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SEMIFLOWS AND INTRINSIC SHAPE IN TOPOLOGICAL SPACES

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Abstract. In this paper we apply the intrinsic approach to shape to study attractors in topological spaces.

1. Introduction

The use of shape theory in the study of dynamical systems was initiated by Hastings in [14]. He was the first to successfully apply the techniques of shape theory to dynamical systems. In his paper [14] he developed an analogue of the Poincare-Bendixson theorem in Euclidean n-space using shape theory and gave several examples. The author duly noted that the usual proof of this theorem breaks down in higher dimensions because the main tool, the Jordan curve theorem (a simple closed curve in the plane divides the plane into two parts) cannot be extended. He circumvent this problem by replacing a geometric description of an invariant set K by a description of its shape.

The result of Hastings was the first one of a series of papers by different authors who analyze similar situations. For example Kapitanski and Rodnianski in [15] studied the shape of attractors of semiflows on complete metric spaces. They proved (using the notion of bounded attraction) that the global attractor of a semiflow shares the shape of the phase space. A slightly generalized version of this result in [28] suggests that the global topological properties of attractors are largely determined by those of their region of attraction. Other authors have also shown how to apply shape theory to obtain global properties of attractors in the papers [2, 11, 12] but always in the framework of metrizable spaces. The attractor theory on metrizable spaces has been fully

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developed for both autonomous and nonautonomous systems. It seems that when the attractor exists, shape theory is specially adequate to get topological invariants shared by a metrizable phase space and the attractor of a semiflow defined on it. Let us also mention that shape theory was related with differential equations in [22] and it is the main tool used in [18, 24] to define a Conley index for discrete dynamical systems.

Initially, shape theory was introduced by Borsuk in [1] in order to study geometric properties of compact metric spaces with not necessarily good local behavior. Mardešić in [17] gave an extension of Borsuk's shape theory to include all topological spaces. Shortly after Rubin and Sanders in [23] gave a different extension to the realm of Hausdorff spaces (as mentioned in [9]) called "compactly generated shape" or shortly H-shape.

On the other hand motivated by the possibility of needing non-metrizable phase spaces in the mathematical formulation of some natural phenomena [3], Li, Wang and Xiong have introduced in [16] the notion of attractor for a semidynamical system defined in a topological space. Based on this notion they developed a basic theory of attractors for local semiflows on topological spaces. Let us only mention that this general context is not vacuous, since semiflows appear naturally in practice, while abstract topological spaces arise, for instance, in compactifications of dynamical systems.

More general results concerning global topological structure of attractors were established in the paper [9] which surely confirms that shape theory is becoming an important tool in the study of topological dynamics. In particular, in this paper the authors A. Giraldo, M.A. Morón, F.R. Ruiz Del Portal and J.M.R. Sanjurjo show how the compactly generated shape, introduced by Rubin and Sanders in [23] can be used to study the topological structure of a global attractor (using the notion of compact attraction) compared to that of the phase space. Namely, they prove a very general result assuring that the compactly generated shape of the global attractor and that of the phase space are the same in the realm of Hausdorff spaces. Also they study the connectedness properties for attractors in Hausdorff k-spaces.

The main aim of this paper is to show how the intrinsic approach to shape developed by Čerin in [5] for topological spaces (which is equivalent to the Mardešić extension of Borsuk's shape theory to topological spaces) can be used to study the topological structure of a global attractor (in the sence of Li, Wang and Xiong) compared to that of the phase space. In particular, we prove a general result claiming that the intrinsic shape of the global attractor and that of the phase space are the same in the realm of perfectly normal Hausdorff spaces hence improving some of the previous results on shape of attractors in metrizable spaces. Also we include some results concerning the connectedness properties for attractors in perfectly normal Hausdorff spaces.

2. Intrinsic shape for topological spaces

The classical homotopy theory studies the equivalence relation of homotopy for maps. The equivalence relation of homotopy for maps leads to a useful and rich theory only when we restrict to spaces with nice local properties like polyhedra and absolute neighborhood retracts. The problem arise when the target space Y is such that there are not many maps from $X \times I$ into Y so that the properties of Y are preventing identification of maps which ought to be identified. In other words the definition of homotopy is too rigid because the map $H : X \times I \to Y$ must be continuous and single-valued and because it must take values in the space Y.

Shape theory was introduced by Borsuk [1] in order to study geometric properties of compact metric spaces with not necessarily good local properties. Namely, homotopy theory turns out to be inappropriate tool for studying spaces with local pathology which appear in the mathematical formulation of many natural phenomena, for example solenoids, attractors etc. Hence, it is natural to look for another adequate tool for handling these problems. Shape theory takes the role in this context because manages to smooth out local pathologies while preserving global properties. Besides, shape theory does not modify homotopy theory in the good framework.

The modification of Borsuk relies on the idea to relinquish the insistence in the definition of homotopy that the map H goes precisely into the space Y. The obvious alternative method which was undertaken by Sanjurjo in [27] and [29] and further followed in the paper [5] is to give up with the requirement that the function H is continuous and (or) single-valued while retaining the desirable condition that it takes values in the space Y. The last one is known as the intrinsic approach to shape theory.

The first intrinsic approach to shape is given in the papers [8] and [26]. In the paper [5] using the notion of a multi-net or M-net over normal coverings intrinsic shape category is constructed for arbitrary topological spaces.

We shall follow the construction given in [5] for topological spaces using the notion of γ - small functions.

Let $\hat{Y}$ denote the collection of all normal covers of a topological space Y. With respect to the refinement relation $\geq$ the set $\hat{Y}$ is a directed set. Two normal covers σ and τ are equivalent provided $\sigma \geq \tau$ and $\tau \geq \sigma$. In order to simplify our notation we denote a normal cover and it's equivalence class by the same symbol. Consequently, $\hat{Y}$ also stands for the associated quotient set.

Let $Inc(Y)$ denote the collection of all finite subsets c of $\hat{Y}$ which have a unique (with respect to the refinement relation) maximal element $\tilde{c} \in \hat{Y}$. We consider $Inc(Y)$ ordered by the inclusion relation and regard $\hat{Y}$ as a subset of single-element subsets of $Inc(Y)$. Notice that $Inc(Y)$ is a cofinite directed set.

Lemma 2.1. *Let $\{f_1, f_2, \dots, f_n\}$ be a finite collections of functions from a cofinite directed set $(M, \leq)$ into a directed set $(L, \leq)$. Then there is an increasing function $g : M \to L$ such that $g(x) \geq f_1(x), \dots, f_n(x)$ for every $x \in M$.*

If τ is a covering of Y and $V \in \tau$ the open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \tau$ such that $W \cap V \neq \emptyset$. We form a new covering of Y, $st(\tau) = \{\mathrm{st}(V) | V \in \tau\}$.

If σ is a normal cover of a space Y let σ^* denotes the set of all normal covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all normal covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

The next two definitions introduce precisely a type of multi-valued functions that we shall use.

Definition 2.2. *Let X and Y be topological spaces. By a multi-valued function or an M-function $F : X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X. Let $M(X, Y)$ denote all M-functions from X into Y.*

For our approach to shape theory the following notion of size for multi-valued functions will play the most important role.

Definition 2.3. *Let $F : X \to Y$ be a multi-valued function and let $\alpha \in \hat{X}$ and $\gamma \in \hat{Y}$. We shall say that F is a multi-valued (α, γ)-function provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subseteq C_A$. On the other hand, F is γ-small function provided there is an $\alpha \in \hat{X}$ such that F is an (α, γ)-function.*

Also important will be the following concept of closeness for two multi-valued functions.

Definition 2.4. *Let $F, G : X \to Y$ be multi-valued functions and let $\gamma \in \hat{Y}$. We shall say that F and G are γ-close and we write $F \overset{\gamma}{=} G$ provided for every $x \in X$ there is $C_x \in \gamma$ with $F(x) \cup G(x) \subseteq C_x$.*

The following definition is the most important for the intrinsic approach to shape theory.

Definition 2.5. *Let $F, G : X \to Y$ be multi-valued functions between topological spaces and let γ be a normal cover of the space Y. We shall say that F and G are γ-homotopic and write $F \overset{\gamma}{\simeq} G$ provided there is a γ-small multi-valued function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) \subseteq H(x, 0)$ and $G(x) \subseteq H(x, 1)$ for every $x \in X$. We shall say that H is a γ-homotopy that joins F and G or that it realizes the relation (or homotopy) $F \overset{\gamma}{\simeq} G$.*

The following lemma is crucial because it provides an adequate substitute for the transitivity of the relation γ-homotopy.

Lemma 2.6. *Let $F, G, H : X \to Y$ be multi-valued functions. Let $\sigma \in \hat{Y}$ and $\tau \in \sigma^*$. If $F \overset{\tau}{\simeq} G$ and $G \overset{\tau}{\simeq} H$, then $F \overset{\sigma}{\simeq} H$.*

2.1. *Multi-nets.* The following two definitions correspond to Borsuk's definitions of fundamental sequences and homotopy for fundamental sequences.

Definition 2.7. *Let X and Y be topological spaces. By a multi-net or an M-net from X into Y we shall mean a collection $\varphi = \{F_c \mid c \in Inc(Y)\}$ of multi-valued functions $F_c : X \to Y$ such that for every $\gamma \in \hat{Y}$ there is a $c \in Inc(Y)$ with $F_d \overset{\gamma}{\simeq} F_c$ for every $d \geq c$. We use functional notation $\varphi : X \to Y$ to indicate that φ is a multi-net from X into Y. Let $MN(X, Y)$ denote all multi-nets $\varphi : X \to Y$.*

Definition 2.8. *Two multi-nets $\varphi = \{F_c\}$ and $\phi = \{G_c\}$ between topological spaces X and Y are homotopic provided for every $\gamma \in \hat{Y}$ there is a $c \in Inc(Y)$ such that $F_d \overset{\gamma}{\simeq} G_d$ for every $d \geq c$.*

It follows from lemma 2.6 that the relation of homotopy is an equivalence relation on the set $MN(X, Y)$. The homotopy class of a multi-net φ is denoted by $[\varphi]$ and the set of all homotopy classes by $HM(X, Y)$.

Our goal now is to define a composition for homotopy classes of multi-nets.

Let $\varphi = \{F_c\} : X \to Y$ be a multi-net. For every $c \in Inc(Y)$ there is an $\bar{f}(c) \in Inc(Y)$ such that for all $d, e \geq \bar{f}(c)$ there is a normal cover $\bar{f}(c, d, e)$ of the cylinder $X \times I$ and an $(\bar{f}(c, d, e), \tilde{c})$-function joining F_d and F_e, where $\tilde{c}$ denotes the unique(with respect to the refinement relation) maximal element of c.

Let $C = \{(c, d, e) \mid c \in Inc(Y), d, e \geq \bar{f}(c)\}$. Then C is a subset of $Inc(Y) \times Inc(Y) \times Inc(Y)$ that becomes a cofinite directed set when we define that $(c, d, e) \geq (c', d', e')$ iff $c \geq c'$, $d \geq d'$ and $e \geq e'$.

Now let $f : Inc(Y) \to Inc(Y)$ be an increasing function such that $f(c) \geq \bar{f}(c), c$ for every $c \in Inc(Y)$. We shall use the same notation f for an increasing function $f : C \to \widehat{X \times I}$ such that $f(c, d, e) \geq \bar{f}(c, d, e)$ for every $(c, d, e) \in C$. Let $(c, d, e) \in C$. For the normal cover $f(c, d, e)$ of $X \times I$ by ([6] p.385), there is a normal cover $\epsilon = \hat{f}(c, d, e)$ of X and a function $r = \tilde{f}(c, d, e) : \epsilon \to \{2, 3, 4, \ldots\}$ such that every set $E \times [(i-1)/r(E), (i+1)/r(E)]$, where $E \in \epsilon$ and $i = 1, 2, \ldots, r(E) - 1$ is contained in a member of $f(c, d, e)$.

Let $\tilde{f} : C \to \hat{X}$ be an increasing function with $\tilde{f}(c, d, e) \geq \hat{f}(c, d, e)$ for every $(c, d, e) \in C$. We shall use the shorter notation $\tilde{f}(c)$ and $f(c)$ for the covers $\tilde{f}(c, f(c), f(c))$ and $f(c, f(c), f(c))$.

Proposition 2.9. *There is an increasing function $f^* : Inc(Y) \to \hat{X}$ such that*

1) *$f^*(c) \geq \tilde{f}(c)$ for every $c \in Inc(Y)$, and*

2) *f^* is cofinal in $\tilde{f}$, i.e., for every $(c, d, e) \in C$ there is an $m \in Inc(Y)$ with $f^*(m) \geq \tilde{f}(c, d, e)$.*

The above discussion shows that every multi-net $\varphi : X \to Y$ determines two increasing functions $f : Inc(Y) \to Inc(Y)$ and $f^* : Inc(Y) \to \hat{X}$. With the help of these functions we shall define the composition of homotopy classes of multi-nets as follows.

Let $\varphi = \{F_c\} : X \to Y$ and $\phi = \{G_s\} : Y \to Z$ be multi-nets. Let $\chi = \{H_s\} : X \to Z$, where $H_s = G_{g(s)} \circ F_{f(g^*(s))}$ for every $s \in Inc(Z)$.

Proposition 2.10. *The collection χ is a multi-net from X into Z.*

Topological spaces and homotopy classes of multi-nets $[(F_c)]$ form the category whose isomorphisms induce classifications which coincide with the standard shape classification, i.e., isomorphic spaces in this category have the same shape.

3. Attractors in topological spaces

The main reference for the elementary concepts of dynamical systems will be [4] but we also recommend [21, 20, 19]. Let us recall some of the basic notions from this theory. Let X be a Hausdorff topological space. A flow in X is a continuous map $\Phi : X \times \mathbb{R} \to X$ that satisfies the following two conditions:

$$\Phi(x, 0) = x, \quad \Phi(\Phi(x, t), s) = \Phi(x, t + s) \qquad \text{for all } x \in X \text{ and } t, s \in \mathbb{R}.$$

The triplet $(X, \mathbb{R}, \Phi)$ forms a dynamical system (flow) with phase map Φ and phase space X. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ we get the corresponding notion of semi-dynamical system (semi-flow).

The trajectory of a point x is the set $\gamma(x) = \{\Phi(x, t) | t \in \mathbb{R}\}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$ we obtain the corresponding notions of positive and negative semi-trajectory. We denote by $\gamma^+(x)$ and $\gamma^-(x)$ correspondingly. For every $t \in \mathbb{R}$ we will consider the map $\Phi_t : X \to X$ defined by $\Phi_t(x) = \Phi(x, t)$.

Definition 3.1. *We say that a given subset $M \subseteq X$ is positively (respectively negatively) invariant under the semi-flow Φ if $\Phi(M, t) \subseteq M$ (respectively $M \subseteq \Phi(M, t)$), for all $t \geq 0$. M is said to be invariant, if it is both negatively and positively invariant.*

Before introducing the concept of a global attractor we need the following definition.

Definition 3.2. *A set $M \subseteq X$ attracts a set $C \subseteq X$ if for every neighborhood U of M there exists $T \in \mathbb{R}$ such that $\Phi_t(C) \subseteq U$, for every $t \geq T$.*

Recall that a set $M \subseteq X$ is said to be sequentially compact (s-compact in short), if each sequence x_n in M has a subsequence converging to a point $x \in M$.

Definition 3.3. *A nonempty compact and s-compact invariant set M is called an attractor of Φ, if there is a neighborhood $\mathcal{U}$ of M such that*

1) *M attracts $\mathcal{U}$; and*
2) *M is the maximal s-compact invariant set in $\mathcal{U}$.*

Such a neighborhood is called an attracting neighborhood for M. Given an attractor M, define

$$\mathcal{A}(M) = \{x \in X \mid M \, attracts \; x\}.$$

$\mathcal{A}(M)$ is called the region of attraction of M. If $\mathcal{A}(M) = X$, then M is simply called a global attractor.

Theorem 3.4. ([16]) *Let M be an attractor of Φ. Then $\mathcal{A}(M)$ is open, and for each compact set $K \subseteq \mathcal{A}(M)$, M attracts a neighborhood $\mathcal{U}$ of K.*

Remark 3.5. Let us note that a global attractor in the above terminology is also a global attractor in the terminology of a compact attraction, i.e. compact invariant set that attracts each compact set.

4. Shape of global attractors in topological spaces

In this paper we apply the theory of intrinsic shape to deduce a result for global attractor properties compared with the properties of the phase space in terms of shape theory.

In order to prove the main theorem we need the following.

A covering $\mathcal{V}$ of M in X is called **regular** if it satisfies the following conditions:

1) if $V \in \mathcal{V}$ then $V \cap M \neq \emptyset$,
2) if $U, V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

For a covering $\mathcal{V}$ of M we introduce the notation $|\mathcal{V}| = \bigcup_{V \in \mathcal{V}} V$.

For a finite regular covering $\mathcal{V}$ we define an M-function $r_{\mathcal{V}} : |\mathcal{V}| \to M$ in the following way:

- for points $y \in M$ we put $r_{\mathcal{V}}(y) = \{y\}$,
- for points $y \in |\mathcal{V}| \backslash M$, by induction we can choose the smallest member $V \in \mathcal{V}$ such that $y \in V$ and put $r_{\mathcal{V}}(y) = V \cap M$.

The function $r_{\mathcal{V}}$ is $\mathcal{V} \cap M$-small.

Lemma 4.1 ([30]). *Let $\mathcal{V}$ be a covering of the space Y. If $f, g : X \to Y$ are $\mathcal{V}$-close and $\mathcal{V}$-small then f and g are $st(\mathcal{V})$-homotopic.*

Lemma 4.2 ([31]). *Let $\mathcal{V}$ be a finite regular covering of M in X. Then $i \circ r_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ and $1_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ are $st(\mathcal{V})$-homotopic by a homotopy $R_{\mathcal{V}} : |\mathcal{V}| \times I \to |\mathcal{V}|$ such that $R_{\mathcal{V}}(x, t) = x$, for $x \in M$.*

We proceed by showing how a semiflow $(\Phi_t : X \to X)$ in a normal T_2- topological space X with a global attractor M induces a shape morphism

$X \to M$ in a natural way. First we shall assume that there exists an attracting neighborhood $\mathcal{U}$ for M such that $\mathcal{U} = X$.

REMARK 4.3. Let us note that in a normal T_2-topological space X the set of all finite regular coverings of the compact M in X is cofinal in the set of all coverings of M in X ([7, p.277]).

We choose an arbitrary increasing sequence $c(1) \leq c(2) \leq \cdots \leq c(n)$ of indices from the index (of covers) set $c(i) \in Inc(X)$ such that $|c(i)| = i$, for every $i = 1, 2, \ldots, n$. By induction we shall define a sequence of regular finite coverings $(\mathcal{V}_{c(i)})$ as follows. Let $c(n) = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\}$ where U_i, for $i = 1, 2, \ldots, n$ are normal coverings of X.

For $k = 1$ we introduce the notation $r(i) = \{\mathcal{U}_i\}$, for every $i = 1, 2, 3, \ldots, n$. According to the previous note there exists a finite sequence of finite regular coverings $\mathcal{W}_{r(1)}, \mathcal{W}_{r(2)}, \mathcal{W}_{r(3)}, \ldots, \mathcal{W}_{r(n)}$ such that $\mathcal{W}_{r(i)} \geq \mathcal{U}_i$, for every $i = 1, 2, \ldots, n$. We define $\mathcal{V}_{r(i)} = \mathcal{W}_{r(i)}$, for every $i = 1, 2, \ldots, n$. Let $c(1) = \{\mathcal{U}_i\}$, for some $i \in \{1, 2, 3, \ldots, n\}$. Then $\mathcal{V}_{c(1)} = \mathcal{V}_{r(i)}$.

Now let us choose an arbitrary sequence of indices $d(1) < d(2) < \cdots < d(n)$ such that $d(n) = c(n)$. We assume that for $k \leq n-1$ the finite sequence of finite regular coverings $\mathcal{V}_{d(1)}, \mathcal{V}_{d(2)}, \ldots, \mathcal{V}_{d(k)}$ is defined. Using again the previous note there exists a finite regular covering $\mathcal{W}$ of M in X such that for an arbitrary sequence of different indices $d(1) \leq d(2) \leq \ldots \leq d(n)$ such that $d(n) = c(n)$ the following holds $\mathcal{W} \geq \mathcal{V}_{d(1)} \cap \mathcal{V}_{d(2)} \cap \cdots \cap \mathcal{V}_{d(n-1)}$. We define $\mathcal{V}_{c(n)} = \mathcal{W}$.

Let us note that in this way for an arbitrary index $c \in Inc(X)$ we have defined a finite regular covering $\mathcal{V}_c$.

Now we are ready to define a net of multi-valued functions in the following manner:

Let $c \in Inc(X)$ be an arbitrary index. We consider the neighborhood $|\mathcal{V}_c| = \bigcup_{V \in \mathcal{V}_c} V$ of M. There exists t_c such that $\Phi(X, (t_c, \infty)) \subseteq |\mathcal{V}_c|$. Hence, we have defined a net of positive real numbers $(t_c | c \in Inc(X))$ such that the following property holds:

$$t_c \to \infty \text{ and } c \geq d \Rightarrow t_c \geq t_d.$$

Namely, let $c \geq d$, $c, d \in Inc(X)$. The induced covering from the index d, $\mathcal{V}_d$ is coarser than the induced covering from the index c, $\mathcal{V}_c$, i.e., $|\mathcal{V}_c| = \bigcup_{V \in \mathcal{V}_c} V \subseteq \bigcup_{V \in \mathcal{V}_d} V = |\mathcal{V}_d|$. Hence the conclusion follows.

CONSTRUCTION OF THE MULTI-NET

We choose an arbitrary index $c \in Inc(X)$. For $x \in X$ and $t \in [t_c, \infty)$ we define:

$$k_c(x, t) = r_{\mathcal{V}_c}(\Phi(x, t)),$$

where $\mathcal{V}_c$ is the finite regular covering adjoined to the index c.

Notice that if we introduce the notation $c = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\}$ then the function is $\mathcal{V}_c \cap M$-small and hence $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \cdots \cap \mathcal{U}_n$-small.

Now we define $f_c : X \to X$ by:

$$f_c(x) = r_{\mathcal{V}_c}(\Phi(x, t_c)).$$

Note that the composition is well defined. Since $\mathcal{V}_c \geq \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \cdots \cap \mathcal{U}_n$ the function f_c is $\tilde{c}$-small.

In this way we have defined a collection of M-functions $f_c : X \to X$ for arbitrary $c \in Inc(X)$ such that $f_c(X) \subseteq M$.

We are ready to state our main theorem.

THEOREM 4.4. *Let X be an arbitrary perfectly normal T_2-topological space and let $(\Phi_t : X \to X|\ t \in \mathbb{R}^+)$ be a semiflow with a global attractor M. Then the inclusion $i : M \to X$ induces a shape equivalence.*

PROOF. We will prove the theorem in two stages.

(i) First we will assume that the attracting neighborhood $\mathcal{U} = X$. Then we can use the previous construction of a family of M-functions $f_c : X \to X$ to formulate the following lemma.

LEMMA 4.5. *The net of functions $\varphi = \{f_c \mid c \in Inc(X)\} : X \to X$ is a multi-net.*

PROOF. We choose an arbitrary normal covering $\mathcal{V}$ of X. We need to find $c \in Inc(X)$ such that $f_d \overset{\mathcal{V}}{\simeq} f_c$ for every $d \geq c$. Let $\mathcal{W} \in \mathcal{V}^{*2}$ and define $c = \{\mathcal{W}\}$. For $d \geq c$ we will define a homotopy $r_{cd} : X \times I \to X$ in the following manner.

We define $R_{cd} : X \times I \to X$ by $R_{cd}(x, t) = k_c(x, (1 - t)t_c + tt_d)$.Then $R_{cd}(x, 0) = f_c(x)$ and $R_{cd}(x, 1) = k_c(x, t_d)$. From $k_c(x, t_d) = r_{\mathcal{V}_c}\Phi(x, t_d)$ and $k_d(x, t_d) = r_{\mathcal{V}_d}\Phi(x, t_d)$ it follows that the two functions $f_d, l : X \to X$ defined by:

$$l(x) = k_c(x, t_d) \text{ and } f_d(x) = k_d(x, t_d),$$

are $\mathcal{W} \cap M$-close and by lemma 4.1 are $st(\mathcal{W} \cap M)$-homotopic say by a homotopy h_c. By $\mathcal{W} \cap M$ is denoted shortly the covering of M defined by $\mathcal{W} \cap M = \{W \cap M | W \in \mathcal{W}\}$(see remark 4.7). Then the concatenation of the homotopies $r_{cd} = R_{cd} * h_c$ is the required $\mathcal{V}$-homotopy satisfying $r_{cd}(x, 0) = f_c(x)$ and $r_{cd}(x, 1) = f_d(x)$. ∎

We will need the following theorem from [13] in the remark that follows.

THEOREM 4.6. *A topological space X is normal if and only if each finite open covering is normal.*

REMARK 4.7. If $\mathcal{V}$ is a normal covering of Y and $M \subseteq Y$ then by $\mathcal{V} \cap M$ we denote the following normal covering of M:

$$\mathcal{V} \cap M = \{V \cap M | V \in \mathcal{V}\}.$$

If a multi-net $(f_c|c \in Inc(Y)) : X \to Y$ satisfies $f_c(X) \subseteq M$, for every $c \in Inc(Y)$ and if for arbitrary normal covering $\mathcal{V}$ of Y, there exists $c \in Inc(Y)$

such that f_d and f_c are $\mathcal{V}$-homotopic in M, for every $d \geq c$ then we can define a multi-net $(f_b | b \in Inc(M)) : X \to M$ in the following way.

For an index $b = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\} \in Inc(M)$ we choose an index $c = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n\} \in Inc(Y)$ such that $\mathcal{V}_i \cap M = \mathcal{U}_i$, for every i=1,2,...,n.

Then the function $f_b : X \to M$ is defined by

$$f_b(x) = f_c(x), \text{ for } x \in X.$$

Then $(f_b) : X \to M$ is a multi-net.

We say that the multi-net $(f_b) : X \to M$ is inherited from $(f_c) : X \to Y$ and the inherited multi-net we denote by $(f_{c \cap M} | c \in Inc(Y))$.

Remark 4.8. Let us note that the inclusion $i : M \to X$ induces a multi-net by setting $i_s(x) = \{x\}$, for every $x \in M$. It is easy to prove that the collection $(i_s \mid s \in Inc(X))$ is a multi-net.

Proposition 4.9. *The shape morphism* $[(f_{c \cap M})] : X \to M$ *is a shape equivalence. Consequently* $Sh(M) = Sh(X)$.

Proof. First we will prove that the composition of the classes of homotopy of the multi-nets of M-functions $\varphi = (f_c \mid c \in Inc(M)) : X \to M$ and $i = (i_s \mid s \in Inc(X)) : M \to X$ satisfy the following equality $[\varphi] \circ [i] = [1_M]$.

According to the definition of composition $[\varphi] \circ [i] = [h]$, where $h = (h_c \mid c \in Inc(M))$ and $h_c = f_{f(c)} \circ i_{g(f^*(c))} : M \to M$, for arbitrary $c \in Inc(M)$, where f, g, f^* are the induced increasing functions from proposition 2.9. Let us note that

$$h_c(x) = f_{f(c)}(i_{g(f^*(c))}(x)) = f_{f(c)}(x) = r_{\mathcal{V}_{f(c)}}(\Phi(x, t_{f(c)})) = \Phi(x, t_{f(c)}).$$

We choose an arbitrary normal covering $\mathcal{V}$ of M. We need to find $c \in Inc(M)$ such that $h_d \overset{\mathcal{V}}{\simeq} 1_d$, for $d \geq c$. Let $c = \{\mathcal{V}\}$. We define a homotopy $H : M \times I \to M$ by

$$H(x, t) = \Phi(x, (1-t)t_{f(d)}).$$

Let us note that H is a $\mathcal{V}$-small homotopy which connects h_d and 1_d, for arbitrary $d \geq c$. Hence $[h] = [1_M]$.

Now we shall prove that $[i] \circ [\varphi] = [1_X]$ which concludes our proof. Again according to the definition of composition $[i] \circ [\varphi] = [h]$, where $h = (h_s \mid s \in Inc(X))$ and $h_s = i_{g(s)} \circ f_{f(g^*(s))} : X \to X$, for arbitrary $s \in Inc(X)$, where g, f, g^* are the induced increasing functions from proposition 2.9.

We choose an arbitrary normal covering $\mathcal{V}$ of X. We need to find $s \in Inc(X)$ such that $h_d \overset{\mathcal{V}}{\simeq} 1_d$, for $d \geq s$. Let $s = \{\mathcal{V}\}$. Let us note that from the construction of the multi-nets φ and i we can choose the functions g, f, g^* by:

For arbitrary $c = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\} \in Inc(M)$ we define $c_n = c$ and $c_i = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_i\}$ for $i = 1, 2, \ldots, n-1$. We introduce the sets $T_i = st^{-2}(\tilde{c}_i) = \{\mathcal{Q} \mid st^2(\mathcal{Q}) \geq \tilde{c}_i\}$, for $i = 1, 2, \ldots, n$. Now we choose $\mathcal{W}_i \in T_i$ and define $f(c) = \mathcal{W}_1 \cap \mathcal{W}_2 \cap \cdots \cap \mathcal{W}_n$.

Further we can define g, g^* by

$$g(p) = p \text{ and if } p = \{\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n\} \text{ then } g^*(p) = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \cdots \cap \mathcal{Q}_n \cap M.$$

Let $d = \{\mathcal{V}, \mathcal{P}_1, \dots, \mathcal{P}_n\}$ be an arbitrary index such that $d \geq s$. We have that

$$h_d = i_d \circ f_{f(g^*(d))} = i \circ f_{f(g^*(d))} = f_{f(\mathcal{V} \cap \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n \cap M)} = f_{\mathcal{W} \cap M},$$

where $\mathcal{W}$ is a normal cover of X which can be chosen such that $st^2(\mathcal{W}) \geq \mathcal{V} \cap \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n$. Let us consider the finite regular covering $\mathcal{V}_{\mathcal{W}}$ adjoined to the index $c = \{\mathcal{W}\}$. From the definition we have that the following inequality holds $\mathcal{V}_{\mathcal{W}} \geq \mathcal{W}$. For simplicity we put $\mathcal{V}_{\mathcal{W}} = \mathcal{G}$.

Mention that since $r_{\mathcal{G}}, R_{\mathcal{G}}$ are $st(\mathcal{G})$- small, then they will be $st(\mathcal{W})$- small as well.

We will define a homotopy $H_{\mathcal{V}} : X \times I \to X$ as concatenation of three homotopies. The first is a continuous map $F : X \times I \to X$, defined by

$$F(x, s) = \Phi(x, st_{\mathcal{W}}).$$

This map satisfies

$$F(x, 0) = x, \; F(x, 1) = \Phi(x, t_{\mathcal{W}}).$$

The third is a function $G : X \times I \to X$ defined by

$$G(x, s) = r_{\mathcal{G}} \Phi(x, t_{\mathcal{W}}).$$

The composition $r_{\mathcal{G}}\Phi$ is $\mathcal{W}$- small and this map satisfies

$$G(x, 0) = r_{\mathcal{G}} \Phi(x, t_{\mathcal{W}}), \qquad G(x, 1) = r_{\mathcal{G}} \Phi(x, t_{\mathcal{W}}) = f_{\mathcal{W} \cap M}(x).$$

The middle homotopy $Q : X \times I \to X$ is defined by

$$Q(x, s) = R_{\mathcal{G}}(\Phi(x, t_{\mathcal{W}}), s),$$

where $R_{\mathcal{G}}$ is the homotopy from lemma 4.2. This homotopy satisfies

$$Q(x, 0) = F(x, 1) = \Phi(x, t_{\mathcal{W}}), \; Q(x, 1) = G(x, 0) = r_{\mathcal{G}} \Phi(x, t_{\mathcal{W}}).$$

Since

$$F(x, 1) = Q(x, 0) \text{ and } Q(x, 1) = G(x, 0),$$

we can define concatenation of the three defined homotopies and finally define the required $\mathcal{V}$- homotopy $H_{\mathcal{V}} : X \times I \to X$ by $H_{\mathcal{V}} = G * Q * F$ and

$$H_{\mathcal{V}}(x, 0) = x = 1_d(x), \; H_{\mathcal{V}}(x, 1) = f_{\mathcal{W} \cap M}(x) = h_d(x).$$

$\square$

(ii) Now let us discuss the second case. We assume that the attracting neighborhood $\mathcal{U} \neq X$. A nonnegative function $\chi \in C(\mathcal{A}(M))$ is called a $\mathcal{K}_0$ function of M if

$$\chi(x) = 0 \text{ if and only if } x \in M.$$

REMARK 4.10. The existence of a $\mathcal{K}_0$ function is a natural equivalent of the distance function in metrizable spaces. For T_6-spaces such functions exist.

We will need the following theorem from [16].

THEOREM 4.11. *Assume that M is closed and has a $\mathcal{K}_0$ function χ on $\mathcal{A}(M)$. Then M has a Lyapunov function L on $\mathcal{A}(M)$.*

If we assume, in addition, that X is normal then for any closed set $K \subseteq \mathcal{A}(M) \setminus M$, M has a Lyapunov function L on $\mathcal{A}(M)$ with $L(x) \geq 1$ for every $x \in K$.

Now using the above theorem 4.11 for $K = X \setminus \mathcal{U} \neq \emptyset$ which is closed there exists a Lyapunov function L such that $L(x) \geq 1$, for every $x \in K$.

Using the standard technique of primitive Lyapunov functions we get a deformation retract from X to a closed and positively invariant neighborhood P of M.

Namely, let $\tau(x) = \sup_{t\geq 0} \chi(\Phi(x,t))$. We define the function

$$L(x) = \int_0^\infty \exp^{-t} \tau(\Phi(x,t))\, dt,$$

which is a primitive Lyapunov function for the semi-dynamical system $(X, \mathbb{R}^+, \Phi)$. We choose $0 < \epsilon < 1 \leq \sup_{x\in X} L(x)$ and define the set

$$P = \{x \in X |\, L(x) \leq \epsilon\},$$

which is positively invariant, closed and is contained in the attracting neighborhood $\mathcal{U}$, i.e. $P \subseteq \mathcal{U}$. Moreover the set M is a global attractor for the semi-dynamical system $(P, \mathbb{R}^+, \Phi)$. Hence the inclusion $i : M \to P$ is a shape equivalence. For arbitrary $x \in X \setminus P$ there is a unique $t = t_x$ such that $L(\Phi(x, t_x)) = \epsilon$.

We define the following map:

$$m(x) = \begin{cases} t_x, & \text{if } x \in X \setminus P, \\ 0, & \text{otherwise} \end{cases}$$

(the map is continuous). Finally we define a map $r : X \to P$ by

$$r(x) = \Phi(x, m(x)).$$

Hence P is a deformation retract of X. □

Now we apply the main theorem to obtain information about the shape of a neighborhood attractor.

DEFINITION 4.12. *A set M is said to be admissible, if for any sequences $x_n \in M$ and $t_n \to \infty$ with $\Phi(x_n, [0, t_n]) \subset M$ for all n, the sequence $\Phi(x_n, t_n)$ has a convergent subsequence.*

Recall that for a given subset $M \subseteq X$ its omega-limit set is defined by:

$$\omega(M) = \{x \in X | \exists x_n \in M, t_n \to \infty, \Phi(x_n, t_n) \to x\}.$$

The alpha limit set $\alpha(M)$ is defined analogous.

COROLLARY 4.13 (Shape of a neighborhood attractor). *Suppose X is normal T_2-space. Let M be a closed subset of X. Assume that M attracts an admissible compact neighborhood N of itself. Then $Sh(\omega(N)) = Sh(N)$.*

PROOF. Let us note that $\omega(N) \neq \emptyset$ which is a simple consequence of the admissibility of N. We will prove that $\omega(N)$ is a compact set. Let $\mathcal{U} = \{U_\alpha | \alpha \in J\}$ be an arbitrary open covering for $\omega(N)$. Using the normality of X there exists an open set V such that $\omega(N) \subset V \subset \overline{V} \subset \bigcup_{\alpha \in J} U_\alpha$. Then the collection $\{X \setminus \overline{V}, U_\alpha | \alpha \in J\}$ is an open covering for N. From the compactness of N there exists a finite subcovering for N, say $X \setminus \overline{V}, U_1, U_2, \ldots, U_n$. Now from the fact that $\omega(N) \subset N$ and $X \setminus \overline{V} \cap \omega(N) = \emptyset$ the conclusion follows. Let us also note that $\omega(N)$ attracts N. Namely, if we suppose the opposite then there exists an open neighborhood V of $\omega(N)$ such that there exists $x_n \in N$ and $t_n \to \infty$ satisfying $\Phi(x_n, t_n) \notin V$ for every $n \in \mathbb{N}$. Since M attracts N there exists $T > 0$ such that $\Phi(N, [T, \infty)) \subset N$. Let us note that $\Phi(x_n, t_n) = \Phi(\Phi(x_n, T), t_n - T)$ for sufficiently large n (we want $t_n - T > 0$ to be fulfilled), so if we choose the sequence $y_n = \Phi(x_n, T)$ and $\tau_n = t_n - T$ then we have that $\Phi(y_n, \tau_n) \notin V$ and $\Phi(y_n, [0, \tau_n]) \subset N$ for every $n \in \mathbb{N}$. Now using the admissibility of N we can assume that $\Phi(y_n, \tau_n)$ converges to a point z, i.e. $\Phi(y_n, \tau_n) \to z$. But $z \in N \setminus V$ which contradicts the fact that $z \in \omega(N)$. Now the conclusion follows from proposition in [16] and our previous theorem. ◻

EXAMPLE 4.14. Let us define a topology on the real line $\mathbb{R}$ with the generalized Sierpinski topology, i.e. let $\tau = \{\emptyset, \mathbb{R}\} \cup \mathcal{P}(\mathbb{Q}_n)$, where $\mathcal{P}(\mathbb{Q}_n)$ is the partitive set of a finite subset $\mathbb{Q}_n = \{0, q_1, q_2, \ldots, q_n\}$ of the rational numbers $\mathbb{Q}$. We define a map $\Phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\Phi(x,t) = \begin{cases} x, & t = 0, x \in \mathbb{R}, \\ -\sqrt{2}, & x \in \mathbb{R} \setminus \mathbb{Q}_n, x < 0, t > 0, \\ \sqrt{2}, & x \in \mathbb{R} \setminus \mathbb{Q}_n, x > 0, t > 0, \\ x, & x \in \mathbb{Q}_n, t \geq 0. \end{cases}$$

Let us note that Φ is a continuous semi-flow on $(\mathbb{R}, \tau)$. We shall consider the following set $M = \{-\sqrt{2}, \sqrt{2}\} \cup \mathbb{Q}_n$. Note that M is compact, s-compact and also invariant set. Also let us notice that M attracts the neighborhood $\mathcal{U} = \mathbb{R}$. Namely, for an arbitrary neighborhood V of M, which according to the topology in this case coincides with $V = \mathbb{R}$, there exists $T > 0$ such that for every $t \geq T$, $\Phi(\mathcal{U}, t) \subset V = \mathbb{R}$. Also let us note that M is maximal s-compact invariant subset in $\mathcal{U} = \mathbb{R}$. Hence M is a global attractor for the semi-flow on $\mathbb{R}$ defined by Φ. Notice that $Sh(\mathbb{R}) = Sh(M)$.

4.1. *Connectedness properties of attractors in topological spaces.* We will give a result concerning connectedness properties of global attractors in perfectly normal or T_6- spaces. Examples of perfectly normal spaces are metric spaces and paracompact Hausdorff $\mathcal{G}_\delta$ spaces. We will use some of the results from [16].

The following theorem is improvement to some of the results given in [10] for metric spaces.

Theorem 4.15. *Let X be a Hausdorff perfectly normal space and let $\{\Phi_t : X \to X | t \in \mathbb{R}^+\}$ be a semi-flow with a global attractor M. Then X is connected if and only if M is connected.*

Proof. Let us suppose that M is not connected. Then $M = M_1 \cup M_2$ with M_1 and M_2 open in M. Then there exist U_1 and U_2 open subsets of X such that $M_1 \subset U_1, M_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Since M is a global attractor, for arbitrary $x \in X$, M attracts $\{x\}$, so for given $U_1 \cup U_2$, neighborhood of M, there exists $T_x \in \mathbb{R}$ such that $\Phi(x,t) \in U_1 \cup U_2$ for every $t \geq T_x$. On the other hand since $\{x\}[T_x, \infty)$ is connected, there exists $i \in \{1, 2\}$ such that $\Phi(x,t) \in U_i$ for every $t \geq T_x$. Consider

$$A_i = \{x \in X | \text{ there exists } T_x \in \mathbb{R} \text{ such that } \Phi(x,t) \in U_i \text{ for every } t \geq T_x\},$$

for $i = 1, 2$. Then $X = A_1 \cup A_2$ and $M_i \subset A_i$ and therefore, $A_i \neq \emptyset$ for $i = 1, 2$. We will prove that A_1 is closed and the same will hold for A_2 as well. Let us choose an arbitrary sequence $x_n \in A_1$ such that converges to p, i.e. $x_n \to p$. Suppose that $p \notin A_1$. Then $p \in A_2$. Hence there exists $T_p \in \mathbb{R}$ such that $\Phi(p,t) \in U_2$ for every $t \geq T_p$. On the other hand using theorem 4.11 for $K = X \setminus (U_1 \cup U_2) \neq \emptyset$ (in the opposite case X is disconnected and the proof is complete) there exists a Lyapunov function $L : X \to \mathbb{R}$ such that $L(x) \geq 1$, for every $x \in K$. Note that M attracts $\{p\}$ hence there exists $t_p \in \mathbb{R}, t_p \geq T_p$ such that $L(\Phi(p, t_p)) < \frac{1}{2}$. From the continuity of L and Φ there exists a neighborhood V of p such that $L(\Phi(x, t_p)) < \frac{1}{2}$ for every $x \in V$. Also from the continuity of Φ there exists a neighborhood W of p such that $\Phi(x, t_p) \in U_2$ for every $x \in W$. Now for sufficiently large n, $x_n \in V \cap W$ hence $L(\Phi(x_n, t_p) < \frac{1}{2}$ and $\Phi(x_n, t_p) \in U_2$. On the other hand from the fact that $x_n \in A_1$ there exists $t \geq t_p$ such that $\Phi(x_n, t) \in U_1$. Let us consider the connected set $Q = x_n[t_p, \infty)$. If $z \in K \cap Q \neq \emptyset$ then from the property of the Lyapunov function we have that $L(z) < \frac{1}{2}$ but from 4.11 we have that $L(z) \geq 1$, a contradiction. So the only possibility left is $Q \subset U_1 \cup U_2$. But from $Q \cap U_1 \neq \emptyset, Q \cap U_2 \neq \emptyset$ we obtain a disconnection of Q. Again a contradiction. So we conclude that A_1 is closed. Hence X is not connected.

Therefore, we have proved that if X is connected then so is M. The converse result is obvious. ∎

Remark 4.16. A slight modification of the above proof permits to obtain a bijection φ between the spaces of components of the global attractor M and X such that the inclusion $i : M_0 \to \varphi(M_0)$ induces a shape equivalence.

References

[1] K. Borsuk, Theory of shape, PWN–Polish Scientific Publishers, Warsaw, 1975.

[2] B. A. Bogatyi and V. I. Gutsu, *On the structure of attracting compacta*, Differentsial'nye Uravneniya **25** (1989), 907–909, 920.

[3] J. M. Ball, *Stability theory for an extensible beam*, J. Differential Equations **14** (1973), 399–418.

[4] N. P. Bhatia and G. P. Szegö, Stability theory of dynamical systems, Springer-Verlag, New York-Berlin, 1970.

[5] Z. Čerin, *Shape via multi-nets*, Tsukuba J. Math. **19** (1995), 245–268.

[6] A. Dold, Lectures on algebraic topology, Springer-Verlag, New York-Berlin, 1972.

[7] S. Eilenberg and N. Steenrod, Foundations of algebraic topology, **15**, Princeton University Press, New Jersey, 1952.

[8] J. E. Felt, *ε-continuity and shape*, Procedings of the American Mathematical society **46** (1974), 426–430.

[9] A. Giraldo, M. A. Morón, F. R. Ruiz Del Portal and J. M. R. Sanjurjo, *Shape of global attractors in topological spaces*, Nonlinear Anal. **60** (2005), 837–847.

[10] M. Gobbino and M. Sardella, *On the connectedness of attractors for dynamical systems*, J. Differential Equations **133** (1997), 1–14.

[11] B. Günter and J. Segal, *Every attractor of a flow on a manifold has the shape of a finite polyhedron*, Proc. Amer. Math. Soc. **119** (1993), 321–329.

[12] A. Giraldo and J. M. R. Sanjurjo, *On the global structure of invariant regions of flows with asymptotically stable attractors*, Math. Z. **232** (1999), 739–746.

[13] N. R. Howes, Modern analysis and topology, Springer-Verlag, New York, 1995.

[14] H. M. Hastings, *A higher-dimensional Poincare-Bendixson theorem*, Glas. Mat. Ser. III **14(34)** (1979), 263–268.

[15] L. Kapitanski and I. Rodnianski, *Shape and Morse theory of attractors*, Comm. Pure Appl. Math. **53** (2000), 218–242.

[16] D. Li, J. Wang and Y. Xiong, *Attractors of local semiflows on topological spaces*, J. Korean Math. Soc. **54** (2017), 773–791.

[17] S. Mardešić, *Shapes for topological spaces*, General Topology and Appl. **3** (1973), 265–282.

[18] M. Mrozek, *Shape index and other indices of Conley type for local maps on locally compact Hausdorff spaces*, Fund. Math. **145** (1994), 15–37.

[19] V. V. Nemytskii and V. V. Stepanov, Kacestvennaya teoriya differencial'nyh urovnenii, OGIZ, Moscow, 1947, Qualitative theory of differential equations, Princeton University Press, Princeton, 1960.

[20] S. Y. Pilyugin, Introduction to structurally stable systems of differential equations, Birkhaüser Verlag, 1992.

[21] J. C. Robinson, Infinite dimensional dynamical systems, Cambridge University Press, Cambridge, 2001.

[22] J. T.Roggers, *The shape of a cross-section of the solution funnel of an ordinary differential equation*, Illinois J. Math. **21** (1977), 420–426.

[23] L. Rubin and J. Sanders, *Compactly generated shape*, General Topology and Appl. **4** (1974), 73–83.

[24] J. W. Robbin and D. Salomon, *Dynamical systems, shape theory and the Conley index*, Ergodic Theory Dynam. Systems 8* (1988), 375–393.

[25] J. J. Sánchez-Gabites, *Dynamical systems and shape*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **102** (2008), 127–159.

[26] J. M. R. Sanjurjo, *A noncontinuous description of the shape category of compacta*, Quart. J. Math. Oxford Ser. (2) **40** (1989), 351–359.

[27] J. M. R. Sanjurjo, *Shape morphisms and small multivalued maps*, Math. Japon. **35** (1990), 713–717.
[28] J. M. R. Sanjurjo, *Stability, attraction and shape: a topological study of flows*, in Topological methods in nonlinear analysis, Juliusz Schauder Cent. Nonlinear Stud., Toruń, 2011, 93–122.
[29] J. M. R. Sanjurjo, *An intrinsic description of shape*, Trans. Amer. Math. Soc. **329** (1992), 625–636.
[30] N. Shekutkovski, *One property of components of chain recurrent set*, Regul. Chaotic Dyn. **20** (2015), 184–188.
[31] N. Shekutkovski and M. Shoptrajanov, *Intrinsic shape of the chain recurrent set*, Topology Appl. **202** (2016), 117–126.

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