\]

\[
+ \int_0^{6k_2r} s^{-\nu(z)/p(z) + \theta} \rho(s) \frac{ds}{s} + \int_0^{6k_2r} s^{-\nu(z)/q(z)} \rho(s) \frac{ds}{s}
\]

\[
+ \nu^{\eta_1} \int_{k_1\tau}^{4k_2\tau} s^{-\nu(x)/p(x) - \eta_2 + \theta} \rho(s) \frac{ds}{s}
\]

\[
+ \nu^{\eta_1} \int_{k_1\tau}^{4k_2\tau} s^{-\nu(x)/q(x) - \eta_2} \rho(s) \frac{ds}{s}
\]

for \( x, z \in G \) and \( 0 < r \leq d_G \), where \( k_1 \) and \( k_2 \) are constants in \( (\rho) \). Assume that \( \psi(x, z, r) < \infty \) for all \( x, z \in G \) and \( 0 < r \leq d_G \). Then there exists a constant \( C > 0 \) such that

\[
|b(x)I_\rho f(x) - b(z)I_\rho f(z)| \leq C\psi(x, z, |x - z|)
\]

for all \( x, z \in G \) and measurable functions \( f \) on \( G \) with \( \|f\|_{L^{\Phi,\nu}(G)} \leq 1 \).
Remark 2.3. Let $\rho(r) = r^\alpha e^{-1/r}$ be as in Example 1.1. Then the mean value property implies that (2.1) holds for $\eta_1 = 1, \eta_2 = 2$ and $\tau = 3/2$. Note here that there exists a constant $C \geq 1$ such that

$$C^{-1} \frac{\rho(r)}{r^{N+2}} \leq \frac{d}{dr} \left( \frac{\rho(r)}{r^N} \right) \leq C \frac{\rho(r)}{r^{N+2}}$$

for all $0 < r < d_G$ and $|x - y|/2 \leq |x - y + t(z - x)| \leq 3|x - y|/2$ for $0 \leq t \leq 1$ and $|x - z| \leq |x - y|/2$.

Remark 2.4. If $\rho(r)^a$ is increasing for some $a \geq 0$ and $\rho(r)/r^b$ is decreasing for some $b \geq 0$, then $\rho$ satisfies the doubling condition and

$$\frac{|\rho(r) - \rho(s)|}{s^N} \leq C_0 |r - s| \rho(r) r^{N+1} \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2.$$ 

See [21].

3. Corollaries

In this section, we give consequences of our theorem.

Corollary 3.1. Let $\rho(r) = r^\alpha (\log(e + 1/r))^{\beta}$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. Suppose $\inf_{x \in G} (\nu(x) - \alpha p(x)) > 0$ and $\inf_{x \in G} ((\alpha + \theta) p(x) - \nu(x)) > 0$. Further suppose $\inf_{x \in G} (\nu(x) - (\alpha - 1)q(x)) > 0$ and $\inf_{x \in G} (\alpha q(x) - \nu(x)) > 0$. Then there exists a constant $C > 0$ such that

$$|b(x)f(x) - b(z)f(z)| \leq C \{ |x - z|^{\alpha - \nu(x)/p(x) + \theta} + |x - z|^{\alpha - \nu(x)/q(x)} \}
\leq \frac{1}{2} \leq \frac{s}{r} \leq 2.$$ 

for all $x, z \in G$ and measurable functions $f$ on $G$ with $\|f\|_{L^{\nu(x)}}(G) \leq 1$.

Proof. Since $\inf_{x \in G} ((\alpha + \theta) p(x) - \nu(x)) > 0$, taking $\varepsilon_1$ such that $0 < \varepsilon_1 < \alpha - \nu(x)/p(x) + \theta$, there exists a constant $c_1 > 0$ such that

$$s_1^{\alpha - \nu(x)/p(x) + \theta - \varepsilon_1} (\log(e + 1/s_1))^{\beta} \leq c_1 s_2^{\alpha - \nu(x)/p(x) + \theta - \varepsilon_1} (\log(e + 1/s_2))^{\beta}$$

whenever $0 < s_1 \leq s_2$ (see e.g. [17, 28]). Therefore we have

$$\int_{s_1}^{4k_2 r} \int_{s_2}^{4k_2 r} s^{\alpha - \nu(x)/p(x) + \theta} (\log(e + 1/s))^\beta \frac{ds}{s}$$

$$\leq c_1 (4k_2 r)^{\alpha - \nu(x)/p(x) + \theta - \varepsilon_1} (\log(e + 1/(4k_2 r)))^{\beta} \int_{s_1}^{4k_2 r} s^{\varepsilon_1} \frac{ds}{s}$$

$$\leq C r^{\alpha - \nu(x)/p(x) + \theta} (\log(e + 1/r))^{\beta}$$

and, similarly we obtain

$$\int_{s_1}^{6k_2 r} \int_{s_2}^{6k_2 r} s^{\alpha - \nu(x)/p(x) + \theta} (\log(e + 1/s))^\beta \frac{ds}{s} \leq C r^{\alpha - \nu(x)/p(x) + \theta} (\log(e + 1/r))^{\beta}.$$
Since $\inf_{x \in G}(\alpha q(x) - \nu(x)) > 0$, we obtain
\[
\int_0^{4k_2r} s^{\alpha - \nu(x)/q(x)}(\log(e + 1/s))^\beta \frac{ds}{s} \leq C r^{\alpha - \nu(x)/q(x)}(\log(e + 1/r))^\beta
\]
and
\[
\int_0^{6k_2r} s^{\alpha - \nu(x)/q(x)}(\log(e + 1/s))^\beta \frac{ds}{s} \leq C r^{\alpha - \nu(x)/q(x)}(\log(e + 1/r))^\beta.
\]
Since $\inf_{x \in G}(\nu(x) - \alpha p(x)) > 0$, taking $\varepsilon_2$ such that $0 < \varepsilon_2 < \nu(z)/p(z) - \alpha$, there exists a constant $c_2 > 0$ such that
\[
s_2^{\alpha - \nu(z)/p(z) + \varepsilon_2}(\log(e + 1/s_2))^\beta \leq c_2 s_1^{\alpha - \nu(z)/p(z) + \varepsilon_2}(\log(e + 1/s_1))^\beta
\]
whenever $0 < s_1 \leq s_2$, so that
\[
n \int_{k_1r}^{4k_2d_G} s^{\alpha - \nu(z)/p(z)}(\log(e + 1/s))^\beta \frac{ds}{s}
\leq c_2 r^{\varepsilon_2}(k_1r)^{\alpha - \nu(z)/p(z) + \varepsilon_2}(\log(e + 1/(k_1r)))^\beta \int_{k_1r}^{4k_2d_G} s^{-\varepsilon_2} \frac{ds}{s}
\leq C r^{\alpha - \nu(z)/p(z) + \varepsilon_2}(\log(e + 1/r))^\beta.
\]
We also have
\[
r \int_{2k_1r}^{4k_2d_G} s^{\alpha - \nu(x)/p(x) - 1 + \theta}(\log(e + 1/s))^\beta \frac{ds}{s}
\leq C r^{\alpha - \nu(x)/p(x) + \theta}(\log(e + 1/r))^\beta,
\]
since $\inf_{x \in G}(\nu(x) - (\alpha + \theta - 1)p(x)) \geq \inf_{x \in G}(\nu(x) - \alpha p(x)) > 0$, and
\[
r \int_{2k_1r}^{4k_2d_G} s^{\alpha - \nu(x)/q(x) - 1}(\log(e + 1/s))^\beta \frac{ds}{s} \leq C r^{\alpha - \nu(x)/q(x)}(\log(e + 1/r))^\beta,
\]
since $\inf_{x \in G}(\nu(x) - (\alpha - 1)q(x)) > 0$.

Collecting these facts, we obtain by our assumptions
\[
\psi(x, z, r) \leq C \left\{ r^{\alpha - \nu(x)/p(x) + \theta} + r^{\alpha - \nu(x)/q(x)} + r^{\alpha - \nu(z)/p(z) + \theta} + r^{\alpha - \nu(z)/q(z)} \right\}(\log(e + 1/r))^\beta < \infty
\]
for $x, z \in G$ and $0 < r \leq d_G$. By Theorem 2.2, we obtain the required result.

\[\square\]

**Corollary 3.2.** Suppose $\inf_{x \in G}(\nu(x) - \alpha p(x)) > 0$ and $\inf_{x \in G}(\nu(x) - \nu(x)) > 0$. Further suppose $\inf_{x \in G}(\nu(x) - (\alpha - 1)q(x)) > 0$ and $\inf_{x \in G}(\alpha q(x) - \nu(x)) > 0$. Then there exists a constant $C > 0$ such that
\[
|b(x)I_\alpha f(x) - b(z)I_\alpha f(z)| \leq C \left\{ |x - z|^{\alpha - \nu(x)/p(x) + \theta} + |x - z|^{\alpha - \nu(x)/q(x)} + |x - z|^{\alpha - \nu(z)/p(z) + \theta} + |x - z|^{\alpha - \nu(z)/q(z)} \right\}
\]
for all $x, z \in G$ and measurable functions $f$ on $G$ with $\|f\|_{L^1(G)} \leq 1$. 

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PROOF. This is the case \( \beta = 0 \) in Corollary 3.1. 

Compare Corollaries 3.1 and 3.2 with [12, Theorem 4.1].

**Corollary 3.3.** Let \( \rho(r) = r^\alpha e^{-1/r} \) be as in Example 1.1. Then there exists a constant \( C > 0 \) such that

\[
|b(x)I_{\rho}f(x) - b(z)I_{\rho}f(z)| \leq C|x - z|^{\theta}
\]

for all \( x, z \in G \) and measurable functions \( f \) on \( G \) with \( \|f\|_{L_{\rho}(G)} \leq 1 \).

**Proof.** Since, for \( a \in \mathbb{R} \), there exists a constant \( c > 0 \) such that

\[
\int_0^r s^\alpha e^{-1/s} \frac{ds}{s} \leq cr^\theta
\]

for all \( 0 < r \leq d_G \), it follows from Remark 2.3 that

\[
\psi(x, z, r) \leq C(r + r^\theta) \leq Cr^\theta
\]

for all \( x, z \in G \) and \( 0 < r \leq d_G \), since \( \theta \in (0, 1] \). Hence, we obtain the required inequality.

**Corollary 3.4.** Let \( \rho(r) = r^\alpha (\log(e + 1/r))^\beta \) for \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). Suppose \( \inf_{x \in G} (\nu(x) - (\alpha - 1)p(x)) > 0 \) and \( \inf_{x \in G} (\alpha p(x) - \nu(x)) > 0 \). Then there exists a constant \( C > 0 \) such that

\[
\left| I_{\rho}f(x) - I_{\rho}f(z) \right| \\
\leq C \left\{ |x - z|^{\alpha - \nu(x)/p(x)} + |x - z|^{\alpha - \nu(z)/p(z)} \right\} \left( \log(e + 1/||x - z||) \right)^\beta
\]

for all \( x, z \in G \) and measurable functions \( f \) on \( G \) with \( \|f\|_{L^p(\nu)(G)} \leq 1 \).

**Proof.** To show this, we take \( b(\cdot) \equiv 1 \) and \( q(\cdot) = p(\cdot) \) in the proof of Theorem 2.2. As in the proof of Corollary 3.1, we obtain the result.

4. Lemmas

Before giving a proof of Theorem 2.2, we prepare two lemmas. To prove the following lemma, (P2) and (Q2) were used.

**Lemma 4.1** ([13, Lemma 2.1], cf. [14, Lemma 2.7]). There exists a constant \( C > 0 \) such that

\[
\frac{\rho^{\nu(x)/p(x)}}{|B(x, r)|} \int_{G \cap B(x, r)} |f(y)| \, dy \leq C
\]

for all \( x \in G \), \( 0 < r < d_G \) and measurable functions \( f \) on \( G \) with \( \|f\|_{L^p(\nu)(G)} \leq 1 \).
Let $\tau > 0$, $\beta \in \mathbb{R}$ and $\rho \in (\rho)$. Let $f$ be a nonnegative function on $G$ such that $\|f\|_{L^p(\nu(G))} \leq 1$. Then there exists a constant $C > 0$ such that

$$
\int_{G \cap B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}} \, dy \leq C \int_{0}^{2k_2r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s}
$$

and

$$
\int_{G \setminus B(x,r)} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}} \, dy \leq C \int_{k_1\tau r}^{4k_2\tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s}
$$

for all $x \in G$ and $0 < r \leq d_G$, where $k_1$ and $k_2$ are constants in $(\rho)$.

Proof. Let $f$ be a nonnegative function on $G$ such that $\|f\|_{L^p(\nu(G))} \leq 1$. Take $\gamma \in \mathbb{R}$ such that $1 < \gamma \leq \min\{1/k, 2\}$. If $y \in G \cap (B(x, \gamma^j r) \setminus B(x, \gamma^{j-1} r))$ for $j \in \mathbb{Z}$, then a geometric observation and (1.1) show

$$
\frac{\rho(\tau|x-y|)}{|x-y|^{N+\beta}} \leq \frac{\max\{1, \gamma^N + \beta\}}{(\gamma^j r)^{N+\beta}} \sup_{\gamma^{j-1} \leq s \leq \gamma^j} \rho(s)
$$

by $\gamma \leq 1/k$. By Lemma 4.1, we have

$$
\frac{1}{|B(x, \gamma^j r)|} \int_{G \cap B(x, \gamma^j r)} f(y) \, dy \leq C_1 (\gamma^j r)^{-\nu(x)/p(x)}
$$

for some constant $C_1 > 0$, so that

$$
\int_{G \cap (B(x, \gamma^j r) \setminus B(x, \gamma^{j-1} r))} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}} \, dy
\leq C_1 \sigma_N \max\{1, \gamma^{2N+\beta}\} \log^{2\gamma^j} \frac{\tau k_2 r}{(\gamma^j r)^{\beta}} \rho(s) \frac{ds}{s} \sup_{G \cap B(x, \gamma^j r)} \frac{1}{|B(x, \gamma^j r)|} \int_{G \cap B(x, \gamma^j r)} f(y) \, dy
\leq C_1 \sigma_N \max\{1, 2^{N+\beta}\} \log^{2\gamma^j} \frac{\tau k_2 r}{(\gamma^j r)^{\beta}} \rho(s) \frac{ds}{s} C_1 (\gamma^j r)^{-\nu(x)/p(x)}
\leq C_1 \sigma_N \max\{1, 2^{N+\beta}\} \log^{2\gamma^j} \frac{\tau k_2 r}{(\gamma^j r)^{\beta}} \rho(s) \frac{ds}{s}
\leq C_1 \sigma_N \max\{1, 2^{N+\beta}\} \log^{2\gamma^j} \frac{\tau k_2 r}{(\gamma^j r)^{\beta}} \rho(s) \frac{ds}{s}
\times \int_{\gamma^{j} k_1 \tau r}^{\gamma^{j} k_2 \tau r} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s}
$$
Therefore we have
\[
\int_{\gamma/k_{1}^{r}} \int_{\gamma/k_{2}^{r}} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s},
\]
which proves (4.1).

Let \( j_{0} \) be the smallest integer such that \( k_{2}/k_{1} \leq \gamma^{j_{0}} \). Using (4.3), we obtain
\[
\int_{G \cap \{B(x, r) \setminus B(x, \gamma^{j_{0}-1}r)\}} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}} \, dy \leq C_{2} \int_{\gamma/k_{1}^{r}} \int_{\gamma/k_{2}^{r}} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s},
\]
Let \( j_{1} \) be the smallest integer such that \( d_{G} \leq \gamma^{j_{1}r} \). If we use (4.3),
\[
\int_{G \setminus \{B(x, r) \setminus B(x, \gamma^{j_{1}-1}r)\}} \frac{\rho(\tau|x-y|)f(y)}{|x-y|^{N+\beta}} \, dy \leq C_{2} \int_{\gamma/k_{1}^{r}} \int_{\gamma/k_{2}^{r}} s^{-\nu(x)/p(x)-\beta} \rho(s) \frac{ds}{s},
\]
which proves (4.1).
Thus, (4.2) follows.

5. PROOF OF THEOREM 2.2

Without loss of generality, we can assume that \( f \) is a nonnegative function on \( G \) such that \( \| f \|_{L^{p,\infty}(G)} \leq 1 \). First note from (2.1) that for \( x, z \in G \) and \( r = |x - z| \)

\[
|b(x)I_{\rho}f(x) - b(z)I_{\rho}f(z)|
\leq b(x) \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)f(y)}{|x - y|^N} \, dy
\]

\[
+ b(z) \int_{G \cap B(x,2r)} \frac{\rho(|z - y|)f(y)}{|z - y|^N} \, dy
\]

\[
+ |b(x) - b(z)| \int_{G \cap B(x,2r)} \frac{\rho(|z - y|)f(y)}{|z - y|^N} \, dy
\]

\[
+ b(x) \int_{G \backslash B(x,3r)} \left| \rho(|x - y|) - \rho(|z - y|) \right| f(y) \, dy
\]

\[
\leq C \left\{ b(x) \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)f(y)}{|x - y|^N} \, dy
\]

\[
+ b(z) \int_{G \cap B(x,2r)} \frac{\rho(|z - y|)f(y)}{|z - y|^N} \, dy + r^\theta \int_{G \backslash B(x,r)} \frac{\rho(|z - y|)f(y)}{|z - y|^N} \, dy
\]

\[
+ r^\eta b(x) \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)f(y)}{|x - y|^{|N + \eta_2|}} \, dy \right\}
\]

\[
= C \left\{ I_1(x) + I_1(z) + I_2(z) + I_3(x) \right\}.
\]

For \( I_1(x) \), we have

\[
I_1(x) \leq \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)}{|x - y|^N} |b(x) - b(y)|f(y) \, dy
\]

\[
+ \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)}{|x - y|^N} b(y)f(y) \, dy
\]

\[
\leq C \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)f(y)}{|x - y|^N} \, dy + \int_{G \cap B(x,2r)} \frac{\rho(|x - y|)b(y)f(y)}{|x - y|^N} \, dy
\]

\[
= CI_{11}(x) + I_{12}(x).
\]

By (4.1), we obtain

\[
I_{11}(x) \leq C \int_0^{4kr} s^{-\nu(x)/p(x)} + \theta \rho(s) \frac{ds}{s},
\]
\[ I_{12}(x) \leq C \int_{0}^{4k_{2}r} s^{-\nu(x)/q(s)} \rho(s) \frac{ds}{s}. \]

For \( \tilde{I}_1(z) \), we have by (4.1)
\[
\tilde{I}_1(z) \leq C \left\{ \int_{0}^{6k_{2}r} s^{-\nu(z)/p(s) + \theta} \rho(s) \frac{ds}{s} + \int_{0}^{6k_{2}r} s^{-\nu(z)/q(s)} \rho(s) \frac{ds}{s} \right\},
\]
as in the estimate of \( I_{11}(x) \) and \( I_{12}(x) \).

For \( I_2(z) \), we have by (4.2)
\[
I_2(z) \leq Cr^{\theta} \int_{k_{1}r}^{4k_{2}dG} s^{-\nu(z)/p(s)} \rho(s) \frac{ds}{s}.
\]

Finally, for \( I_3(x) \) we have
\[
I_3(x) \leq r^{m_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|)}{|x-y|^{N+\eta_2}} |b(x) - b(y)| f(y) dy
+ r^{m_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|)}{|x-y|^{N+\eta_2}} b(y) f(y) dy
\leq Cr^{m_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|) f(y)}{|x-y|^{N-\theta+\eta_2}} dy
+ r^{m_1} \int_{G \setminus B(x,2r)} \frac{\rho(\tau|x-y|) \{b(y) f(y)\}}{|x-y|^{N+\eta_2}} dy
= CI_{31}(x) + I_{32}(x).
\]

Note from (4.2) that
\[ I_{31}(x) \leq Cr^{m_1} \int_{2k_{1}r}^{4k_{2}dG} s^{-\nu(x)/p(x)-\eta_2 + \theta} \rho(s) \frac{ds}{s} \]
and
\[ I_{32}(x) \leq Cr^{m_1} \int_{2k_{1}r}^{4k_{2}dG} s^{-\nu(x)/q(x)-\eta_2} \rho(s) \frac{ds}{s} \]

Collecting these facts, we obtain
\[ |b(x)I_{\rho}f(x) - b(z)I_{\rho}f(z)| \leq C\psi(x,z,r). \]

Thus this theorem is proved.

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